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On two sufficient conditions for univalency of real coefficient functions

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Abstract

It is well known that if the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic in \(|z| < 1\) and satisfies one of the following conditions

\[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2} \quad \text{in} \ |z| < 1
\]

or

\[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad \text{in} \ |z| < 1,
\]

then \( f(z) \) is univalent in \(|z| < 1\). In this paper, we improve the above conditions for the function \( f(z) \) whose coefficients are all real.

1. Introduction

Let \( \mathcal{A} \) be the set of analytic functions defined in the unit disk \( \mathbb{E} = \{ z \mid |z| < 1 \} \) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

and let

\[
\mathcal{S} = \{ f(z) \mid f(z) \in \mathcal{A} \text{ and } f(z) \text{ is univalent in } \mathbb{E} \}.
\]

The late professor Ozaki [1] proved the following theorem.

**Theorem A.** Let \( f(z) \in \mathcal{A} \) and if \( f(z) \) satisfies one of the following conditions

(i) \[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2} \quad \text{in} \ |z| < 1
\]

or

(ii) \[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad \text{in} \ |z| < 1,
\]

then we have \( f(z) \in \mathcal{S} \).

2. Theorems

First our theorem is contained in
**Theorem 1.** Let \( f(z) \in A \), all the coefficients \( a_n, 2 \leq n \in \mathbb{N} = \{1, 2, 3, \ldots\} \) are real and suppose that
\[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > -1 \quad (z \in E).
\]
Then we have \( f(z) \in S \).

**Proof.** Suppose that if there exists a positive real number \( r, 0 < r < 1 \) for which \( f(z) \) is univalent in \( |z| < r \) but \( f(z) \) is not univalent in \( |z| \leq r \), then from the hypothesis, there exists two points \( z_1 = re^{i\theta_1}, z_2 = re^{i\theta_2}, \theta_1 < \theta_2 \) and \( \theta_2 - \theta_1 < \pi \) for which \( f(z_1) = f(z_2) \).

From the hypothesis (1), we have \( f'(z) \neq 0 \) in \( E \), because if \( f'(z) \) has a zero in \( E \), then it is impossible that \( f(z) \) satisfies the condition (1).

Let us put \( C = \{ z \mid z = re^{i\theta}, \theta_1 \leq \theta \leq \theta_2 \} \)
and \( C_{f(z)} = \{ f(z) \mid z \in C \} \).

Then we have
\[
\int_{C_{f(z)}} \text{darg} f(z) = -\pi
\]
\[
= \int_C \text{darg} f'(z)dz
\]
\[
= \int_{\theta_1}^{\theta_2} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta
\]
\[
> \int_{\theta_1}^{\theta_2} (-1)d\theta = \theta_1 - \theta_2 > -\pi.
\]
This is a contradiction and therefore it completes the proof.

**Theorem 2.** Let \( f(z) \in A \), all the coefficients \( a_n, 2 \leq n \in \mathbb{N} = \{1, 2, 3, \ldots\} \) are real and suppose that
\[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < 2 \quad (z \in E).
\]
Then we have \( f(z) \in S \).

**Proof.** Applying the same method as the proof of Theorem 1, if there exists a positive real number \( r, 0 < r < 1 \) for which \( f(z) \) is univalent in \( |z| < r \) but \( f(z) \) is not univalent in \( |z| \leq r \), then there are four points such as the proof of Theorem 1, \( z_1 = re^{i\theta_1}, z_2 = re^{i\theta_2}, z_3 = re^{i(2\pi - \theta_2)} \)
and \( z_4 = re^{i(2\pi - \theta_1)}, 0 < \theta_1 < \theta_2 < \pi \) for which we have \( f(z_1) = f(z_2) \) and \( f(z_3) = f(z_4) \).

From the hypothesis, the tangent line at the point \( f(z_1) \) and \( f(z_2) \) is the common tangent and it is the same for the points \( f(z_3) \) and \( f(z_4) \).

Therefore, we have
\[
\int_{\theta_1}^{\theta_2} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta = -\pi
\]
and
\[ \int_{2\pi - \theta_{1}}^{2\pi - \theta_{2}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta = -\pi, \]
where \( z = re^{i\theta} \).

From the hypothesis (2) and the same reason as the proof of Theorem 1, we have \( f'(z) \neq 0 \) in \( E \).

From the hypothesis (2), we have
\[
\int_{|z|=r} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta = 2\pi
\]
\[
= \int_{\theta_{1}}^{\theta_{2}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta + \int_{\theta_{1}}^{2\pi - \theta_{1}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta
+ \int_{2\pi - \theta_{1}}^{2\pi - \theta_{2}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta
+ \int_{2\pi - \theta_{2}}^{2\pi - \theta_{1}} \left( 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) \right) d\theta
< -\pi + \int_{\theta_{1}}^{2\pi - \theta_{2}} 2d\theta - \pi + \int_{2\pi - \theta_{1}}^{2\pi - \theta_{2}} 2d\theta
= \{4\pi - 2(\theta_{2} - \theta_{1})\} - 2\pi
< 4\pi - 2\pi = 2\pi.
\]

This is a contradiction and so, it completes the proof.

**Remark.** A function \( f(z) \in A \) is typically real in \( E \) if \( \text{Im} f(z) \) \( \text{Im} z > 0 \) for \( E/R = \{ z \mid z \in E \cap z \notin \mathbb{R} \} \). In Theorem 1, if \( f(z) \) is typically real and satisfies (1), then the conclusion continues to hold true.

**References**