<table>
<thead>
<tr>
<th>Title</th>
<th>On two sufficient conditions for univalency of real coefficient functions (Study on Geometric Univalent Function Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nunokawa, Mamoru</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2008), 1579: 98-100</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/81388">http://hdl.handle.net/2433/81388</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On two sufficient conditions for univalency of real coefficient functions

Mamoru Nunokawa

Abstract

It is well known that if the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic in \( |z| < 1 \) and satisfies one of the following conditions

\[
1 + \Re \left( \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2} \quad \text{in} \ |z| < 1
\]

or

\[
1 + \Re \left( \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad \text{in} \ |z| < 1,
\]

then \( f(z) \) is univalent in \( |z| < 1 \). In this paper, we improve the above conditions for the function \( f(z) \) whose coefficients are all real.

1. Introduction

Let \( \mathcal{A} \) be the set of analytic functions defined in the unit disk \( E = \{ z \mid |z| < 1 \} \) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

and let

\[
\mathcal{S} = \{ f(z) \mid f(z) \in \mathcal{A} \text{ and } f(z) \text{ is univalent in } E \}.
\]

The late professor Ozaki [1] proved the following theorem.

**Theorem A.** Let \( f(z) \in \mathcal{A} \) and if \( f(z) \) satisfies one of the following conditions

(i) \[ 1 + \Re \left( \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2} \quad \text{in} \ |z| < 1 \]

or

(ii) \[ 1 + \Re \left( \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad \text{in} \ |z| < 1, \]

then we have \( f(z) \in \mathcal{S} \).

2. Theorems

First our theorem is contained in
Theorem 1. Let \( f(z) \in A \), all the coefficients \( a_n, 2 \leq n \in \mathbb{N} = \{1, 2, 3, \ldots\} \) are real and suppose that

\[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > -1 \quad (z \in E).
\]

Then we have \( f(z) \in S \).

Proof. Suppose that if there exists a positive real number \( r, 0 < r < 1 \) for which \( f(z) \) is univalent in \( |z| < r \) but \( f(z) \) is not univalent in \( |z| \leq r \), then from the hypothesis, there exists two points \( z_1 = re^{i\theta_1}, z_2 = re^{i\theta_2}, \theta_1 < \theta_2 \) and \( \theta_3 - \theta_1 < \pi \) for which \( f(z_1) = f(z_2) \).

From the hypothesis (1), we have \( f'(z) \neq 0 \) in \( E \), because if \( f'(z) \) has a zero in \( E \), then it is impossible that \( f(z) \) satisfies the condition (1).

Let us put
\[
C = \{ z | z = re^{i\theta}, \theta_1 \leq \theta \leq \theta_2 \}
\]
and
\[
C_{f(z)} = \{ f(z) | z \in C \}.
\]

Then we have
\[
\int_{C_{f(z)}} \text{arg} df(z) = -\pi
\]
\[
= \int_{C} \text{arg} f'(z) dz
\]
\[
= \int_{\theta_1}^{\theta_2} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta
\]
\[
> \int_{\theta_1}^{\theta_2} (-1) d\theta = \theta_1 - \theta_2 > -\pi.
\]

This is a contradiction and therefore it completes the proof.

Theorem 2. Let \( f(z) \in A \), all the coefficients \( a_n, 2 \leq n \in \mathbb{N} = \{1, 2, 3, \ldots\} \) are real and suppose that

\[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < 2 \quad (z \in E).
\]

Then we have \( f(z) \in S \).

Proof. Applying the same method as the proof of Theorem 1, if there exists a positive real number \( r, 0 < r < 1 \) for which \( f(z) \) is univalent in \( |z| < r \) but \( f(z) \) is not univalent in \( |z| \leq r \), then there are four points such as the proof of Theorem 1, \( z_1 = re^{i\theta_1}, z_2 = re^{i\theta_2}, z_3 = re^{i(2\pi - \theta_2)} \) and \( z_4 = re^{i(2\pi - \theta_1)}, 0 < \theta_1 < \theta_2 < \pi \) for which we have \( f(z_1) = f(z_3) \) and \( f(z_2) = f(z_4) \). From the hypothesis, the tangent line at the point \( f(z_1) \) and \( f(z_2) \) is the common tangent and it is the same for the points \( f(z_3) \) and \( f(z_4) \).

Therefore, we have
\[
\int_{\theta_1}^{\theta_2} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta = -\pi
\]
and

\[ \int_{2\pi-\theta_{1}}^{2\pi-\theta_{2}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta = -\pi, \]

where \( z = re^{i\theta} \).

From the hypothesis (2) and the same reason as the proof of Theorem 1, we have \( f'(z) \neq 0 \) in \( E \).

From the hypothesis (2), we have

\[
\int_{|z|=r} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta = 2\pi
\]

\[
= \int_{\theta_{1}}^{\theta_{2}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta + \int_{\theta_{2}}^{2\pi-\theta_{1}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta
\]

\[
+ \int_{2\pi-\theta_{1}}^{2\pi-\theta_{2}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta + \int_{2\pi-\theta_{2}}^{2\pi-\theta_{1}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta
\]

\[
< -\pi + \int_{\theta_{1}}^{\theta_{2}} 2d\theta - \pi + \int_{\theta_{2}}^{2\pi-\theta_{1}} 2d\theta
\]

\[
= \{4\pi - 2(\theta_{2} - \theta_{1})\} - 2\pi
\]

\[
< 4\pi - 2\pi = 2\pi.
\]

This is a contradiction and so, it completes the proof.

**Remark.** A function \( f(z) \in A \) is typically real in \( E \) if \((\text{Im} f(z))(\text{Im} z) > 0 \) for \( E/R = \{z \in E \cap z \notin \mathbb{R}\} \). In Theorem 1, if \( f(z) \) is typically real and satisfies (1), then the conclusion continues to hold true.

**References**


Mamoru Nunokawa  
Emeritus Professor of University of Gunma  
Hoshikuki-Cho 798 - 8  
Chou-Ward, Chiba City 260 - 0808, Japan  
E-mail: mamoru.nuno@doctor.nifty.jp