数理解析研究所講究録
第 1578 巻 2008 年 4-10

On special metrics characterizing ω_1-strongly countable-dimensional spaces

Masahiro Matsumoto

1 Introduction

In this note, we characterize the class of ω_1-strongly countable-dimensional metrizable spaces by a special metric. A characterization of locally finite-dimensional metrizable spaces is also obtained.

If every finite open cover of a metrizable space X has a finite open refinement of order $\leq n+1$, then X has *covering dimension* $\leq n$, $\dim X \leq n$. For $\varepsilon > 0$, we let $S_\varepsilon(x)$ denote the ε-ball $\{y \in X \mid \rho(x,y) < \varepsilon\}$ about x.

In [5], [6] and [7], J. Nagata gave a characterization of metrizable spaces of $\dim \leq n$ by a special metric.

Theorem 1.1 (J. Nagata [5], [6], [7]) *The following conditions are equivarent for a metrizable space X:*

(1) $\dim X \leq n$.

(2) *There is an admissible metric ρ satisfying the following condition: for every $\varepsilon > 0$, every point x of X and every $n+2$ many points $y_1, \ldots, y_{n+2}$ of X with $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, \ldots, n+2$, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) < \varepsilon$.*

(3) *There is an admissible metric ρ satisfying the following condition: for every point x of X and every $n+2$ many points $y_1, \ldots, y_{n+2}$ of X, there are ditinct natural numbers i and j such that $\rho(y_i,\ y_j) \leq \rho(x,\ y_j)$.*

For the case of the separable metrizable spaces, J. de Groot [2] gave the following characterization.

Theorem 1.2 (J. de Groot [2]) *A separable metrizable space X has* $\dim X \leq n$ *if and only if X can introduce an admissible totally bounded metric satisfying the following condition:*

For every point x of X and every $n+2$ many points $y_1, \ldots, y_{n+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x,\ y_k)$.

Let $\mathbb{N}$ denote the set of all natural numbers. A metrizable space X is the countable sum of finite-dimensional closed sets, we call X a *strongly countable-dimensional*.

In [8], J. Nagata extended Theorems 1.1 and 1.2 to strongly countable-dimensional metrizable spaces.

Theorem 1.3 (J. Nagata [8]) *The following conditions are equivarent for a metrizable space X:*

(1) *X is strongly countable-dimensional.*

(2) *There is an admissible metric ρ satisfying the following condition: for every point x of X, there is an $n(x) \in \mathbb{N}$ such that for every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X, there are ditinct natural numbers i and j such that $\rho(y_i,\ y_j) \leq \rho(x,\ y_j)$.*

(3) *There is an admissible metric ρ satisfying the following condition: for every point x of X, there is an $n(x) \in \mathbb{N}$ such that for every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x,\ y_k)$.*

In [3], Y. Hattori characterized the class of strongly countable-dimensional spaces by extending the condition (2) of Theorem 1.1.

Theorem 1.4 (Y. Hattori [3]) *A metrizable space X is strongly countable-dimensional if and only if X can introduce an admissible metric ρ satisfying the following condition:*

For every point x of X, there is an $n(x) \in \mathbb{N}$ such that for every $\varepsilon > 0$, and every $n+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X with $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, \ldots, n(x)+2$, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) < \varepsilon$.

2 A characterization of ω_1-strongly countable -dimensional spaces

In this section, we characterize the class of ω_1-strongly countable-dimensional metrizable spaces by a special metric. A characterization of locally finite-dimensional metrizable spaces is also obtained. Theorem 2.4 and Theorem 2.5 are main theorems.

Definition 2.1 A metrizable space X is *locally finite-dimensional* if for every point $x \in X$ there exists an open subspace U of X such that $x \in U$ and $\dim U < \infty$.

The first infinite ordinal number is denoted by ω and ω_1 is the first uncountable ordinal number.

Definition 2.2 A metrizable space X is called an *ω_1-strongly countable-dimensional space* if $X = \bigcup\{P_\xi \mid 0 \leq \xi < \xi_0\}$, $\xi_0 < \omega_1$, where P_ξ is an open subset of $X - \bigcup\{P_\eta \mid 0 \leq \eta < \xi\}$ and $\dim P_\xi < \infty$.

For a metrizable space X and a non-negative integer n, we put

$$P_n(X) = \bigcup\{U \mid U \text{ is an open subspace of } X \text{ and } \dim U \leq n\}.$$

We notice that for each ordinal number α, we can put $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number or 0 and $n(\alpha)$ is a non-negative integer.

Definition 2.3 Let X be a metrizable space and α either an ordinal number ≥ 0 or the integer -1. Then *strong small transfinite dimension sind* of X is defined as follows:

(1) $\operatorname{sind} X = -1$ if and only if $X = \emptyset$.

(2) $\operatorname{sind} X \leq \alpha$ if X is expressed in the form $X = \bigcup\{P_\xi \mid \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X - \bigcup\{P_\eta \mid \eta < \lambda(\xi)\})$.

Furthermore, if $\operatorname{sind} X$ is defined, we say that X has *strong small transfinite dimension*.

Clearly, a metrizable space X is locally finite-dimensional if and only if $\operatorname{sind} X \leq \omega$ (R. Engelking [1]). And X is ω_1-strongly countable-dimensional if and only if there is a $\xi_0 < \omega_1$ such that $\operatorname{sind} X \leq \xi_0$.

Theorem 2.4 is one of main theorems. Thus we characterize the class of ω_1-strongly countable-dimensional metrizable spaces by a special metric.

Theorem 2.4 *The following conditions are equivalent for a metrizable space X:*

(a) X is an ω_1-strongly countable-dimensional space.

(b) There are an admissible metric ρ for X, an ordinal number $\alpha < \omega_1$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions: (b-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ if β is a limit. (b-2) For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in \mathbb{N}$ such that for every $\varepsilon > 0$, every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X with $\rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon$ for each $i = 1, \ldots, n(x)+2$, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) < \varepsilon$.

(c) There are an admissible metric ρ for X, an ordinal number $\alpha < \omega_1$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions: (c-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ if β is a limit. (c-2) For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in \mathbb{N}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) \leq \rho(x',\ y_j)$.

(d) There are an admissible metric ρ for X, an ordinal number $\alpha < \omega_1$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions: (d-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ if β is a limit. (d-2)

For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in \mathbb{N}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x',\ y_k)$.

Also Theorem 2.5 is one of main theorems. We characterize the class of locally finite-dimensional metrizable spaces by a special metric.

Theorem 2.5 *The following conditions are equivalent for a metrizable space X:*

(a) X is a locally finite-dimensional space.

(b) There is an admissible metric ρ for X satisfing the following conditions: For every point x of X, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of x in X such that for every $\varepsilon > 0$, every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X with $\rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon$ for each $i = 1, ..., n(x)+2$, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) < \varepsilon$.

(c) There is an admissible metric ρ for X satisfing the following conditions: For every point x of X, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of x in X such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) \leq \rho(x',\ y_j)$.

(d) There is an admissible metric ρ for X satisfing the following conditions: For every point x of X, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of x in X such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x',\ y_k)$.

To obtain those theorems, we need the following lemmas and theorems. Essentially, the following lemma is the same as [3; Lemma 1.5]. By a minor modification in the proof of [3; Lemma 1.5], we obtain the following lemma.

Lemma 2.6 ([3; Lemma 2.5], [8; Lemma 1]) *Let n be a non-negative integer and let $\{F_m \mid m = 0,\ 1, ...\}$ be a closed cover of a metrizable space X such that $\dim F_m \leq (n-1)+m$, $F_m \subset F_{m+1}$ for $m = 0,\ 1, ...$ Then for every open cover $\mathcal{U}$ of X, there are a sequence $\mathcal{V}_1,\ \mathcal{V}_2,\ ...$ of discrete families of open sets of X and an open cover $\mathcal{W}$ of X which satisfy the following conditions:*

(1) $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ *is a cover of X.*

(2) $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ *refines $\mathcal{U}$.*

(3) *If $W \in \mathcal{W}$ satisfies $W \cap F_m \neq \emptyset$, then W meets at most one member of $\mathcal{V}_k$ for $k \leq (n+0)+(n+1)+...+(n+m)$ and meets no member of $\mathcal{V}_k$ for $k > (n+0)+(n+1)+...+(n+m)$.*

Let Q^* denote the set of all rational numbers of the form $2^{-m_1}+...+2^{-m_t}$, where $m_1,...,m_t$ are natural numbers satisfying $1 \leq m_1 < ... < m_t$.

Essentially, the following lemma is the same as [3; Lemma 1.6]. By a minor modification in the proof of [3; Lemma 1.6], we obtain the following lemma.

Lemma 2.7 ([3; Lemma 2.6], [8; Lemma 3]) *Let n be a non-negative integer and let $\{F_m \mid m = 0, 1, \ldots\}$ be a closed cover of a metrizable space X such that* $\dim F_m \leq (n-1)+m$, $F_m \subset F_{m+1}$ *for* $m = 0, 1, \ldots$. *Then for every $q \in Q^*$, there is an open cover $\mathcal{S}(q)$ which satisfies the following conditions:*

(1) $\mathcal{S}(q) = \bigcup_{i=1}^{\infty} \mathcal{S}^i(q)$, *where each $\mathcal{S}^i(q)$ is discrete in X.*

(2) $\{St(x, \mathcal{S}(q)) \mid q \in Q^*\}$ *is a neighborhood base at $x \in X$.*

(3) *Let p, $q \in Q^*$ and $p < q$. Then $\mathcal{S}(p)$ refines $\mathcal{S}(q)$.*

(4) *Let p, $q \in Q^*$ and $p < q$. If $S_1 \in \mathcal{S}^i(p)$ and $S_2 \in \mathcal{S}^i(q)$, then $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$.*

(5) *Let p, $q \in Q^*$ and $p + q < 1$. Let $S_1 \in \mathcal{S}(p)$, $S_2 \in \mathcal{S}(q)$ and $S_1 \cap S_2 \neq \emptyset$. Then there is an $S_3 \in \mathcal{S}(p+q)$ such that $S_1 \cup S_2 \subset S_3$.*

(6) *For every $q \in Q^*$ and every $S \in \bigcup\{\mathcal{S}^i(q) \mid i > (n+0)+(n+1)+\ldots+(n+m)\}$, $S \cap F_m = \emptyset$.*

By Lemma 2.6 and Lemma 2.7, we obtain the following theorem.

Theorem 2.8 *Let α be an ordinal number with $\alpha < \omega_1$ and let n be a non-negative integer. The following conditions are equivarent for a metrizable space X:*

(*a*) $\operatorname{sind} X \leq \omega\alpha + n$.

(*b*) *There are an admissible metric ρ for X and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions:* (*b*-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ *for* $\beta \leq \beta' \leq \alpha$, $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ *if β is a limit, and $X_\alpha = \emptyset$ if $n = 0$.* (*b*-2) *For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in N_{\beta(x)}$ such that for every $\varepsilon > 0$, every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X with $\rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon$ for each $i = 1, \ldots, n(x)+2$, there are distinct natural numbers i and j such that $\rho(y_i, y_j) < \varepsilon$, where*

$$N_{\beta(x)} = \begin{cases} \mathbb{N}, & \text{if } \beta(x) < \alpha, \\ \{n-1\}, & \text{if } \beta(x) = \alpha. \end{cases}$$

(*c*) *There are an admissible metric ρ for X and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions:* (*c*-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ *for* $\beta \leq \beta' \leq \alpha$, $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ *if β is a limit, and $X_\alpha = \emptyset$ if $n = 0$.* (*c*-2) *For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in N_{\beta(x)}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X, there are distinct natural numbers i and j such that $\rho(y_i, y_j) \leq \rho(x', y_j)$, where*

$$N_{\beta(x)} = \begin{cases} \mathbb{N}, & \text{if } \beta(x) < \alpha, \\ \{n-1\}, & \text{if } \beta(x) = \alpha. \end{cases}$$

Remark 2.9 Let $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a family of closed sets of X satisfing the condition (b-1). Then we shall show that for every point x of X, there is a maximum element $\beta(x)$ of $\{\beta \mid x \in X_\beta\}$. Indeed, if $x \in X_{\lambda(\alpha)}$, then $\beta(x) = \max\{\beta \mid x \in X_\beta,\ \lambda(\alpha) \leq \beta \leq \alpha\}$. Now, we suppose that $x \in X_{\lambda(\alpha)}$, there is a minimum element $\beta_0 > 0$ of $\{\beta \mid x \notin X_\beta\}$. Assume that β_0 is limit. By the condition (b-1), $x \in \bigcap\{X_\beta \mid \beta < \beta_0\} = X_{\beta_0}$. This contradicts the definition of β_0. Therefore β_0 is not limit and hence $\beta(x) = \beta_0 - 1$.

By Theorems 1.2 and 2.8, we obtain the following theorem.

Theorem 2.10 *Let α be an ordinal number with $\alpha < \omega_1$ and let n be a non-negative integer. The following conditions are equivarent for a compact metrizable space X:*

(*a*) $\operatorname{sind} X \leq \omega\alpha + n$.

(*d*) *There are an admissible totally bounded metric ρ for X and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions:* (*d*-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ *for* $\beta \leq \beta' \leq \alpha$, $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ *if β is a limit, and* $X_\alpha = \emptyset$ *if* $n = 0$. (*d*-2) *For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in N_{\beta(x)}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x', y_k)$, where*

$$N_{\beta(x)} = \begin{cases} \mathbb{N}, & \text{if } \beta(x) < \alpha, \\ \{n-1\}, & \text{if } \beta(x) = \alpha. \end{cases}$$

By Theorems 2.8 and 2.10, we obtain the Main Theorem 2.4 and Theorem 2.5.

References

[1] R. Engelking, *Theory of Dimensions Finite and Infinite*, Heldermann Verlag (1995).

[2] J. de Groot, *On a metric that characterizes dimension*, Canad. J. Math. 9 (1957), 511-514.

[3] Y. Hattori, *On special metrics characterizing topological properties*, Fund. Math. 126 (1986), 133-145.

[4] M. Katětov, *On the relations between the metric and topological dimensions*, Czech. Math. J. 8 (1958) 163-166.

[5] J. Nagata, *On a relation between dimension and metrization*, Proc. Jap. Ac. 32 (1956), 237-240.

[6] J. Nagata, *On a special metric characterizing a metric space of* dim $\leq n$, Proc. Japan Acad. 39 (1963), 278-282.

[7] J. Nagata, *On a special metric and dimension*, Fund. Math. 55 (1964), 181-194.

[8] J. Nagata, *Topics in dimension theory*, General Topology and its Relations to Modern Analysis and Algebra (Proc. Fifth Prague Topology Symposium 1981), Heldermann Verlag, Berlin (1982), 497-507.

[9] J. Nagata, *Modern Dimension Theory, revised and extended edn.*, Heldermann, Berlin, 1983.

[10] J. Nagata, *Open problems left in my wake of research*, Topology and its Appl. 146-147 (2005), 5-13.

[11] Z. Shmuely, *On strongly countable-dimensional sets*, Duke Math. J. 38 (1971), 169-173.

Masahiro Matsumoto
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571 Japan
e-mail: masahiro@math.tsukuba.ac.jp