CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

OH SANG KWON AND BYUNG GU PARK

ABSTRACT. The object of the present paper is to drive some properties of certain class $K_{n,p}(A,B)$ of multivalent analytic functions in the open unit disk $E$.

1. Introduction

Let $A_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$$

which are analytic in the open unit disk $E = \{ z \in \mathbb{C} : |z| < 1 \}$. A function $f \in A_p$ is said to be $p$-valently starlike functions of order $\alpha$ of it satisfies the condition

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p, z \in E).$$

We denote by $S^*_p(\alpha)$.

On the other hand, a function $f \in A_p$ is said to be $p$-valently close-to-convex functions of order $\alpha$ if it satisfies the condition

$$\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > \alpha \quad (0 \leq \alpha < p, z \in E),$$

for some starlike function $g(z)$. We denote by $C_p(\alpha)$.

2000 Mathematics Subject Classification. 30C45.

Key words and phrases. $p$-valently starlike functions of order $\alpha$, $p$-valently close-to-convex functions of order $\alpha$, subordination, hypergeometric series.
For \( f \in A_p \) given by (1.1), the generalized Bernardi integral operator \( F_c \) is defined by
\[
F_c(z) = \frac{c+p}{z^c} \int_0^z f(t)t^{c-1}dt = z^p + \sum_{k=1}^{\infty} \frac{c+p}{c+p+k} a_{p+k}z^{p+k} \quad (c+p > 0, z \in E).
\]

For an analytic function \( g \), defined in \( E \) by
\[
g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k}z^{p+k}
\]
and Flett [3] defined the multiplier transform \( I^n \) for a real number \( \eta \) by
\[
I^n g(z) = \sum_{k=0}^{\infty} (p+k+1)^{-\eta} b_{p+k}z^{p+k} \quad (z \in E).
\]
Clearly, the function \( I^n g \) is analytic in \( E \) and
\[
I^n(J^\mu g(z)) = I^{n+\mu}g(z)
\]
for all real number \( \eta \) and \( \mu \).

For any integer \( n \), J. Patel and P. Sahoo [5] also defined the operator \( D^n \), for an analytic function \( f \) given by (1.1), by
\[
D^n f(z) = z^p + \sum_{k=1}^{\infty} \left( \frac{p+k+1}{1+p} \right)^{-n} a_{p+k}z^{p+k} = f(z) \ast z^{p-1} \left[ z + \sum_{k=1}^{\infty} \left( \frac{k+1+p}{1+p} \right)^{-n} z^{k+1} \right] \quad (z \in E)
\]
where \( \ast \) stands for the Hadamard product or convolution.

It follows from (1.3) that
\[
z(D^n f(z))' = (p+1)D^{n-1}f(z) - D^n f(z). \tag{1.4}
\]

We also have
\[
D^0 f(z) = f(z) \quad \text{and} \quad D^{-1} f(z) = \frac{zf'(z) + f(z)}{p+1}.
\]
CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

If \( f \) and \( g \) are analytic functions in \( E \), then we say that \( f \) is subordinate to \( g \) written \( f \prec g \) or \( f(z) \prec g(z) \), if there is a function \( w \) analytic in \( E \), with \( w(0) = 0 \), \( |w(z)| < 1 \) for \( z \in E \), such that \( f(z) = g(w(z)) \), for \( z \in E \). If \( g \) is univalent then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(E) \subset g(E) \).

Making use of the operator notation \( D^n \), we introduce a subclass of \( A_p \) as follows:

**Definition 1.1.** For any integer \( n \) and \(-1 \leq B < A \leq 1\), a function \( f \in A_p \) is said to be in the class \( K_{n,p}(A, B) \) if

\[
\frac{z(D^n f(z))'}{z^p} \prec \frac{p(1 + Az)}{1 + Bz}
\]  

(1.5)

where \( \prec \) denotes subordination.

For convenience, we write

\[
K_{n,p} \left( 1 - \frac{2\alpha}{p}, -1 \right) = K_{n,p}(\alpha),
\]

where \( K_{n,p}(\alpha) \) denote the class of function \( f \in A_p \) satisfying the inequality

\[
\text{Re} \left\{ \frac{z(D^n f(z))'}{z^p} \right\} > \alpha \quad (0 \leq \alpha < p, \ z \in E).
\]

We also note that \( K_{0,p}(\alpha) \equiv C_p(\alpha) \) is the class of \( p \)-valently close-to-convex functions of order \( \alpha \).

In this present paper, we derive some properties of certain class \( K_{n,p}(A, B) \) by using the differential subordination.

2. Preliminaries and Main Results

In our present investigation of the general class \( K_{n,p}(A, B) \), we shall require the following lemmas.
Lemma 1 [4]. If the function \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is analytic in \( E \), \( h(z) \) is convex in \( E \) with \( h(0) = 1 \), and \( \gamma \) is complex number such that \( \text{Re} \ \gamma > 0 \). Then the Briot-Bouquet differential subordination

\[
p(z) + \frac{zp'(z)}{\gamma} \prec h(z)
\]

implies

\[
p(z) < q(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t) dt \prec h(z) \quad (z \in E)
\]

and \( q(z) \) is the best dominant.

For complex number \( a, b \) and \( c \neq 0, -1, -2, \cdots \), the hypergeometric series

\[
2F_1(a, b; c; z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \cdots
\] (2.1)

represents an analytic function in \( E \). It is well known by [1] that

Lemma 2. Let \( a, b \) and \( c \) be real \( c \neq 0, -1, -2, \cdots \) and \( c > b > 0 \). Then

\[
\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} 2F_1(a, b; c; z),
\] (2.2)

\[
2F_1(a, b; c; z) = (1-z)^{-a} 2F_1 \left( a, c-b; c; \frac{z}{z-1} \right)
\]

and

\[
2F_1(a, b; c; z) = 2F_1(b, a; c; z).
\] (2.3)

Lemma 3 [6]. Let \( \phi(z) \) be convex and \( g(z) \) is starlike in \( E \). Then for \( F \) analytic in \( E \) with \( F(0) = 1 \), \( \frac{\phi*Fg}{\phi*g}(E) \) is contained in the convex hull of \( F(E) \).

Lemma 4 [2]. Let \( \phi(z) = 1 + \sum_{k=1}^\infty c_k z^k \) and \( \phi(z) \prec \frac{1 + Az}{1 + Bz} \). Then

\[
|c_k| \leq (A - B).
\]
CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

Theorem 1. Let $n$ be any integer and $-1 \leq B < A \leq 1$. If $f \in K_{n,p}(A, B)$, then

$$\frac{z(D^{n+1}f(z))'}{z^p} \prec q(z) \prec \frac{p(1+Az)}{1+Bz} \quad (z \in E), \quad (2.4)$$

where

$$q(z) = \begin{cases} 2F_1(1,p+1;p+2;-Bz) + \frac{p+1}{p+2}Az \frac{2F_1(1,p+2;p+3;-Bz)}{A}, & B \neq 0 \\ 1 + \frac{p+1}{p+2}Az, & B = 0 \end{cases} \quad (2.5)$$

and $q(z)$ is the best dominant of (2.4). Furthermore, $f \in K_{n+1,p}(\rho(p,A,B))$, where

$$\rho(p,A,B) = \begin{cases} p_2F_1(1,p+1;p+2;B) - \frac{p(p+1)}{p+2}A_2F_1(1,p+2;p+3;B), & B \neq 0 \\ 1 - \frac{p+1}{p+2}A, & B = 0. \end{cases} \quad (2.6)$$

Proof. Let

$$p(z) = \frac{z(D^{n+1}f(z))'}{pz^p} \quad (2.7)$$

where $p(z)$ is an analytic function with $p(0) = 1$.

Using the identity (1.4) in (2.7) and differentiating the resulting equation, we get

$$\frac{z(D^n f(z))'}{pz^p} = p(z) + \frac{zp'(z)}{p+1} \prec \frac{1+Az}{1+Bz} (\equiv h(z)). \quad (2.8)$$

Thus, by using Lemma 1 (for $\gamma = p+1$), we deduce that

$$p(z) \prec (p+1)z^{-(p+1)} \int_0^z \frac{t^p(1+At)}{1+Bt}dt (\equiv q(z))$$

$$= (p+1) \int_0^1 \frac{s^p(1+Asz)}{1+Bsz}ds = (p+1)Az \int_0^1 \frac{s^{p+1}}{1+Bsz}ds. \quad (2.9)$$
By using (2.2) in (2.9), we obtain

\[
p(z) \prec q(z) = \begin{cases} 
2F_1(1,p+1;p+2;-Bz) \\
\frac{p+1}{p+2}Az_2F_1(1,p+2;p+3;-Bz), & B \neq 0 \\
1 + \frac{p+1}{p+2}Az, & B = 0.
\end{cases}
\]

Thus, this proves (2.5).

Now, we show that

\[
\text{Re} \; q(z) \geq q(-r) \quad (|z| = r < 1).
\]  \hspace{1cm} (2.10)

Since \(-1 \leq B < A \leq 1\), the function \((1 + Az)/(1 + Bz)\) is convex (univalent) in \(E\) and

\[
\text{Re} \left( \frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br} > 0 \quad (|z| = r < 1).
\]

Setting

\[
g(s,z) = \frac{1 + Asz}{1 + Bsiz} \quad (0 \leq s \leq 1, \; z \in E)
\]

and \(d\mu(s) = (p+1)s^p ds\), which is a positive measure on \([0, 1]\), we obtain from (2.9) that

\[
q(z) = \int_0^1 g(s, z)d\mu(s) \quad (z \in E).
\]

Therefore, we have

\[
\text{Re} \; q(z) = \int_0^1 \text{Re} \; g(s, z)d\mu(s) \geq \int_0^1 \frac{1 - Asr}{1 - Brs}d\mu(s)
\]

which proves the inequality (2.10).

Now, using (2.10) in (2.9) and letting \(r \to 1^-\), we obtain

\[
\text{Re} \left\{ \frac{z(D^{n+1}f(z))'}{z^p} \right\} > \rho(p, A, B),
\]
CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

where

\[
\rho(p, A, B) = \begin{cases} 
p_2F_1(1,p+1;p+2;B) \\
p(p+1) - \frac{p(p+1)}{p+2} A_2F_1(1,p+2;p+3;B), & B \neq 0 \\
p - \frac{p(p+1)}{p+2} A, & B = 0.
\end{cases}
\]

This proves the assertion of Theorem 1. The result is best possible because of the best dominant property of \(q(z)\).

Putting \(A = 1 - \frac{2\alpha}{p}\) and \(B = -1\) in Theorem 1, we have the following:

**Corollary 1.** For any integer \(n\) and \(0 \leq \alpha < p\), we have

\[K_{n,p}(\alpha) \subset K_{n+1,p}(\rho(p, \alpha)),\]

where

\[
\rho(p, \alpha) = p_2F_1(1,p+1;p+2;-1) - \frac{p(p+1)}{p+2}(1-2\alpha)_2F_1(1,p+2;p+3;-1).
\]

The result is best possible.

Taking \(p = 1\) in Corollary 1, we have the following:

**Corollary 2.** For any integer \(n\) and \(0 \leq \alpha < 1\), we have

\[K_n(\delta) \subset K_{n+1}(\delta(\alpha))\]

where

\[
\delta(\alpha) = 1 + 4(1-2\alpha) \sum_{k=1}^{\infty} \frac{1}{k+2}(-1)^k.
\]

Theorem 2. For any integer \(n\) and \(0 \leq \alpha < p\), if \(f(z) \in K_{n+1,p}(\alpha)\) then \(f \in K_{n,p}(\alpha)\) for \(|z| < R(p)\), where \(R(p) = \frac{-1 + \sqrt{1 + (p + 1)^2}}{p + 1}\).

The result is best possible.

**Proof.** Since \(f(z) \in K_{n+1,p}(\alpha)\), we have

\[
\frac{z(D^{n+1}f(z))'}{z^p} = \alpha + (p - \alpha)w(z), \quad (0 \leq \alpha < p),
\]

(2.13)
where \( w(z) = 1 + w_1 z + w_2 z + \cdots \) is analytic and has a positive real part in \( E \). Making use of the logarithmic differentiation and using identity (1.4) in (2.13), we get
\[
\frac{z(D^n f(z))'}{z^p} - \alpha = (p - \alpha) \left[ w(z) + \frac{zw'(z)}{p+1} \right].
\]
(2.14)

Now, using the well-known by [5],
\[
\frac{|zw'(z)|}{\text{Re } w(z)} \leq \frac{2r}{1-r^2} \quad \text{and} \quad \text{Re } w(z) \geq \frac{1-r}{1+r} \quad (|z| = r < 1),
\]
in (2.14). We get
\[
\text{Re} \left\{ \frac{z(D^n f(z))'}{z^p} - \alpha \right\} = (p - \alpha) \text{Re } w(z) \left\{ 1 + \frac{1}{p+1} \frac{\text{Re } zw'(z)}{\text{Re } w(z)} \right\}
\geq (p - \alpha) \text{Re } w(z) \left\{ 1 - \frac{1}{p+1} \frac{|zw'(z)|}{\text{Re } w(z)} \right\}
\geq (p - \alpha) \frac{1-r}{1+r} \left\{ 1 - \frac{1}{p+1} \frac{2r}{1-r^2} \right\}.
\]

It is easily seen that the right-hand side of the above expression is positive if \(|z| < R(p) = \frac{-1 + \sqrt{1+(p+1)^2}}{p+1}\). Hence \( f \in K_{n,p}(\alpha) \) for \(|z| < R(p)\).

To show that the bound \( R(p) \) is best possible, we consider the function \( f \in A_p \) defined by
\[
\frac{z(D^{n+1} f(z))'}{z^p} = \alpha + (p - \alpha) \frac{1-z}{1+z} \quad (z \in E).
\]
Noting that
\[
\frac{z(D^n f(z))'}{z^p} - \alpha = (p - \alpha) \cdot \frac{1-z}{1+z} \left\{ 1 + \frac{1}{p+1} \frac{z}{(p+1)(1-z^2)} \right\}
= (p - \alpha) \cdot \frac{1-z}{1+z} \left\{ \frac{(p+1) + (p+1)z^2 - 2z}{(p+1) - (p+1)z^2} \right\}
= 0
\]
for \( z = \frac{-1 + \sqrt{1+(p+1)^2}}{p+1} \), we complete the proof of Theorem 2.

Putting \( n = -1, \ p = 1 \) and \( 0 \leq \alpha < 1 \) in Theorem 2, we have the following:
CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

Corollary 3. If $\text{Re } f'(z) > \alpha$, then $\text{Re } \{zf''(z) + 2f'(z)\} > \alpha$ for $|z| < \frac{-1 + \sqrt{5}}{2}$.

Theorem 3. (a) If $f \in K_{n,p}(A,B)$, then the function $F_c$ defined by (1.2) belongs to $K_{n,p}(A,B)$.

(b) $f \in K_{n,p}(A,B)$ implies that $F_c \in K_{n,p}(\eta(p, c, A, B))$ where

$$
\eta(p, c, A, B) = \left\{ \begin{array}{ll}
    p_2F_1(1, p + c; p + c + 1; B) & 
    \quad B \neq 0 \\
    -\frac{p(p + c)}{p + c + 1} A_2F_1(1, p + c + 1; p + c + 2; B), & 
    \quad B \neq 0 \\
    p - \frac{p(p + c)}{p + c + 1} A, & 
    \quad B = 0.
\end{array} \right.
$$

Proof. Let

$$
\phi(z) = \frac{z(D^n F_c(z))'}{pz^p}, \quad (2.15)
$$

where $\phi(z)$ is analytic function with $\phi(0) = 1$. Using the identity

$$
z(D^n F_c(z))' = (p + c) D^n f(z) - cD^n F_c(z) \quad (2.16)
$$
in (2.15) and differentiating the resulting equation, we get

$$
\frac{z(D^n f(z))'}{pz^p} = \phi(z) + \frac{z\phi'(z)}{p + c}.
$$

Since $f \in K_{n,p}(A,B)$,

$$
\phi(z) + \frac{z\phi'(z)}{p + c} < \frac{1 + Az}{1 + Bz}.
$$

By Lemma 1, we obtain $F_c(z) \in K_{n,p}(A,B)$. We deduce that

$$
\phi(z) < q(z) < \frac{1 + Az}{1 + Bz} \quad (2.17)
$$

where $q(z)$ is given (2.5) and $q(z)$ is best deminent of (2.17).

This proves the (a) part of theorem. Proceeding as in Theorem 3, the (b) part follows.

Putting $A = 1 - \frac{2\alpha}{p}$ and $B = -1$ in Theorem 2, we have the following:
Corollary 4. If \( f \in K_{n,p}(A,B) \) for \( 0 \leq \alpha < p \), then \( F_c \in K_{n,p} \mathcal{H}(p,c,\alpha) \) where

\[
\mathcal{H}(p,c,\alpha) = p \cdot 2F_1(1,p+c;p+c+1;-1) - \frac{p+c}{p+c+1}(p-2\alpha)2F_1(1,p+c;p+c+1;-1).
\]

Setting \( c = p = 1 \) in Theorem 3, we get the following result.

Corollary 4. If \( f \in K_{n,p}(\alpha) \) for \( 0 \leq \alpha < 1 \), then the function

\[
G(z) = \frac{2}{z} \int_0^z f(t)dt
\]

belongs to the class \( K_n(\delta(\alpha)) \), where \( \delta(\alpha) \) is given by (2.12).

Theorem 4. For any integer \( n \) and \( 0 \leq \alpha < p \) and \( c > -p \), if \( F_c \in K_{n,p}(\alpha) \) then the function \( f \) defined by (1.1) belongs to \( K_{n,p}(\alpha) \) for \(|z| < R(p,c) = \frac{-1 + \sqrt{1 + (p+c)^2}}{p+c}\). The result is best possible.

Proof. Since \( F_c \in K_{n,p}(\alpha) \), we write

\[
\frac{z(D^nF_c)'}{z^p} = \alpha + (p-\alpha)w(z), \tag{2.18}
\]

where \( w(z) \) is analytic, \( w(0) = 1 \) and \( \text{Re } w(z) > 0 \) in \( E \). Using (2.16) in (2.18) and differentiating the resulting equation, we obtain

\[
\text{Re } \left\{ \frac{z(D^n f(z))'}{z^p} - \alpha \right\} = (p-\alpha)\text{Re } \left\{ w(z) + \frac{zw'(z)}{p+c} \right\}. \tag{2.19}
\]

Now, by following the line of proof of Theorem 2, we get the assertion of Theorem 4.
CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

**Theorem 5.** Let \( f \in K_{n,p}(A,B) \) and \( \phi(z) \in A_p \) convex in \( E \). Then

\[
(f \ast \phi(z))(z) \in K_{n,p}(A,B).
\]

**Proof.** Since \( f(z) \in K_{n,p}(A,B) \),

\[
\frac{z(D^n f(z))'}{pz^p} \prec \frac{1+Az}{1+Bz}.
\]

Now

\[
\frac{z(D^n(f \ast \phi)(z))'}{pz^p \ast \phi(z)} = \frac{\phi(z) \ast z(D^n f)'}{\phi(z) \ast pz^p} = \frac{\phi(z) \ast \frac{z(D^n f(z))'}{pz^p}pz^p}{\phi(z) \ast pz^p}.
\]

Then applying Lemma 3, we deduce that

\[
\frac{\phi(z) \ast \frac{z(D^n f(z))'}{pz^p}pz^p}{\phi(z) \ast pz^p} \prec \frac{1+Az}{1+Bz}.
\]

Hence \((f \ast \phi(z))(z) \in K_{n,p}(A,B)\).

**Theorem 6.** Let a function \( f(z) \) defined by (1.1) be in the class \( K_{n,p}(A,B) \). Then

\[
|a_{p+k}| \leq \frac{p(A-B)(p+k+1)^n}{(1+p)^n(p+k)} \quad \text{for} \quad k = 1, 2, \ldots.
\]

The result is sharp.

**Proof.** Since \( f(z) \in K_{n,p}(A,B) \), we have

\[
\frac{z(D^n f(z))'}{pz^p} \equiv \phi(z) \quad \text{and} \quad \phi(z) \prec \frac{1+Az}{1+Bz}.
\]

Hence

\[
z(D^n f(z))' = pz^p \phi(z) \quad \text{and} \quad \phi(z) = 1 + \sum_{k=1}^{\infty} c_k z^k.
\]
From (2.22), we have

$$z(D^n f(z))' = z \left( z^p + \sum_{k=1}^{\infty} \left( \frac{1+p}{p+k+1} \right)^n a_{p+k} z^{p+k} \right)'$$

$$= pz^p + \sum_{k=1}^{\infty} \left( \frac{1+p}{p+k+1} \right)^n (p+k)a_{p+k} z^{p+k}$$

$$= pz^p \left( 1 + \sum_{k=1}^{\infty} c_k z^k \right).$$

Therefore

$$\left( \frac{1+p}{p+k+1} \right)^n (p+k)a_{p+k} = pc_k. \quad (2.23)$$

By using Lemma 4 in (2.23),

$$\left( \frac{1+p}{p+k+1} \right)^n (p+k)|a_{p+k}| \leq \frac{p(A-B)(p+k+1)^n}{p} = |c_k| \leq A - B.$$

Hence

$$|a_{p+k}| \leq \frac{p(A-B)(p+k+1)^n}{(1+p)^n(p+k)}. \quad (2.23)$$

The equality sign in (2.21) holds for the function $f$ given by

$$(D^n f(z))' = \frac{pz^{p-1} + p(A-B-1)z^p}{1-z}. \quad (2.24)$$

Hence

$$\frac{z(D^n f(z))'}{pz^p} = \frac{1 + (A-B-1)z}{1-z} < \frac{1+A}{1+B}z \quad \text{for } k=1,2,\ldots.$$
CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

REFERENCES


Oh Sang Kwon
Department of Mathematics, Kyungsung University
Busan 608-736, Korea
oskwon@ks.ac.kr