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On special metrics characterizing \( \omega_1 \)-strongly countable-dimensional spaces

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1 Introduction

In this note, we characterize the class of \( \omega_1 \)-strongly countable-dimensional metrizable spaces by a special metric. A characterization of locally finite-dimensional metrizable spaces is also obtained.

If every finite open cover of a metrizable space \( X \) has a finite open refinement of order \( \leq n + 1 \), then \( X \) has covering dimension \( \leq n \), \( \dim X \leq n \). For \( \varepsilon > 0 \), we let \( S_\varepsilon(x) \) denote the \( \varepsilon \)-ball \( \{ y \in X \mid \rho(x, y) < \varepsilon \} \) about \( x \).

In [5], [6] and [7], J. Nagata gave a characterization of metrizable spaces of \( \dim \leq n \) by a special metric.

**Theorem 1.1** (J. Nagata [5], [6], [7]) The following conditions are equivalent for a metrizable space \( X \):

1. \( \dim X \leq n \).
2. There is an admissible metric \( \rho \) satisfying the following condition: for every \( \varepsilon > 0 \), every point \( x \) of \( X \) and every \( n + 2 \) many points \( y_1, \ldots, y_{n+2} \) of \( X \) with \( \rho(S_{\varepsilon/2}(x), y_i) < \varepsilon \) for each \( i = 1, \ldots, n + 2 \), there are distinct natural numbers \( i \) and \( j \) such that \( \rho(y_i, y_j) < \varepsilon \).
3. There is an admissible metric \( \rho \) satisfying the following condition: for every point \( x \) of \( X \) and every \( n + 2 \) many points \( y_1, \ldots, y_{n+2} \) of \( X \), there are distinct natural numbers \( i \) and \( j \) such that \( \rho(y_i, y_j) \leq \rho(x, y_k) \).

For the case of the separable metrizable spaces, J. de Groot [2] gave the following characterization.

**Theorem 1.2** (J. de Groot [2]) A separable metrizable space \( X \) has \( \dim X \leq n \) if and only if \( X \) can introduce an admissible totally bounded metric satisfying the following condition:

For every point \( x \) of \( X \) and every \( n + 2 \) many points \( y_1, \ldots, y_{n+2} \) of \( X \), there are natural numbers \( i, j \) and \( k \) such that \( i \neq j \) and \( \rho(y_i, y_j) \leq \rho(x, y_k) \).

Let \( \mathbb{N} \) denote the set of all natural numbers. A metrizable space \( X \) is the countable sum of finite-dimensional closed sets, we call \( X \) a strongly countable-dimensional.

In [8], J. Nagata extended Theorems 1.1 and 1.2 to strongly countable-dimensional metrizable spaces.
Theorem 1.3 (J. Nagata [8]) The following conditions are equivalent for a metrizable space $X$:

1. $X$ is strongly countable-dimensional.
2. There is an admissible metric $\rho$ satisfying the following condition: for every point $x$ of $X$, there is an $n(x) \in \mathbb{N}$ such that for every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) \leq \rho(x, y_j)$.
3. There is an admissible metric $\rho$ satisfying the following condition: for every point $x$ of $X$, there is an $n(x) \in \mathbb{N}$ such that for every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are natural numbers $i$, $j$ and $k$ such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x, y_k)$.

In [3], Y. Hattori characterized the class of strongly countable-dimensional spaces by extending the condition (2) of Theorem 1.1.

Theorem 1.4 (Y. Hattori [3]) A metrizable space $X$ is strongly countable-dimensional if and only if $X$ can introduce an admissible metric $\rho$ satisfying the following condition:

For every point $x$ of $X$, there is an $n(x) \in \mathbb{N}$ such that for every $\epsilon > 0$, and every $n+2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$ with $\rho(S_{\epsilon/2}(x), y_i) < \epsilon$ for each $i = 1, \ldots, n(x) + 2$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) < \epsilon$.

2 A characterization of $\omega_1$-strongly countable-dimensional spaces

In this section, we characterize the class of $\omega_1$-strongly countable-dimensional metrizable spaces by a special metric. A characterization of locally finite-dimensional metrizable spaces is also obtained. Theorem 2.4 and Theorem 2.5 are main theorems.

Definition 2.1 A metrizable space $X$ is locally finite-dimensional if for every point $x \in X$ there exists an open subspace $U$ of $X$ such that $x \in U$ and $\dim U < \infty$.

The first infinite ordinal number is denoted by $\omega$ and $\omega_1$ is the first uncountable ordinal number.

Definition 2.2 A metrizable space $X$ is called an $\omega_1$-strongly countable-dimensional space if $X = \bigcup \{P_\xi \mid 0 \leq \xi < \xi_0\}$, $\xi_0 < \omega_1$, where $P_\xi$ is an open subset of $X - \bigcup \{P_\eta \mid 0 \leq \eta < \xi\}$ and $\dim P_\xi < \infty$. 

For a metrizable space $X$ and a non-negative integer $n$, we put

$$P_n(X) = \bigcup \{U \mid U \text{ is an open subspace of } X \text{ and } \dim U \leq n\}.$$ 

We notice that for each ordinal number $\alpha$, we can put $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number or 0 and $n(\alpha)$ is a non-negative integer.

**Definition 2.3** Let $X$ be a metrizable space and $\alpha$ either an ordinal number $\geq 0$ or the integer $-1$. Then *strong small transfinite dimension* $\text{sind}$ of $X$ is defined as follows:

1. $\text{sind}X = -1$ if and only if $X = \emptyset$.
2. $\text{sind}X \leq \alpha$ if $X$ is expressed in the form $X = \bigcup \{P_{\xi} \mid \xi < \alpha\}$, where $P_{\xi} = P:\{P_{\eta} \mid \eta < \lambda(\xi)\}$.

Furthermore, if $\text{sind}X$ is defined, we say that $X$ has *strong small transfinite dimension*.

Clearly, a metrizable space $X$ is locally finite-dimensional if and only if $\text{sind}X \leq \omega$ (R. Engelking [1]). And $X$ is $\omega_1$-strongly countable-dimensional if and only if there is a $\xi_0 < \omega_1$ such that $\text{sind}X \leq \xi_0$.

Theorem 2.4 is one of the main theorems. Thus we characterize the class of $\omega_1$-strongly countable-dimensional metrizable spaces by a special metric.

**Theorem 2.4** The following conditions are equivalent for a metrizable space $X$:

(a) $X$ is an $\omega_1$-strongly countable-dimensional space.

(b) There are an admissible metric $\rho$ for $X$, an ordinal number $\alpha < \omega_1$ and a family $\{X_{\beta} \mid 0 \leq \beta \leq \alpha\}$ of closed sets of $X$ satisfying the following conditions: (b-1) $X_0 = X$, $X_{\beta} \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_{\beta} = \bigcap \{X_{\beta'} \mid \beta' < \beta\}$ if $\beta$ is a limit. (b-2) For every point $x$ of $X$ there is an open neighborhood $U(x)$ of $x$ in $X_{\beta(x)}$, where $\beta(x) = \max \{\beta \mid x \in X_{\beta}\}$, and an $n(x) \in \mathbb{N}$ such that for every $\epsilon > 0$, every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, ..., y_{n(x)+2}$ of $X$ with $\rho(S_{\epsilon/2}(x'), y_i) < \epsilon$ for each $i = 1, ..., n(x) + 2$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) < \epsilon$.

(c) There are an admissible metric $\rho$ for $X$, an ordinal number $\alpha < \omega_1$ and a family $\{X_{\beta} \mid 0 \leq \beta \leq \alpha\}$ of closed sets of $X$ satisfying the following conditions: (c-1) $X_0 = X$, $X_{\beta} \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_{\beta} = \bigcap \{X_{\beta'} \mid \beta' < \beta\}$ if $\beta$ is a limit. (c-2) For every point $x$ of $X$ there are an open neighborhood $U(x)$ of $x$ in $X_{\beta(x)}$, where $\beta(x) = \max \{\beta \mid x \in X_{\beta}\}$, and an $n(x) \in \mathbb{N}$ such that for every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, ..., y_{n(x)+2}$ of $X$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) \leq \rho(x', y_j)$.

(d) There are an admissible metric $\rho$ for $X$, an ordinal number $\alpha < \omega_1$ and a family $\{X_{\beta} \mid 0 \leq \beta \leq \alpha\}$ of closed sets of $X$ satisfying the following conditions: (d-1) $X_0 = X$, $X_{\beta} \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_{\beta} = \bigcap \{X_{\beta'} \mid \beta' < \beta\}$ if $\beta$ is a limit. (d-2)
For every point $x$ of $X$ there are an open neighborhood $U(x)$ of $x$ in $X_{\beta(x)}$, where 
$\beta(x) = \max\{\beta \mid x \in X_{\beta}\}$, and an $n(x) \in \mathbb{N}$ such that for every point $x'$ of $U(x)$ and 
every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are natural numbers $i, j$ and 
k such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x', y_k)$. 

Also Theorem 2.5 is one of main theorems. We characterize the class of locally 
finit-dimensional metrizable spaces by a special metric.

**Theorem 2.5** The following conditions are equivalent for a metrizable space $X$:

(a) $X$ is a locally finite-dimensional space.

(b) There is an admissible metric $\rho$ for $X$ satisfying the following conditions: For every point $x$ of $X$, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of $x$ in $X$ such that for every $\epsilon > 0$, every point $x'$ of $U(x)$ and every $n(x) + 2$ many 
points $y_1, \ldots, y_{n(x)+2}$ of $X$ with $\rho(S_{\epsilon/2}(x'), y_i) < \epsilon$ for each $i = 1, \ldots, n(x) + 2$, there 
are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) < \epsilon$.

(c) There is an admissible metric $\rho$ for $X$ satisfying the following conditions: For every point $x$ of $X$, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of $x$ in $X$ such that for every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) \leq \rho(x', y_j)$.

(d) There is an admissible metric $\rho$ for $X$ satisfying the following conditions: For every point $x$ of $X$, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of $x$ in $X$ such that for every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are natural numbers $i, j$ and $k$ such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x', y_k)$.

To obtain those theorems, we need the following lemmas and theorems. Essentially, the following lemma is the same as [3; Lemma 1.5]. By a minor modification in the proof of [3; Lemma 1.5], we obtain the following lemma.

**Lemma 2.6** ([3; Lemma 2.5], [8; Lemma 1]) Let $n$ be a non-negative integer and 
let $\{F_m \mid m = 0, 1, \ldots\}$ be a closed cover of a metrizable space $X$ such that $\dim F_m \leq (n - 1) + m, F_m \subset F_{m+1}$ for $m = 0, 1, \ldots$. Then for every open cover $\mathcal{U}$ of $X$, there 
are a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of discrete families of open sets of $X$ and an open cover 
$\mathcal{W}$ of $X$ which satisfy the following conditions:

1. $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ is a cover of $X$.
2. $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ refines $\mathcal{U}$.
3. If $W \in \mathcal{W}$ satisfies $W \cap F_m \neq \emptyset$, then $W$ meets at most one member of 
$\mathcal{V}_k$ for $k \leq (n + 0) + (n + 1) + \ldots + (n + m)$ and meets no member of $\mathcal{V}_k$ for 
k > $(n + 0) + (n + 1) + \ldots + (n + m)$.

Let $Q^*$ denote the set of all rational numbers of the form $2^{-m_1} + \ldots + 2^{-m_t}$, where 
m_1, \ldots, m_t are natural numbers satisfying $1 \leq m_1 < \ldots < m_t$.

Essentially, the following lemma is the same as [3; Lemma 1.6]. By a minor modification in the proof of [3; Lemma 1.6], we obtain the following lemma.
Lemma 2.7 ([3; Lemma 2.6], [8; Lemma 3]) Let $n$ be a non-negative integer and let $\{F_m \mid m = 0, 1, \ldots \}$ be a closed cover of a metrizable space $X$ such that $\dim F_m \leq (n-1) + m$, $F_m \subset F_{m+1}$ for $m = 0, 1, \ldots$. Then for every $q \in Q^*$, there is an open cover $S(q)$ which satisfies the following conditions:

1. $S(q) = \bigcup_{i=1}^{\infty} S^i(q)$, where each $S^i(q)$ is discrete in $X$.
2. $\{St(x, S(q)) \mid q \in Q^* \}$ is a neighborhood base at $x \in X$.
3. Let $p$, $q \in Q^*$ and $p < q$. Then $S(p)$ refines $S(q)$.
4. Let $p$, $q \in Q^*$ and $p < q$. If $S_1 \in S^i(p)$ and $S_2 \in S^i(q)$, then $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$.
5. Let $p$, $q \in Q^*$ and $p + q < 1$. Let $S_1 \in S(p)$, $S_2 \in S(q)$ and $S_1 \cap S_2 \neq \emptyset$. Then there is an $S_3 \subset S(p+q)$ such that $S_1 \cup S_2 \subset S_3$.
6. For every $q \in Q^*$ and every $S \in \bigcup\{S^i(q) \mid i > (n+0)+(n+1)+\ldots+(n+m)\}$, $S \cap F_m = \emptyset$.

By Lemma 2.6 and Lemma 2.7, we obtain the following theorem.

Theorem 2.8 Let $\alpha$ be an ordinal number with $\alpha < \omega_1$ and let $n$ be a non-negative integer. The following conditions are equivalent for a metrizable space $X$:

(a) $\text{sind } X \leq \omega \alpha + n$.

(b) There are an admissible metric $\rho$ for $X$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of $X$ satisfying the following conditions: (b-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$, $X_\beta = \bigcap \{X_{\beta'} \mid \beta' < \beta\}$ if $\beta$ is a limit, and $X_\alpha = \emptyset$ if $n = 0$. (b-2) For every point $x$ of $X$ there are an open neighborhood $U(x)$ of $x$ in $X_\beta(x)$, where $\beta(x) = \max \{\beta \mid x \in X_\beta\}$, and an $n(x) \in N_{\beta(x)}$ such that for every $\epsilon > 0$, every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$ with $\rho(S_{\beta(x)}(x'), y_i) < \epsilon$ for each $i = 1, \ldots, n(x) + 2$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) < \epsilon$, where

$$N_{\beta(x)} = \begin{cases} N, & \text{if } \beta(x) < \alpha, \\ \{n-1\}, & \text{if } \beta(x) = \alpha. \end{cases}$$

(c) There are an admissible metric $\rho$ for $X$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of $X$ satisfying the following conditions: (c-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$, $X_\beta = \bigcap \{X_{\beta'} \mid \beta' < \beta\}$ if $\beta$ is a limit, and $X_\alpha = \emptyset$ if $n = 0$. (c-2) For every point $x$ of $X$ there are an open neighborhood $U(x)$ of $x$ in $X_\beta(x)$, where $\beta(x) = \max \{\beta \mid x \in X_\beta\}$, and an $n(x) \in N_{\beta(x)}$ such that for every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) \leq \rho(x', y_j)$, where

$$N_{\beta(x)} = \begin{cases} N, & \text{if } \beta(x) < \alpha, \\ \{n-1\}, & \text{if } \beta(x) = \alpha. \end{cases}$$
Remark 2.9 Let \( \{X_\beta \mid 0 \leq \beta \leq \alpha \} \) be a family of closed sets of \( X \) satisfying the condition (b-1). Then we shall show that for every point \( x \) of \( X \), there is a maximum element \( \beta(x) \) of \( \{\beta \mid x \in X_\beta\} \). Indeed, if \( x \in X_{\lambda(\alpha)} \), then \( \beta(x) = \max\{\beta \mid x \in X_\beta, \lambda(\alpha) \leq \beta \leq \alpha\} \). Now, we suppose that \( x \in X_{\lambda(\alpha)} \), there is a minimum element \( \beta_0 > 0 \) of \( \{\beta \mid x \not\in X_\beta\} \). Assume that \( \beta_0 \) is limit. By the condition (b-1), \( x \in \cap\{X_\beta \mid \beta < \beta_0\} = X_{\beta_0} \). This contradicts the definition of \( \beta_0 \). Therefore \( \beta_0 \) is not limit and hence \( \beta(x) = \beta_0 - 1 \).

By Theorems 1.2 and 2.8, we obtain the following theorem.

Theorem 2.10 Let \( \alpha \) be an ordinal number with \( \alpha < \omega_1 \) and let \( n \) be a non-negative integer. The following conditions are equivalent for a compact metrizable space \( X \):

(a) \( \mathrm{ind} \ X \leq \omega\alpha + n \).

(d) There are an admissible totally bounded metric \( \rho \) for \( X \) and a family \( \{X_\beta \mid 0 \leq \beta \leq \alpha\} \) of closed sets of \( X \) satisfying the following conditions: \( \mathrm{(d-1)} \) \( X_0 = X \), \( X_\beta \supset X_\beta' \) for \( \beta \leq \beta' \leq \alpha \), \( X_\beta = \cap\{X_\beta' \mid \beta' < \beta\} \) if \( \beta \) is a limit, and \( X_\alpha = \emptyset \) if \( n = 0 \). \( \mathrm{(d-2)} \) For every point \( x \) of \( X \) there are an open neighborhood \( U(x) \) of \( x \) in \( X_{\beta(x)} \), where \( \beta(x) = \max\{\beta \mid x \in X_\beta\} \), and an \( n(x) \in N_{\beta(x)} \) such that for every point \( x' \) of \( U(x) \) and every \( n(x) + 2 \) many points \( y_1, \ldots, y_{n(x)+2} \) of \( X \), there are natural numbers \( i, j \) and \( k \) such that \( i \neq j \) and \( \rho(y_i, y_j) \leq \rho(x', y_k) \), where

\[
N_{\beta(x)} = \begin{cases} 
\mathbb{N}, & \text{if } \beta(x) < \alpha, \\
\{n - 1\}, & \text{if } \beta(x) = \alpha.
\end{cases}
\]

By Theorems 2.8 and 2.10, we obtain the Main Theorem 2.4 and Theorem 2.5.

References


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