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# Geometric properties of certain analytic functions with real coefficients

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**Abstract**

Let $\mathcal{T}$ be the class of analytic functions with real coefficients in the open unit disk $\mathbb{U}$. For $f(z)$ belonging to the class $\mathcal{T}$, some sufficient conditions for starlikeness and convexity are discussed. Furthermore, for $f(z)$ in the class $\mathcal{T}$, we prove the starlikeness of $f(z)$ having property $\mathrm{Re}\{f'(z)\} > 0$.

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## 1 Introduction

Let $\mathcal{A}$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

We denote by $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ the subclasses of $\mathcal{A}$ whose members map $\mathbb{U}$ onto domain which are univalent, starlike, convex and close-to-comvex.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order $\alpha$ ($\alpha < 1$) in $\mathbb{U}$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathbb{U}). \tag{1.2}$$

Similarly, $f(z) \in \mathcal{A}$ is said to be convex of order $\alpha$ ($\alpha < 1$) in $\mathbb{U}$ if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathbb{U}). \tag{1.3}$$

We shall denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclassses of $\mathcal{A}$ whose members satisfy (1.2) and (1.3), respectively.

It is known that for $0 \leqq \alpha < 1$, $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*$, $\mathcal{K}(\alpha) \subset \mathcal{K}$ and that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, $\mathcal{K}(0) \equiv \mathcal{K}$. Chichra [2] showed that for $f(z) \in \mathcal{A}$ and $\alpha \geqq 0$ the following implication holds in $\mathbb{U}$:

$$\text{(1.4)} \qquad \mathrm{Re}\{f'(z) + \alpha f''(z)\} > 0 \Rightarrow \mathrm{Re} f'(z) > 0.$$

On the other hand, Singh and Singh [13], Mocanu [5] have the following results for $f(z) \in \mathcal{A}$, respectively.

$$\text{(1.5)} \qquad \mathrm{Re}\{f'(z) + zf''(z)\} > -\frac{1}{4} \Rightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0,$$

$$\text{(1.6)} \qquad \mathrm{Re}\left\{f'(z) + \frac{1}{2}zf''(z)\right\} > 0 \Rightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

Furthermore, Salagean [11] defined $\mathcal{N}$ the class of functions with negative coefficient, that is,

$$\text{(1.7)} \qquad \mathcal{N} = \left\{f(z) \in \mathcal{A} | f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ a_n \geqq 0\right\}$$

and obtained the following implications that are that if $f(z) \in \mathcal{N}$ in $\mathbb{U}$, then

$$\text{(1.8)} \qquad \mathrm{Re}\{f'(z) + zf''(z)\} > -1 \Rightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0,$$

$$\text{(1.9)} \qquad \mathrm{Re}\{f'(z) + zf''(z)\} > 0 \Rightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2}.$$

## 2 Pleliminaries

Recently, we prove the following Lemma in [9].

**Lemma 1.** [9, Nunokawa et al] *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\text{(2.1)} \qquad \mathrm{Re}\{f'(z) + \alpha f''(z)\} > -\frac{\alpha}{2} \quad \text{in } \mathbb{U}$$

*for some $\alpha$ $(\alpha > 0)$. Then we have $\mathrm{Re} f'(z) > 0$ in $\mathbb{U}$.*

Next lemma was given by Nunokawa in 1993.

**Lemma 2.** [8] *Let $p(z)$ be analytic in $\mathbb{U}$, $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{U}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi\alpha}{2} \quad \text{for } |z| < |z_0|$$

*and*

$$|\arg p(z_0)| = \frac{\pi\alpha}{2}$$

*where* $\alpha > 0$. *Then we have* $\dfrac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$ *where*

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \ \arg p(z_0) = \frac{\pi\alpha}{2}$$

*and*

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \ \arg p(z_0) = -\frac{\pi\alpha}{2}$$

*where*

$$p(z_0)^{\frac{1}{\alpha}} = \pm i\alpha, \ \textit{and} \ a > 0.$$

Let us define $T$ the class of analytic functions with real coefficients, that is,

$$\mathcal{T} = \left\{ f(z) \in \mathcal{A} | f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ a_n \in \mathbb{R} \right\} \tag{2.2}$$

where $\mathbb{R}$ is the set of real numbers. Then it follows that

$$\mathcal{N} \subset \mathcal{T} \subset \mathcal{A}.$$

In [9], we have the following theorem.

**Theorem A.** [9] *Let* $f(z) \in \mathcal{T}$ *and suppose that*

$$\operatorname{Re}\{f'(z) + \alpha z f''(z)\} > 0 \quad \text{in } \mathbb{U}$$

*where* $\alpha \geqq 1$. *Then we have*

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \frac{\alpha - 1}{\alpha} \quad \text{in } \mathbb{U},$$

*or* $f(z)$ *is convex of order* $\dfrac{\alpha-1}{\alpha}$.

**Remark 1.** Putting $\alpha = 1$ in Theorem 1, we hae

$$f(z) \in \mathcal{T}, \quad \operatorname{Re}\{f'(z) + zf''(z)\} > 0$$

$$\Rightarrow 1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathbb{U})$$

$$\Rightarrow \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \qquad (z \in \mathbb{U}).$$

Let $\mathcal{P}'$ be the subclass of $\mathcal{A}$ whose members $f(z)$ satisfy $\operatorname{Re} f'(z) > 0$ in $\mathbb{U}$. It is well-known that $\mathcal{P}'$ is a subclass of $\mathcal{C}$ whose elements are close-to-convex in $\mathbb{U}$.

## 3 Main results

**Theorem 1.** *Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in $\mathbb{U}$ and all coefficients $p_i$ are real numbers. Suppose that*

$$\operatorname{Re}\{p(z) + \alpha z p'(z)\} > 0 \quad \text{in } \mathbb{U} \tag{3.1}$$

*where $\alpha \geqq 1$. Then we have*

$$1 + \operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} > 0 \quad \text{in } \mathbb{U}. \tag{3.2}$$

*Proof.* Using assumption (3.1) and Lemma1, we have

$$\operatorname{Re}\{p(z)\} > 0 \quad \text{in } \mathbb{U}.$$

Therefore, we hae

$$\left|\arg p(z) + \arg\left(1 + \alpha\frac{zp'(z)}{p(z)}\right)\right| < \frac{\pi}{2} \quad \text{in } \mathbb{U}. \tag{3.3}$$

for a sufficiently small and positive $\epsilon$, there exists a point $z_1 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\epsilon \quad \text{for } |z| < |z_1|$$

and

$$|\arg p(z_1)| = \frac{\pi}{2}\epsilon,$$

then from Lemma 2, we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = i\epsilon k$$

where

$$k \geqq 1 \text{ when } \arg p(z_1) = \frac{\pi}{2}\epsilon$$

and

$$k \leqq -1 \text{ when } \arg p(z_1) = -\frac{\pi}{2}\epsilon.$$

Then it follows that for the case $\arg p(z_1) = \frac{\pi}{2}\epsilon$, we have

$$\begin{aligned} \arg\left(1 + \alpha\frac{z_1 p'(z_1)}{p(z_1)}\right) &= \arg(1 + i\alpha\epsilon k) \\ &= \tan^{-1}\alpha\epsilon k \geqq \tan^{-1}\alpha\epsilon > 0. \end{aligned} \tag{3.4}$$

And for the case $\arg p(z_1) = -\frac{\pi}{2}\epsilon$, we also have

$$\begin{aligned} \arg\left(1 + \alpha\frac{z_1 p'(z_1)}{p(z_1)}\right) &= \arg(1 + i\alpha\epsilon k) \\ &= \tan^{-1}\alpha\epsilon k \leqq \tan^{-1}(-\alpha\epsilon) < 0. \end{aligned} \tag{3.5}$$

From the assumption of Theorem 1, the image domains of the open unit disk $\mathbb{U}$ under the mapping $w = p(z)$ and $w = 1 + \alpha\frac{zp'(z)}{p(z)}$ are symmetric with respect to the real axis. Therefore, from above properties (3.4) and (3.5), it shows that the image domains of the open unit disk $\mathbb{U}$ under the mapping $w = p(z)$ and $w = 1 + \alpha\frac{zp'(z)}{p(z)}$ are the same side of the complex plane which is devided into two parts by the real axis.

Now then, if there exists a point $z_0 \in \mathbb{U}$ such that

$$\left|\arg\left(1 + \alpha\frac{zp'(z)}{p(z)}\right)\right| < \frac{\pi}{2} \quad \text{for } |z| < |z_0|$$

and

$$\left|\arg\left(1 + \alpha\frac{z_0 p'(z_0)}{p(z_0)}\right)\right| = \frac{\pi}{2},$$

then for the case

$$\arg\left(1 + \alpha\frac{z_0 p'(z)}{p(z)}\right) = \frac{\pi}{2}$$

we have $\arg p(z_0) > 0$. This contradicts (3.3) and for the case

$$\arg\left(1 + \alpha\frac{z_0 p'(z_0)}{p(z_0)}\right) = -\frac{\pi}{2},$$

we have $\arg p(z_0) < 0$. This contradicts (3.3) and therefore, we have

$$1 + \alpha \mathrm{Re}\frac{zp'(z)}{p(z)} > 0 \quad \text{in } \mathbb{U}.$$

□

Letting $p(z) = f'(z)$, we have Theorem A. Furthermore, putting $p(z) = \frac{f(z)}{z}$ for $f(z) \in \mathcal{A}$, we have

**Corollary 1.** *Let $f(z) \in \mathcal{T}$ and suppose that*

$$\mathrm{Re}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > 0 \qquad (z \in \mathbb{U})$$

*and $\alpha \geqq 1$. Then we have*

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \frac{\alpha - 1}{\alpha} \qquad (z \in \mathbb{U}),$$

*that is, $f(z)$ is starlike of order $\frac{\alpha-1}{\alpha}$.*

**Remark 2.** In 1962, Krzyz [3] gave an example of a function $f(z) \in \mathcal{P}'$ such that $f(z) \notin \mathcal{S}^*$

However, in the case of $f(z) \in \mathcal{T}$, we have

**Theorem 2.** *Let $f(z) \in \mathcal{T}$. If $\mathrm{Re}f'(z) > 0$ in $\mathbb{U}$, then we have $f(z) \in \mathcal{S}^*$.*

*Proof.* Putting $\alpha = 1$ in Corollary 1, we prove Theorem 2. □

Using our results, we have many starlike functions and convex functions.

**Example 1.** *Let $f(z) \in \mathcal{T}$ and $\alpha \geqq 1$. If*

$$f'(z) + \alpha z f''(z) = \frac{1+z}{1-z},$$

*then we have*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2}{n(1+(n-1)\alpha)} z^n \in \mathcal{K}\left(\frac{\alpha-1}{\alpha}\right).$$

**Example 2.** *Putting $\alpha = 1$ in Example 1, we have*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n \in \mathcal{K}, \qquad |f(z)| < \frac{\pi^2 - 3}{3} = 2.289\cdots.$$

**Example 3.** *Let $f(z) \in \mathcal{T}$ and $\alpha \geqq 1$. If*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \frac{1+z}{1-z},$$

*then we have*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2}{1+(n-1)\alpha} z^n \in \mathcal{S}^*\left(\frac{\alpha-1}{\alpha}\right).$$

**Example 4.** *Letting $\alpha = 1$ in Example 3, we have*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2}{n} z^n \in \mathcal{S}^*.$$

Next result is well-known. Let

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt \qquad (c > -1)$$

that is, Libera transform. If $f(z) \in \mathcal{P}'$, then $F(z) \in \mathcal{P}'$.
A natural question arise, that is, If $f(z) \in \mathcal{P}'$, is the Libera transform of $f(z)$ starlike in

$\mathbb{U}$ ?

Singh and Singh [12] answered.

**Theorem B.** ([12]) *If $f(z) \in \mathcal{P}'$, then the function $F(z)$, defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \qquad (c > -1)$$

*belongs to $\mathcal{S}^*$ for all c* $(-1 < c \leqq 0)$.

We consider the next question, that is,
" *If $f(z) \in \mathcal{T}$ and* $\operatorname{Re} f'(z) > 0$, *is the Libera transform of $f(z)$ convex in* $\mathbb{U}$ ?"

**Theorem 3.** *$f(z) \in \mathcal{T}$ and* $\operatorname{Re} f'(z) > 0$, *then the function*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \qquad (c > -1) \tag{3.6}$$

*belongs $\mathcal{K}(-c)$ for all c* $(-1 < c \leqq 0)$.

*Proof.* By diffentiating (3.6), we have

$$F'(z) + \frac{1}{c+1} zF''(z) = f'(z).$$

Therefore,

$$\operatorname{Re}\left\{F'(z) + \frac{1}{c+1} zF''(z)\right\} = \operatorname{Re} f'(z) > 0$$

and $\dfrac{1}{c+1} \geqq 1$ $(1 < c \leqq 0)$. Using Theorem A, we have

$$1 + \operatorname{Re}\left\{\frac{zF''(z)}{F'(z)}\right\} > -c \qquad (0 \leqq -c < 1).$$

That is, $F(z) \in \mathcal{K}(-c)$. □

Putting $c = 0$ in Theorem 3, we have

**Corollary 2.** *If $f(z) \in \mathcal{T}$ and* $\operatorname{Re} f'(z) > 0$, *then the function*

$$g(z) = \int_0^z \frac{f(t)}{t} dt$$

*belongs to $\mathcal{K}$, that is, $g(z) \in \mathcal{K}$.*

To prove our next result, we prepare the following lemma due to Owa and Nunokawa [10].

**Lemma 3.** [10] *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0)=1$, $p'(0)=\cdots=p^{(n-1)}(0)=0$. If*

$$\mathrm{Re}\{p(z)+\alpha zp'(z)\}>\beta \qquad (z\in\mathbb{U}),$$

*then*

$$\mathrm{Re}\{p(z)\}>\beta+(1-\beta)\left\{2\int_0^1\frac{1}{1+\rho^{n\mathrm{Re}(\alpha)}}d\rho-1\right\} \qquad (z\in\mathbb{U})$$

*where $\alpha\neq 0$, $\mathrm{Re}(\alpha)\geqq 0$ and $\beta<1$.*

Letting $\beta=0$, $n=1$ in Lemma3, and applying Theorem 3, we can prove next Theorem.

**Theorem 4.** *If $f(z)\in\mathcal{T}$ and $\mathrm{Re}f'(z)>0$, let the function $F(z)$ given by (3.6), then we have*

$$\mathrm{Re}F'(z)>2\int_0^1\frac{1}{1+\rho^{\frac{1}{c+1}}}d\rho-1>0.$$

Putting $c=0$ in Theorem 4, we have

**Corollary 3.** *If $f(z)\in\mathcal{T}$ and $\mathrm{Re}f'(z)>0$, and let the function*

$$g(z)=\int_0^z\frac{f(t)}{t}dt,$$

*then we have*

$$\mathrm{Re}g'(z)>2\log 2-1.$$

Letting $c=1$ in Theorem 4, we can get

**Corollary 4.** *If $f(z)\in\mathcal{T}$ and $\mathrm{Re}f'(z)>0$, and let the function*

$$s(z)=\frac{2}{z}\int_0^z f(t)dt,$$

*then we have*

$$\mathrm{Re}s'(z)>3-4\log 2.$$

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