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Convolutions and Hölder inequality for certain analytic functions

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Abstract

Applying the coefficient inequalities of functions $f(z)$ belonging to the subclass $\mathcal{MD}(\alpha,\beta)$ of certain analytic functions in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{M}_1(\alpha,\beta)$ and $\mathcal{M}_2(\alpha,\beta)$ are defined. The object of the present paper is to derive some properties for functions $f(z)$ in the classes $\mathcal{M}_1(\alpha,\beta)$ and $\mathcal{M}_2(\alpha,\beta)$ involving their generalized convolution by utilizing methods on the basis of the Hölder inequalities.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}\,|\,|z|<1\}$. Nishiwaki and Owa [2], [4] have considered the subclass $\mathcal{MD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)<\alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta \qquad (z\in\mathbb{U})$$

for some $\alpha(\alpha\leqq 0)$ and $\beta(\beta>1)$. We discuss some properties of functions $f(z)$ belonging to the class $\mathcal{MD}(\alpha,\beta)$.

We note if $f(z)\in\mathcal{MD}(\alpha,\beta)$, then $\dfrac{zf'(z)}{f(z)}=u+iv$ maps $\mathbb{U}$ onto the elliptic domain such that

$$\left(u-\frac{\alpha^2-\beta}{\alpha^2-1}\right)^2+\frac{\alpha^2}{\alpha^2-1}v^2<\frac{\alpha^2(\beta-1)^2}{(\alpha^2-1)^2}$$

for $\alpha<-1$, the parabolic domain such that

$$u<-\frac{1}{2(\beta-1)}v^2+\frac{\beta+1}{2}$$

for $\alpha=-1$, and the hyperbolic domain such that

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$$\left(u-\frac{\alpha^2-\beta}{\alpha^2-1}\right)^2-\frac{\alpha^2}{1-\alpha^2}v^2>\frac{\alpha^2(\beta-1)^2}{(\alpha^2-1)^2}$$

for $-1<\alpha\leqq 0$.

Recently, Nishiwaki and Owa [2] have given the following coefficient inequality for $f(z)$ belonging to the class $\mathcal{MD}(\alpha,\beta)$.

Lemma 1.1. *If $f(z)\in\mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}\{|n-\beta+1|+|n-\beta-1|-2\alpha(n-1)\}|a_n|\leqq\beta-|2-\beta| \tag{1.1}$$

for some $\alpha(\alpha\leqq 0)$ and $\beta(\beta>1)$, then $f(z)\in\mathcal{MD}(\alpha,\beta)$.

From the above lemma, we easily know

$$\sum_{n=2}^{\infty}\frac{(n-\beta+1)+|n-\beta-1|-2\alpha(n-1)}{2(\beta-1)}|a_n|\leqq\sum_{n=2}^{\infty}\frac{(n-\beta+1)+(n+\beta-3)-2\alpha(n-1)}{2(\beta-1)}|a_n|$$
$$\leqq 1$$

for some $\beta(1<\beta\leqq 2)$ and

$$\sum_{n=2}^{\infty}\frac{1}{2}\{|n-\beta+1|+|n-\beta-1|-2\alpha(n-1)\}|a_n|\leqq\sum_{n=2}^{\infty}\frac{1}{2}\{(n+\beta-3)+(n+\beta-3)-2\alpha(n-1)\}|a_n|$$
$$\leqq 1$$

for some $\beta(\beta\geqq 2)$. In view of these inequalities, we define the subclass $\mathcal{M}_1(\alpha,\beta)$ of $\mathcal{MD}(\alpha,\beta)$ consisting of functions $f(z)$ which satisfy the condition

$$\sum_{n=2}^{\infty}\frac{(n-1)(1-\alpha)}{\beta-1}|a_n|\leqq 1 \tag{1.2}$$

for some $\alpha(\alpha\leqq 0)$ and $\beta(1<\beta\leqq 2)$, and also the subclass $\mathcal{M}_2(\alpha,\beta)$ of $\mathcal{MD}(\alpha,\beta)$ consisting of functions $f(z)$ which satisfy the condition

$$\sum_{n=2}^{\infty}\{n(1-\alpha)-3+\alpha+\beta\}|a_n|\leqq 1 \tag{1.3}$$

for some $\alpha(\alpha\leqq 0)$ and $\beta(\beta\geqq 2)$.

2 Generalizations of the Convolutions for the classes $\mathcal{M}_1(\alpha,\beta)$ and $\mathcal{M}_2(\alpha,\beta)$

In this section, some convolution properties of $f(z)$ belonging to the classes $\mathcal{M}_1(\alpha,\beta)$ and $\mathcal{M}_2(\alpha,\beta)$ are discussed.

For functions $f_j(z) \in \mathcal{A}$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (j = 1, 2, \cdots, m),$$

we define

$$H_m(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^{m} a_{n,j}^{p_j} \right) z^n \qquad (p_j > 0).$$

Then $H_m(z)$ denotes the generalization of the convolutions. It was considered by Choi, Kim and Owa [1]. Lately, it was studied by Srivastava and Owa [5] (also see [3]).

For functions $f_j(z) \in \mathcal{A}$, Hölder inequality is given by

$$\sum_{n=2}^{\infty} \left(\prod_{j=1}^{m} |a_{n,j}| \right) \leqq \prod_{j=1}^{m} \left(\sum_{n=2}^{\infty} |a_{n,j}|^{p_j} \right)^{\frac{1}{p_j}},$$

where $p_j > 1$ and $\sum_{j=1}^{m} \frac{1}{p_j} \geqq 1$.

Our first result for $H_m(z)$ is contained in

Theorem 2.1. *If $f_j(z) \in \mathcal{M}_1(\alpha,\beta_j)$ for each $j = 1, 2, \cdots, m$ $(\alpha \leqq 0,\ 1 < \beta_j \leqq 2)$, then $H_m(z) \in \mathcal{M}_1(\alpha,\beta^*)$ with*

$$\beta^* = 1 + \frac{\prod_{j=1}^{m} (\beta_j - 1)^{p_j}}{(1-\alpha)^{s-1}},$$

where $s = \sum_{j=1}^{m} p_j \geqq 1$, $p_j \geqq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1$.

Proof. Let $f_j(z) \in \mathcal{M}_1(\alpha,\beta_j)$, then the inequality (1.2) gives us that

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta_j - 1} |a_{n,j}| \leqq 1 \qquad (j = 1, 2, \cdots, m),$$

which implies

$$\left\{ \sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta_j - 1} |a_{n,j}| \right\}^{\frac{1}{q_j}} \leqq 1$$

with $q_j > 1$ and $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1$. Applying the Hölder inequality, we have the following inequality

$$\sum_{n=2}^{\infty} \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} |a_{n,j}|^{\frac{1}{q_j}} \leqq 1.$$

Then we have to find the largest β such that

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^{m} |a_{n,j}|^{p_j} \right) \leqq 1,$$

that is,

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^{m} |a_{n,j}|^{p_j} \right) \leqq \sum_{n=2}^{\infty} \left\{ \prod_{j=1}^{m} \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} |a_{n,j}|^{\frac{1}{q_j}} \right\}.$$

Therefore, we need to find the largest β such that

$$\frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^{m} |a_{n,j}|^{p_j} \right) \leqq \prod_{j=1}^{m} \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} |a_{n,j}|^{\frac{1}{q_j}}$$

which is equivalent to

$$\frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^{m} |a_{n,j}|^{p_j - \frac{1}{q_j}} \right) \leqq \prod_{j=1}^{m} \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}}$$

for all $n \geqq 2$. Since

$$\prod_{j=1}^{m} \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{p_j - \frac{1}{q_j}} |a_{n,j}|^{p_j - \frac{1}{q_j}} \leqq 1 \qquad \left(p_j - \frac{1}{q_j} \geqq 0 \right),$$

we see that

$$\prod_{j=1}^{m} |a_{n,j}|^{p_j - \frac{1}{q_j}} \leqq \frac{1}{\prod_{j=1}^{m} \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{p_j - \frac{1}{q_j}}}.$$

This implies that

$$\frac{(n-1)(1-\alpha)}{\beta^* - 1} \leqq \prod_{j=1}^{m} \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{p_j}$$

for all $n \geqq 2$. Therefore, β^* should be

$$\beta^* \geqq 1 + \frac{\prod_{j=1}^{m} (\beta_j - 1)^{p_j}}{(1-\alpha)^{s-1} (n-1)^{s-1}} \qquad \left(s = \sum_{j=1}^{m} p_j \right),$$

so that, the right hand side of the last inequality is a decreasing function for $n \geqq 2$. This means

$$\beta^* = \max_{n\geqq 2}\left\{1 + \frac{\prod_{j=1}^{m}(\beta_j - 1)}{(1-\alpha)^{s-1}(n-1)^{s-1}}\right\}$$

$$= 1 + \frac{\prod_{j=1}^{m}(\beta_j - 1)^{p_j}}{(1-\alpha)^{s-1}}.$$

This completes the proof of the theorem. □

Example 2.1. Let us define

$$f_j(z) = z + \sum_{n=2}^{\infty} \frac{(\beta_j - 1)\varepsilon_j}{n(n-1)^2(1-\alpha)} z^n \qquad (|\varepsilon_j| = 1)$$

for each j $(j = 1, 2, 3, \cdots, m)$. It is easy to see that $f_j(z) \in \mathcal{M}_1(\alpha, \beta_j)$. Then we have

$$H_m(z) = z + \sum_{n=2}^{\infty}\left(\prod_{j=1}^{m}\left(\frac{(\beta_j - 1)\varepsilon_j}{n(n-1)^2(1-\alpha)}\right)^{p_j}\right) z^n.$$

For this function $H_m(z)$, we calculate that

$$\sum_{n=2}^{\infty}\left(\frac{(n-1)(1-\alpha)}{\beta^* - 1}\right)\left|\prod_{j=1}^{m}\left(\frac{(\beta_j - 1)\varepsilon_j}{n(n-1)^2(1-\alpha)}\right)^{p_j}\right|$$

$$= \sum_{n=2}^{\infty}\frac{1}{n^s(n-1)^{2s-1}} \leqq \sum_{n=2}^{\infty}\frac{1}{n(n-1)} = 1.$$

Thus we know that $H_m(z) \in \mathcal{M}_1(\alpha, \beta^*)$.

Taking $\beta_j = \beta$ $(j = 1, 2, \cdots, m)$ in Theorem 2.1, we obtain

Corollary 2.1. *If $f_j(z) \in \mathcal{M}_1(\alpha, \beta)$ for each $j = 1, 2, \cdots, m$ $(\alpha \leqq 0,\ 1 < \beta \leqq 2)$, then $H_m(z) \in \mathcal{M}_1(\alpha, \beta^*)$ with*

$$\beta^* = 1 + \frac{(\beta - 1)^s}{(1-\alpha)^{s-1}},$$

where $s = \sum_{j=1}^{m} p_j \geqq 1$, $p_j \geqq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^{m}\frac{1}{q_j} \geqq 1$.

By using $\mathcal{M}_1(\alpha_j, \beta)$ instead of $\mathcal{M}_1(\alpha, \beta_j)$ in Theorem 2.1, we also derive the next result.

Theorem 2.2. *If $f_j(z) \in \mathcal{M}_1(\alpha_j,\beta)$ for each $j = 1,2,\cdots,m$ ($\alpha_j \leqq 0,\ 1 < \beta \leqq 2$), then $H_m(z) \in \mathcal{M}_1(\alpha^*,\beta)$ with*

$$\alpha^* = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}},$$

where $s = \sum_{j=1}^{m} p_j \geqq 1$, $p_j \geqq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^{m}\frac{1}{q_j} \geqq 1$.

Proof. Using the same method as in the proof of Theorem 2.1, we have

$$\frac{(n-1)(1-\alpha^*)}{\beta-1} \leqq \frac{(n-1)^s\prod_{j=1}^{m}(1-\alpha_j)^{p_j}}{(\beta-1)^s},$$

which implies that

$$\alpha^* \geqq 1 - \frac{(n-1)^{s-1}\prod_{j=1}^{m}(1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}}$$

for all $n \geqq 2$, so that, the right hand side of the last inequality is a decreasing for $n \geqq 2$. This means

$$\alpha^* = \max_{n\geqq 2}\left\{1 - \frac{(n-1)^{s-1}\prod_{j=1}^{m}(1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}}\right\}$$

$$= 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}},$$

which proves the theorem. □

Example 2.2. Let us consider

$$f_j(z) = z + \sum_{n=2}^{\infty}\frac{(\beta-1)\varepsilon_j}{n(n-1)^2(1-\alpha_j)}z^n \qquad (|\varepsilon_j| = 1)$$

for each j ($j = 1,2,3,\cdots,m$). Then we see that $f_j(z) \in \mathcal{M}_1(\alpha_j,\beta)$. Also we have that

$$H_m(z) = z + \sum_{n=2}^{\infty}\left(\prod_{j=1}^{m}\left(\frac{(\beta-1)\varepsilon_j}{n(n-1)^2(1-\alpha_j)}\right)^{p_j}\right)z^n.$$

It follows from the function $H_m(z)$ that

$$\sum_{n=2}^{\infty}\left(\frac{(n-1)(1-\alpha^*)}{\beta-1}\right)\left|\prod_{j=1}^{m}\left(\frac{(\beta-1)\varepsilon_j}{n(n-1)^2(1-\alpha_j)}\right)^{p_j}\right|$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^s(n-1)^{2s-1}} \leqq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1.$$

This implies that $H_m(z) \in \mathcal{M}_1(\alpha^*, \beta)$.

Letting $\alpha_j = \alpha$ $(j = 1, 2, \cdots, m)$ in Theorem 2.2, we obtain

Corollary 2.2. *If* $f_j(z) \in \mathcal{M}_1(\alpha, \beta)$ *for each* $j = 1, 2, \cdots, m$ $(\alpha \leqq 0,\ 1 < \beta \leqq 2)$, *then* $H_m(z) \in \mathcal{M}_1(\alpha^*, \beta)$ *with*

$$\alpha^* = 1 - \frac{(1-\alpha)^s}{(\beta-1)^{s-1}},$$

where $s = \sum_{j=1}^{m} p_j \geqq 1$, $p_j \geqq \frac{1}{q_j}$, $q_j > 1$ *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1$.

Next, for the generalized Hadamard product (or Convolution) of funcitons in the class $\mathcal{M}_2(\alpha, \beta)$, we also derive

Theorem 2.3. *If* $f_j(z) \in \mathcal{M}_2(\alpha, \beta_j)$ *for each* $j = 1, 2, \cdots, m$ $(\alpha \leqq 0,\ \beta_j \geqq 2)$, *then* $H_m(z) \in \mathcal{M}_2(\alpha, \beta^*)$ *with*

$$\beta^* = 1 + \alpha + \prod_{j=1}^{m} (\beta_j - 1 - \alpha)^{p_j},$$

where $p_j \geqq \frac{1}{q_j}$, $q_j > 1$ *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1$.

Proof. In the same manner as in the proof of Theorem 2.1, we obtain

$$\beta^* + (n-1)(1-\alpha) - 2 \leqq \prod_{j=1}^{m} \{n(1-\alpha) - 3 + \alpha + \beta_j\}^{p_j}.$$

The left hand side of the above inequality is a increasing function for $n \geqq 2$. Then we get

$$\beta^* - 1 - \alpha \leqq \prod_{j=1}^{m} \{n(1-\alpha) - 3 + \alpha + \beta_j\}^{p_j}.$$

Also the right hand side of it is a increasing function for $n \geqq 2$, so that, we have

$$\beta^* \leqq 1 + \alpha + \prod_{j=1}^{m} (\beta_j - 1 - \alpha)^{p_j}.$$

This completes the proof of the theorem. □

If we take $\beta_j = \beta$ $(j = 1, 2, \cdots, m)$ in Theorem 2.3, then we obtain

Corollary 2.3. *If $f_j(z) \in \mathcal{M}_2(\alpha, \beta)$ for each $j = 1, 2, \cdots, m$ $(\alpha \leqq 0,\ \beta \geqq 2)$, then $H_m(z) \in \mathcal{M}_2(\alpha, \beta^*)$ with*

$$\beta^* = 1 + \alpha + (\beta - 1 - \alpha)^s,$$

where $s = \sum_{j=1}^{m} p_j \geqq 1$, $p_j \geqq \dfrac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^{m} \dfrac{1}{q_j} \geqq 1$.

Using $\mathcal{M}_2(\alpha_j, \beta)$ instead of $\mathcal{M}_2(\alpha, \beta_j)$ in Theorem 2.3, we also derive the next result.

Theorem 2.4. *If $f_j(z) \in \mathcal{M}_2(\alpha_j, \beta)$ for each $j = 1, 2, \cdots, m$ $(\alpha_j \leqq 0,\ \beta \geqq 2)$, then $H_m(z) \in \mathcal{M}_2(\alpha^*, \beta)$ with*

$$\alpha^* = \max_{n \geqq 2} \left\{ 1 - \frac{(\beta - 2) + \prod_{j=1}^{m} (n(1 - \alpha_j) - 3 + \alpha_j + \beta)^{p_j}}{n - 1} \right\},$$

where $p_j \geqq \dfrac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^{m} \dfrac{1}{q_j} \geqq 1$.

Proof. By using the same method as in the proof of Theorem 2.1, we see that

$$n - 3 - \alpha^*(n - 1) + \beta \leqq \prod_{j=1}^{m} \{n(1 - \alpha_j) - 3 + \alpha_j + \beta\}^{p_j}$$

which implies that

$$\alpha^* \geqq 1 - \frac{(\beta - 2) + \prod_{j=1}^{m} \{n(1 - \alpha_j) - 3 + \alpha_j + \beta\}^{p_j}}{n - 1}.$$

Therefore, we prove the theorem. □

Finally, taking $\alpha_j = \alpha$ $(j = 1, 2, \cdots, m)$ in Theorem 2.4, we obtain

Corollary 2.4. *If $f_j(z) \in \mathcal{M}_2(\alpha, \beta)$ for each $j = 1, 2, \cdots, m$ $(\alpha \leqq 0,\ \beta \geqq 2)$, then $H_m(z) \in \mathcal{M}_2(\alpha^*, \beta)$ with*

$$\alpha^* = 3 - \beta - (\beta - 1 - \alpha)^s,$$

where $s = \sum_{j=1}^{m} p_j \geqq 1 + \dfrac{2(\beta - 2)}{1 - \alpha}$, $p_j \geqq \dfrac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^{m} \dfrac{1}{q_j} \geqq 1$.

Proof. In view of Theorem 2.4, we obtain

$$\alpha^* \geqq 1 - \frac{(\beta-2)+\{n(1-\alpha)-3+\alpha+\beta\}^s}{n-1}.$$

Let $F(n)$ be the right hand side of the above inequality. Further, let us define $G(n)$ by the numerator of $F'(n)$, so that

$$\begin{aligned} G(n) &= -(n(1-\alpha)-3+\alpha+\beta)^{s-1}\{n(1-\alpha)(s-1)-s(1-\alpha)+3-\alpha-\beta\}+(\beta-2) \\ &\leqq -(\beta-1-\alpha)^{s-1}\{2(1-\alpha)(s-1)-s(1-\alpha)+3-\alpha-\beta\}+(\beta-2) \\ &\leqq 2\beta-3-\alpha-s(1-\alpha) \\ &\leqq 0 \qquad \left(s \geqq 1+\frac{2(\beta-2)}{1-\alpha}\right) \end{aligned}$$

which implies that

$$\begin{aligned} \alpha^* &= \max_{n \geqq 2}\left\{1-\frac{(\beta-2)+\prod_{j=1}^{m}(n(1-\alpha_j)-3+\alpha_j+\beta)^{p_j}}{n-1}\right\} \\ &= 3-\beta-(\beta-1-\alpha)^s. \end{aligned}$$

This completes proof of the corollary. □

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