Mapping theorems for $C$-spaces

We assume that all spaces are normal unless otherwise stated. We refer the readers to [2] for dimension theory.

In this note we study mapping theorems for $C$-spaces.

A space $X$ is a $C$-space (an $A$-weakly infinite-dimensional space) if for every countable collection $\{G_i : i \in \mathbb{N}\}$ of open covers (two-element open covers, respectively) of $X$ there exists a countable collection $\{H_i : i \in \mathbb{N}\}$ of collections of pairwise disjoint open subsets of $X$ such that $H_i < G_i$ for every $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} H_i$ covers $X$ (cf. [1]).

Evidently, every $C$-space is $A$-weakly infinite-dimensional. However, it is not known whether the converse is true.

Polkowski [5] proved the following theorem.

**Theorem 1** (Polkowski [5]). If $f : X \to Y$ is a closed mapping of an $A$-weakly infinite-dimensional countably paracompact space $X$ onto a space $Y$ and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $Y$ is $A$-weakly infinite-dimensional.

We proved that the following theorem, which is an analogous result for $C$-spaces.
Theorem 2. If \( f : X \rightarrow Y \) is a closed mapping of a countably paracompact C-space \( X \) onto a paracompact space \( Y \) and there exists an integer \( k \geq 1 \) such that \( |f^{-1}(y)| \leq k \) for every \( y \in Y \), then \( Y \) is a C-space.

Problem. Does theorem 1 (or theorem 2) hold for closed mappings with finite fibers?

In [4], Pol proved the following theorem.

Theorem 3 (Pol [4]). If \( f : X \rightarrow Y \) is a continuous mapping of a compact metrizable space \( X \) onto a metrizable space \( Y \) such that \( |f^{-1}(y)| \leq \aleph_0 \) for every \( y \in Y \), then \( X \) is an A-weakly infinite-dimensional space (resp. a C-space) if and only if \( Y \) is an A-weakly infinite-dimensional space (resp. a C-space).

Does Theorem 3 remain true if we replace \(|f^{-1}(y)| \leq \aleph_0\) by \(|f^{-1}(y)| < c\)? In [5], Polkowski proved the following theorem.

Theorem 4 (Polkowski [5]). If \( f : X \rightarrow Y \) is a continuous mapping of a compact A-weakly infinite-dimensional space \( X \) onto a space \( Y \) such that \( |f^{-1}(y)| < c \) for every \( y \in Y \), then \( Y \) is A-weakly infinite-dimensional.

Similarly, the following theorem holds.

Theorem 5. If \( f : X \rightarrow Y \) is a continuous mapping of a compact C-space \( X \) onto a space \( Y \) such that \( |f^{-1}(y)| < c \) for every \( y \in Y \), then \( Y \) is a C-space.

On the other hand, Hattori and Yamada proved that the following theorem.

Theorem 6 (Hattori and Yamada [3]).

(i) If \( f : X \rightarrow Y \) is a closed mapping of a countably paracompact (or hereditarily normal) space \( X \) onto a C-space \( Y \) such that \( f^{-1}(y) \) is A-weakly infinite-dimensional for every \( y \in Y \), then \( X \) is A-weakly infinite-dimensional.

(ii) If \( f : X \rightarrow Y \) is a closed mapping of a paracompact space \( X \) onto a C-space \( Y \) such that \( f^{-1}(y) \) is a C-space for every \( y \in Y \), then \( X \) is a C-space.

Problem. Does Theorem 6(i) remain true if we replace '\( f^{-1}(y) \) is a C-space' by '\( f^{-1}(y) \) is A-weakly infinite-dimensional'?
References


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