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<td>著者</td>
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<tr>
<td>引用</td>
<td>数理解析研究所講究録 1579: 98-100</td>
</tr>
<tr>
<td>発行年月</td>
<td>2008-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/81388">http://hdl.handle.net/2433/81388</a></td>
</tr>
<tr>
<td>機関</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>版式</td>
<td>publisher</td>
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On two sufficient conditions for univalency of real coefficient functions

Mamoru Nunokawa

Abstract

It is well known that if the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic in \( |z| < 1 \) and satisfies one of the following conditions

\[
1 + \operatorname{Re} \left( \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2} \quad \text{in} \ |z| < 1
\]

or

\[
1 + \operatorname{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad \text{in} \ |z| < 1,
\]

then \( f(z) \) is univalent in \( |z| < 1 \). In this paper, we improve the above conditions for the function \( f(z) \) whose coefficients are all real.

1. Introduction

Let \( \mathcal{A} \) be the set of analytic functions defined in the unit disk \( E = \{ z \mid |z| < 1 \} \) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

and let

\[
S = \{ f(z) \mid f(z) \in \mathcal{A} \text{ and } f(z) \text{ is univalent in } E \}.
\]

The late professor Ozaki [1] proved the following theorem.

**Theorem A.** Let \( f(z) \in \mathcal{A} \) and if \( f(z) \) satisfies one of the following conditions

(i) \[ 1 + \operatorname{Re} \left( \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2} \quad \text{in} \ |z| < 1 \]

or

(ii) \[ 1 + \operatorname{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad \text{in} \ |z| < 1, \]

then we have \( f(z) \in S \).

2. Theorems

First our theorem is contained in
Theorem 1. Let \( f(z) \in A \), all the coefficients \( a_n, 2 \leq n \in \mathbb{N} = \{1, 2, 3, \ldots\} \) are real and suppose that

\[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > -1 \quad (z \in \mathbb{E}).
\]

Then we have \( f(z) \in S \).

\textbf{Proof.} Suppose that if there exists a positive real number \( r, 0 < r < 1 \) for which \( f(z) \) is univalent in \( |z| < r \) but \( f(z) \) is not univalent in \( |z| \leq r \), then from the hypothesis, there exists two points \( z_1 = re^{i\theta_1}, z_2 = re^{i\theta_2}, \theta_1 < \theta_2 \) and \( \theta_2 - \theta_1 < \pi \) for which \( f(z_1) = f(z_2) \).

From the hypothesis (1), we have \( f'(z) \neq 0 \) in \( \mathbb{E} \), because if \( f'(z) \) has a zero in \( \mathbb{E} \), then it is impossible that \( f(z) \) satisfies the condition (1).

Let us put

\[
C = \{ z \mid z = re^{i\theta}, \theta_1 \leq \theta \leq \theta_2 \}
\]

and

\[
C_{f(z)} = \{ f(z) \mid z \in C \}.
\]

Then we have

\[
\int_{C_{f(z)}} \text{darg} df(z) = -\pi
\]

\[
= \int_{C} \text{darg} f'(z) \text{dz}
\]

\[
= \int_{\theta_1}^{\theta_2} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta
\]

\[
> \int_{\theta_1}^{\theta_2} (-1) d\theta = \theta_1 - \theta_2 > -\pi.
\]

This is a contradiction and therefore it completes the proof.

Theorem 2. Let \( f(z) \in A \), all the coefficients \( a_n, 2 \leq n \in \mathbb{N} = \{1, 2, 3, \ldots\} \) are real and suppose that

\[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < 2 \quad (z \in \mathbb{E}).
\]

Then we have \( f(z) \in S \).

\textbf{Proof.} Applying the same method as the proof of Theorem 1, if there exists a positive real number \( r, 0 < r < 1 \) for which \( f(z) \) is univalent in \( |z| < r \) but \( f(z) \) is not univalent in \( |z| \leq r \), then there are four points such as the proof of Theorem 1, \( z_1 = re^{i\theta_1}, z_2 = re^{i\theta_2}, z_3 = re^{i(2\pi - \theta_2)} \) and \( z_4 = re^{i(2\pi - \theta_1)} \), \( 0 < \theta_1 < \theta_2 < \pi \) for which we have \( f(z_1) = f(z_2) \) and \( f(z_3) = f(z_4) \). From the hypothesis, the tangent line at the point \( f(z_1) \) and \( f(z_2) \) is the common tangent and it is the same for the points \( f(z_3) \) and \( f(z_4) \).

Therefore, we have

\[
\int_{\theta_1}^{\theta_2} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta = -\pi
\]
and
\[ \int_{2\pi-\theta_{1}}^{2\pi-\theta_{2}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta = -\pi, \]
where \( z = re^{i\theta}. \)

From the hypothesis (2) and the same reason as the proof of Theorem 1, we have \( f'(z) \neq 0 \) in \( E. \)

From the hypothesis (2), we have
\[ \int_{|z|=r} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta = 2\pi = \int_{l_{1}}^{b} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f(z)} \right) \right) d\theta + \int_{\theta}^{2\pi-\theta} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta + \int_{l-l}^{*-l_{1}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f^{l}(z)} \right) \right) d\theta + \int_{2l-\theta_{1}}^{2+l_{1}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f(z)} \right) \right) d\theta \]
\[ < -\pi + \int_{\theta}^{2\pi-\theta} 2d\theta - \pi + \int_{2\pi-\theta_{1}}^{2\pi+\theta_{1}} 2d\theta \]
\[ = \{4\pi - 2(\theta_{2} - \theta_{1})\} - 2\pi \]
\[ < 4\pi - 2\pi = 2\pi. \]

This is a contradiction and so, it completes the proof.

**Remark.** A function \( f(z) \in \mathcal{A} \) is typically real in \( E \) if \( (\text{Im} f(z)) (\text{Im} z) > 0 \) for \( E/\mathbb{R} = \{z \in E \cap z \notin \mathbb{R}\}. \) In Theorem 1, if \( f(z) \) is typically real and satisfies (1), then the conclusion continues to hold true.

**References**


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