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A CLASS OF DOUBLE SUBORDINATION-PRESERVING INTEGRAL OPERATORS FOR MULTIVALENT FUNCTIONS

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In the present paper, we obtain some subordination- and superordination- preserving properties for certain integral operators defined on the space of multivalent functions in the open unit disk. The sandwich-type theorems for these integral operators are also considered. Moreover, we consider applications of the subordination and superordination theorems to the Gauss hypergeometric function.

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1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}).$$

In such a case, we write

$$f \prec F \quad (z \in \mathbb{U}) \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \mathbb{U}).$$

If the function F is univalent in $\mathbb{U}$, then we have (cf. [10,17])

$$f \prec F \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Definition 1.1 (Miller and Mocanu [10]). Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination:

$$\phi(p(z), zp'(z)) \prec h(z), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

Recently, Miller and Mocanu [11] introduced the following differential superordinations, as the dual concept of differential subordinations.

Definition 1.2 (Miller and Mocanu [11]). Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination:

$$h(z) \prec \varphi(p(z), zp'(z)), \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 1.3 [11]. We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} (z) = \infty \right\}, \tag{1.3}$$

and are such that

$$f'(\zeta) \neq 0 \quad (\zeta \in \partial\mathbb{U} \setminus E(f)).$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, \cdots\}) \tag{1.4}$$

which are analytic and p-valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}_p^*(A, B)$ be the subclass of $\mathcal{A}_p$ satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec p\frac{1+Az}{1+Bz} \quad (f^{p+1}(0) \neq 0;\ -1 \leq B < A \leq 1;\ z \in \mathbb{U}).$$

We note that $\mathcal{S}_p^*(1, -1) \equiv \mathcal{S}_p^*$ is the class of all functions which are p-valent starlike in $\mathbb{U}$.

For a function $f \in \mathcal{A}_p$, we introduce the following integral operator $I_{\alpha,\beta}$ defined by

$$I_{\alpha,\beta}(f)(z) := \left(\frac{p\alpha+\beta}{z^\beta}\int_0^z t^{\beta-1} f^\alpha(t)dt\right)^{1/\alpha} \tag{1.5}$$

$$(f \in \mathcal{A}_p;\ \alpha \in \mathbb{C}\setminus\{0\};\ \beta \in \mathbb{C};\ \mathrm{Re}\{p\alpha+\beta\} > 0).$$

The two-parameter integral operator defined by (1.5) have been extensively studied by many authors [1-3,5-6,9,12,14] with suitable restriction on the parameters α and β, and for f belonging to some favoured classes of analytic functions. In particular, Kumar and Shukla [5] showed that the integral operator $I_{\alpha,\beta}(f)$ belongs to the class $\mathcal{S}_p^*(A, B)$ for $\alpha > 0$ and $\beta \geq -p\alpha(1-A)/(1-B)$, whenever f belongs to the class $\mathcal{S}_p^*(A, B)$, which include the results earlier by Bernardi [1] and Libera [6].

Making use of the principle of subordination between analytic functions, Miller et al. [13] obtained some subordination theorems involving certain integral operators for analytic functions in $\mathbb{U}$ (see, also [2,15]). Moreover, Bulboacă [3] investigated the superordination-preserving properties of the integral operator defined by (1.5) with some conditions on the parameters p, α and β. In the present paper, we obtain the subordination- and superordination-preserving properties of the integral operator $I_{\alpha,\beta}$ defined by (1.5) with the sandwich-type theorems. We also consider some interesting applications of our main results to the Gauss hypergeometric function.

The following lemmas will be required in our present investigation.

Lemma 1.1 (Miller and Mocanu [7]). *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the following condition:*

$$\mathrm{Re}\{H(is, t)\} \leq 0$$

for all real s *and* $t \leq -n(1+s^2)/2$, *where* n *is a positive integer. If the function*

$$p(z) = 1 + p_n z^n + \cdots$$

is analytic in $\mathbb{U}$ *and*

$$\text{Re}\{H(p(z), zp'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then

$$\text{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Lemma 1.2 (Miller and Mocanu [8]). *Let* $\alpha, \beta \in \mathbb{C}$ *with* $\alpha \neq 0$ *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$. *If*

$$\text{Re}\{\alpha h(z) + \beta\} > 0 \quad (z \in \mathbb{U}),$$

then the solution of the differential equation:

$$q(z) + \frac{zq'(z)}{\alpha q(z) + \beta} = h(z) \quad (z \in \mathbb{U};\ q(0) = c)$$

is analytic in $\mathbb{U}$ *and satisfies the inequality given by*

$$\text{Re}\{\alpha q(z) + \beta\} > 0 \quad (z \in \mathbb{U}).$$

Lemma 1.3 (Miller and Mocanu [10]). *Let* $p \in \mathcal{Q}$ *with* $p(0) = a$ *and let*

$$q(z) = a + a_n z^n + \cdots$$

be analytic in $\mathbb{U}$ *with* $q(z) \not\equiv a$ *and a positive integer* n. *If* q *is not subordinate to* p, *then there exist points*

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad \textit{and} \quad \zeta_0 \in \partial\mathbb{U} \backslash E(f),$$

for which

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}),\ q(z_0) = p(\zeta_0) \quad \textit{and} \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ for $z \in \mathbb{U}$ and $0 \leq s < t$.

Lemma 1.4 (Miller and Mocanu [11]). *Let* $q \in \mathcal{H}[a,1]$ *and* $\varphi : \mathbb{C}^2 \to \mathbb{C}$. *Also set*

$$\varphi(q(z), zq'(z)) \equiv h(z) \quad (z \in \mathbb{U}).$$

If

$$L(z,t) = \varphi(q(z), tzq'(z))$$

is a subordination chain and $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, *then*

$$h(z) \prec \varphi(p(z), zp'(z)) \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec p(z) \quad (z \in \mathbb{U}).$$

Furthermore, if

$$\varphi(q(z), zq'(z)) = h(z)$$

has a univalent solution $q \in \mathcal{Q}$, *then* q *is the best subordinant.*

Lemma 1.5 (Pommerenke [16]). *The function*

$$L(z,t) = a_1(t)z + \cdots$$

with $a_1(t) \neq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$ *is a subordination chain if and only if*

$$\mathrm{Re}\left\{ \frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

2. Main Results

Subordination theorem involving the integral operator $I_{\alpha,\beta}$ defined by (1.5) is contained in Theorem 2.1 below.

Theorem 2.1. *Let* $f, g \in S_p^*(A,B)$. *Suppose that*

$$\mathrm{Re}\left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\delta \tag{2.1}$$

$$\left(z \in \mathbb{U};\ \phi(z) := \left(\frac{g(z)}{z^p} \right)^{\alpha} \right),$$

where

$$\delta = \frac{1+|p\alpha+\beta|^2-|1-(p\alpha+\beta)^2|}{4\mathrm{Re}\{p\alpha+\beta\}} \quad \left(\alpha>0;\ \beta>-\frac{p\alpha(1-A)}{1-B}\right). \tag{2.2}$$

Then the subordination:

$$\left(\frac{f(z)}{z^p}\right)^\alpha \prec \left(\frac{g(z)}{z^p}\right)^\alpha \quad (z\in\mathbb{U}), \tag{2.3}$$

implies that

$$\left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^\alpha \prec \left(\frac{I_{\alpha,\beta}(g)(z)}{z^p}\right)^\alpha \quad (z\in\mathbb{U}), \tag{2.4}$$

where $I_{\alpha,\beta}$ is the integral operator defined by (1.5). *Moreover, the function*

$$\left(\frac{I_{\alpha,\beta}(g)(z)}{z^p}\right)^\alpha$$

is the best dominant.

Proof. Let us define the functions F and G by

$$F(z) := \left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^\alpha \quad \text{and} \quad G(z) := \left(\frac{I_{\alpha,\beta}(g)(z)}{z^p}\right)^\alpha, \tag{2.5}$$

respectively. Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and that

$$G'(\zeta)\neq 0 \quad (|\zeta|=1).$$

Otherwise, we replace F and G by

$$F_r(z)=F(rz) \quad \text{and} \quad G_r(z)=G(rz) \quad (0<r<1),$$

respectively. Then these functions satisfy the conditions of the theorem on $\overline{\mathbb{U}}$. We can prove that

$$F_r(z)\prec G_r(z),$$

which enables us to obtain (2.4) on letting $r\to 1$.

We first show that, if the function q is defined by

$$q(z) := 1+\frac{zG''(z)}{G'(z)} \quad (z\in\mathbb{U}), \tag{2.6}$$

then

$$\mathrm{Re}\{q(z)\}>0 \quad (z\in\mathbb{U}).$$

From the definition of (1.5), we obtain

$$\alpha\frac{z(I_{\alpha,\beta}(g)(z))'}{I_{\alpha,\beta}(g)(z)} = -\beta + (p\alpha+\beta)\frac{\phi(z)}{G(z)}. \tag{2.7}$$

We also have

$$\alpha\frac{z(I_{\alpha,\beta}(g)(z))'}{I_{\alpha,\beta}(g)(z)} = p\alpha + \frac{zG'(z)}{G(z)}. \tag{2.8}$$

By a simple calculation in conjuction with (2.7) and (2.8), we obtain the following relationship:

$$\begin{aligned} 1+\frac{z\phi''(z)}{\phi'(z)} &= 1+\frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z)+p\alpha+\beta} \\ &= q(z) + \frac{zq'(z)}{q(z)+p\alpha+\beta} \equiv h(z). \end{aligned} \tag{2.9}$$

We also see from (2.1) that

$$\mathrm{Re}\{h(z)+p\alpha+\beta\} > 0 \ \ (z\in\mathbb{U}),$$

and by using Lemma 1.2, we conclude that the differential equation (2.9) has a solution $q\in\mathcal{H}(\mathbb{U})$ with

$$q(0) = h(0) = 1.$$

Let us put

$$H(u,v) = u + \frac{v}{u+p\alpha+\beta} + \delta, \tag{2.10}$$

where δ is given by (2.2). From (2.1), (2.9) and (2.10), we obtain

$$\mathrm{Re}\{H(q(z), zq'(z))\} > 0 \quad (z\in\mathbb{U}).$$

Now we proceed to show that

$$\mathrm{Re}\{H(is,t)\} \leq 0 \tag{2.11}$$

for all real s and $t \leq -(1+s^2)/2$. Indeed, from (2.10), we have

$$\begin{aligned}\mathrm{Re}\{H(is,t)\} &= \mathrm{Re}\left\{is + \frac{t}{is+p\alpha+\beta}+\delta\right\} \\ &= \frac{t\mathrm{Re}\{p\alpha+\beta\}}{|p\alpha+\beta+is|^2}+\delta \\ &\leq -\frac{E_\delta(s)}{2|p\alpha+\beta+is|^2},\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}E_\delta(s) :=&(\mathrm{Re}\{p\alpha+\beta\}-2\delta)s^2 - 4\delta\mathrm{Im}\{p\alpha+\beta\}s \\ &-2\delta|p\alpha+\beta|^2+\mathrm{Re}\{p\alpha+\beta\}.\end{aligned} \tag{2.13}$$

For δ given by (2.2), the coefficient of s^2 in the quadratic expression $E_\delta(s)$ given by (2.13) is positive or equal to zero. Moreover, the quadratic expression $E_\delta(s)$ by s in (2.13) is a perfect square. Hence from (2.12), we obtain the inequality given by (2.11). Thus, by using Lemma 1.1, we conclude that

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}),$$

that is, that G defined by (2.5) is convex in $\mathbb{U}$.

Next, we prove that the subordination condition (2.3) implies that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}) \tag{2.14}$$

for the functions F and G defined by (2.5). For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{1+t}{p\alpha+\beta}zG'(z) \;\; (z \in \mathbb{U};\; 0 \leq t < \infty).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(1+\frac{1+t}{p\alpha+\beta}\right) \neq 0 \;\; (0 \leq t < \infty;\; \mathrm{Re}\{p\alpha+\beta\} > 0).$$

This shows that the function

$$L(z,t) = a_1(t)z + \cdots$$

satisfies the condition $a_1(t) \neq 0$ for all $t \in [0,\infty)$. Furthermore, we have

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \mathrm{Re}\left\{p\alpha+\beta+(1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right\} > 0,$$

since G is convex and $\mathrm{Re}\{p\alpha+\beta\} > 0$. Therefore, by virtue of Lemma 1.5, $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$\phi(z) = G(z) + \frac{1}{p\alpha+\beta} zG'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \quad (z \in \mathbb{U};\ 0 \le t < \infty).$$

This implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \quad (\zeta \in \partial\mathbb{U};\ 0 \le t < \infty).$$

Now suppose that F is not subordinate to G, then by Lemma 1.3, there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1+t)\zeta_0 G'(\zeta_0) \ \ (0 \le t < \infty).$$

Hence we have

$$\begin{aligned} L(\zeta_0,t) &= G(\zeta_0) + \frac{1+t}{p\alpha+\beta}\zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{1}{p\alpha+\beta} z_0 F'(z_0) \\ &= \left(\frac{f(z_0)}{z_0^p}\right)^\alpha \in \phi(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition (2.3). This contradicts the above observation that $L(\zeta_0,t) \notin \phi(\mathbb{U})$. Therefore, the subordination condition (2.3) must imply the subordination given by (2.14). Considering $F(z) = G(z)$, we see that the function G is the best dominant. This evidently completes the proof of Theorem 2.1.

Remark 2.1. We note that δ given by (2.2) in Theorem 2.1 satisfies the inequality $0 < \delta \le 1/2$.

We next prove a dual problem of Theorem 2.1, in the sense that the subordinations are replaced by superordinations.

Theorem 2.2. *Let $f, g \in S_p^*(A,B)$. Suppose that*

$$\operatorname{Re}\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$

$$\left(z \in \mathbb{U};\ \phi(z) := \left(\frac{g(z)}{z^p}\right)^{\alpha}\right),$$

where δ is given by (2.2), *and the function $(f(z)/z^p)^{\alpha}$ is univalent in* $\mathbb{U}$ *and*

$$\left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^{\alpha} \in \mathcal{Q},$$

where $I_{\alpha,\beta}$ is the integral operator defined by (1.5). *Then the superordination:*

$$\left(\frac{g(z)}{z^p}\right)^{\alpha} \prec \left(\frac{f(z)}{z^p}\right)^{\alpha} \quad (z \in \mathbb{U}) \tag{2.15}$$

implies that

$$\left(\frac{I_{\alpha,\beta}(g)(z)}{z^p}\right)^{\alpha} \prec \left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^{\alpha} \quad (z \in \mathbb{U}).$$

Moreover, the function

$$\left(\frac{I_{\alpha,\beta}(g)(z)}{z^p}\right)^{\alpha}$$

is the best subordinant.

Proof. The first part of the proof is similar to that of Theorem 2.1 and so we will use the same notation as in the proof of Theorem 2.1.

Now let us define the functions F and G, respectively, by (2.5). We first note that from (2.7) and (2.8), we have

$$\begin{aligned} \phi(z) &= G(z) + \frac{1}{p\alpha+\beta} zG'(z) \\ &=: \varphi(G(z), zG'(z)). \end{aligned} \tag{2.16}$$

After a simple calculation, the equation (2.16) yields the relationship:

$$1+\frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z)+p\alpha+\beta}.$$

where the function q is defined by (2.6). Then by using the same method as in the proof of Theorem 2.1, we can prove that

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}),$$

that is, that G defined by (2.5) is convex(univalent) in $\mathbb{U}$.

Next, we prove that the superordination condition (2.16) implies that

$$G(z) \prec F(z) \quad (z \in \mathbb{U}) \tag{2.17}$$

for the functions F and G defined by (2.5). Now consider the function $L(z,t)$ defined by

$$L(z,t) := G(z) + \frac{t}{p\alpha + \beta} zG'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

Since G is convex and $\mathrm{Re}\{p\alpha + \beta\} > 0$, we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 2.1. Therefore according to Lemma 1.4, we conclude that the superordination condition (2.15) must imply the superordination given by (2.17). Furthermore, since the differential equation (2.16) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore we complete the proof of Theorem 2.2.

If we combine Theorem 2.1 and Theorem 2.2, then we obtain the following sandwich-type theorem.

Theorem 2.3. *Let $f, g_k \in \mathcal{S}_p^*(A,B)(k=1,2)$. Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta \tag{2.18}$$

$$\left(z \in \mathbb{U};\ \phi_k(z) := \left(\frac{g_k(z)}{z^p}\right)^\alpha;\ k = 1,2\right),$$

where δ is given by (2.2), *and the function $(f(z)/z^p)^\alpha$ is univalent in $\mathbb{U}$ and*

$$\left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^\alpha \in \mathcal{Q},$$

where $I_{\alpha,\beta}$ is the integral operator defined by (1.5). *Then the subordination relation:*

$$\left(\frac{g_1(z)}{z^p}\right)^\alpha \prec \left(\frac{f(z)}{z^p}\right)^\alpha \prec \left(\frac{g_2(z)}{z^p}\right)^\alpha \quad (z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\alpha,\beta}(g_1)(z)}{z^p}\right)^\alpha \prec \left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^\alpha \prec \left(\frac{I_{\alpha,\beta}(g_2)(z)}{z^p}\right)^\alpha \quad (z \in \mathbb{U}).$$

Moreover, the functions

$$\left(\frac{I_{\alpha,\beta}(g_1)(z)}{z^p}\right)^\alpha \quad \text{and} \quad \left(\frac{I_{\alpha,\beta}(g_2)(z)}{z^p}\right)^\alpha$$

are the best subordinant and the best dominant, respectively.

The assumption of Theorem 2.3, that the functions

$$\left(\frac{f(z)}{z^p}\right)^\alpha \quad \text{and} \quad \left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^\alpha$$

need to be univalent in $\mathbb{U}$, may be replaced by another condition in the following result.

Corollary 2.1. *Let $f, g_k \in S_p^*(A, B)(k = 1, 2)$. Suppose that the condition* (2.18) *is satisfied and*

$$\operatorname{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta \tag{2.19}$$

$$\left(z \in \mathbb{U};\ \psi(z) := \left(\frac{f(z)}{z^p}\right)^\alpha;\ \frac{f(z)}{z^p} \in \mathcal{Q}\right),$$

where δ is given by (2.2). *Then the subordination relation:*

$$\left(\frac{g_1(z)}{z^p}\right)^\alpha \prec \left(\frac{f(z)}{z^p}\right)^\alpha \prec \left(\frac{g_2(z)}{z^p}\right)^\alpha \quad (z \in \mathbb{U}),$$

implies that

$$\left(\frac{I_{\alpha,\beta}(g_1)(z)}{z^p}\right)^\alpha \prec \left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^\alpha \prec \left(\frac{I_{\alpha,\beta}(g_2)(z)}{z^p}\right)^\alpha \quad (z \in \mathbb{U}),$$

where $I_{\alpha,\beta}$ is the integral operator defined by (1.5). *Moreover, the functions*

$$\left(\frac{I_{\alpha,\beta}(g_1)(z)}{z^p}\right)^\alpha \quad \text{and} \quad \left(\frac{I_{\alpha,\beta}(g_2)(z)}{z^p}\right)^\alpha$$

are the best subordinant and the best dominant, respectively.

Proof. In order to prove Corollary 2.1, we have to show that the condition (2.19) implies the univalence of $\psi(z)$ and

$$F(z) := \left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^\alpha.$$

Since $0 < \delta \leq 1/2$ from Remark 2.1, the condition (2.20) means that ψ is a close-to-convex function in $\mathbb{U}$ (see [4]) and hence ψ is univalent in $\mathbb{U}$. Furthermore, by using the same tecnniques as in the proof of Theorem 2.1, we can prove the convexity(univalence) of F and so the details may be omitted. Therefore, by applying Theorem 2.3, we obtain Corollary 2.1.

Taking $p\alpha + \beta = 1 (0 < \alpha \leq 1/p)$ with $A = 1$ and $B = -1$ in Theorem 2.3, we have the following result.

Corollary 2.2. *Let $f, g_k \in \mathcal{S}_p^*(k = 1, 2)$. Suppose that*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\frac{1}{2}$$

$$\left(z \in \mathbb{U};\ \phi(z) := \left(\frac{g_k(z)}{z^p}\right)^{\alpha};\ k = 1, 2\right)$$

and the function $(f(z)/z^p)^{\alpha}$ is univalent in $\mathbb{U}$ and

$$\left(\frac{I_{\alpha,1-p\alpha}f(z)}{z^p}\right)^{\alpha} \in \mathcal{Q},$$

where the integral operator $I_{\alpha,1-p\alpha}$ is defined by (1.5) with $\beta = 1 - p\alpha (0 < \alpha \leq 1/p)$. Then the subordination relation:

$$\left(\frac{g_1(z)}{z^p}\right)^{\alpha} \prec \left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \left(\frac{g_2(z)}{z^p}\right)^{\alpha} \quad (z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\alpha,1-p\alpha}(g_1)(z)}{z^p}\right)^{\alpha} \prec \left(\frac{I_{\alpha,1-p\alpha}(f)(z)}{z^p}\right)^{\alpha} \prec \left(\frac{I_{\alpha,1-p\alpha}(g_2)(z)}{z^p}\right)^{\alpha} \quad (z \in \mathbb{U}).$$

Moreover, the functions

$$\left(\frac{I_{\alpha,1-p\alpha}(g_1)(z)}{z^p}\right)^{\alpha} \quad \text{and} \quad \left(\frac{I_{\alpha,1-p\alpha}(g_2)(z)}{z^p}\right)^{\alpha}$$

are the best subordinant and the best dominant, respectively.

3. Applications to the Gauss Hypergeometric Function

We begin by recalling that the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by (see, for details, [14] and [18, Chapter 14])

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

$$(z \in \mathbb{U};\ b \in \mathbb{C};\ c \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \cdots\}),$$

where $(\lambda)_{\nu}$ denotes the Pochhammer symbol (or the shifted factorial) defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+\nu-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases}$$

For this useful special function, the following Eulerian integral representation is fairly well-known [18, p. 293]:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt \tag{3.1}$$

$$(\mathrm{Re}\{c\} > \mathrm{Re}\{a\} > 0;\ |\arg(1-z)| \leq \pi - \epsilon;\ 0 < \epsilon < \pi).$$

In view of (3.1), we set

$$\tau(z) = \frac{z^p}{(1-z)^\kappa} \quad (\kappa > 0) \tag{3.2}$$

so that the definition (1.5) yield

$$\begin{aligned} I_{\alpha,\beta}(\tau)(z) &= \left(\frac{p\alpha+\beta}{z^\beta}\int_0^z t^{p\alpha+\beta-1}(1-t)^{-\kappa\alpha}dt\right)^{1/\alpha} \\ &= \left((p\alpha+\beta)z^{p\alpha}\int_0^1 u^{p\alpha+\beta-1}(1-zu)^{-\kappa\alpha}du\right)^{1/\alpha} \\ &= z^p[{}_2F_1(p\alpha+\beta, \kappa\alpha; p\alpha+\beta+1; z)]^{1/\alpha} \quad (\mathrm{Re}\{p\alpha+\beta\} > 0). \end{aligned}$$

Moreover, we note from the definition (3.2) that

$$\frac{\tau(z)}{z^p} = \frac{1}{(1-z)^\kappa} \neq 0 \quad (z \in \mathbb{U}).$$

Thus, by applying to Theorem 2.1 with $g(z)$ replaced by the function $\tau(z)$ defined by (3.2), we obtain the following result involving the Gauss hypergeometric function.

Theorem 3.1. *Let $f \in S_p^*$. Suppose that*

$$0 < \kappa\alpha \leq 2(1+\delta) - 1 \quad (0 < \kappa \leq 2p;\ \alpha > 0;\ \beta \geq 0),$$

where δ is given by (2.2). Then the subordination:

$$\left(\frac{f(z)}{z^p}\right)^\alpha \prec \frac{1}{(1-z)^{\kappa\alpha}} \quad (z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\alpha,\beta}(f)(z)}{z^p}\right)^\alpha \prec {}_2F_1(p\alpha+\beta, \kappa\alpha; p\alpha+\beta+1; z) \quad (z \in \mathbb{U}),$$

where $I_{\alpha,\beta}$ *is the integral operator defined by* (1.5). *Moreover, the function*

$${}_2F_1(p\alpha+\beta,\kappa\alpha;p\alpha+\beta+1;z)$$

is the best dominant.

By setting $\beta = 1 - p\alpha (0 < \alpha \leq 1/p)$ in Theorem 3.1, we are led to the following Corollary 3.1.

Corollary 3.1. *Let* $f \in \mathcal{A}_{\alpha,\beta}$ *and* $0 < \kappa \leq 2p,\ 0 < \alpha \leq 1/p$. *Then the subordination:*

$$\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \frac{1}{(1-z)^{\kappa}} \quad (z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\alpha,1-p\alpha}(f)(z)}{z^p}\right)^{\alpha} \prec {}_2F_1(1,\kappa\alpha;2;z) \quad (z \in \mathbb{U}),$$

where $I_{\alpha,1-p\alpha}$ *is the integral operator defined by* (1.5) *with* $\beta = 1 - p\alpha$.

Remark 3.1. We note that we can obtain the dual result corresponding to Theorem 3.1 by using Theorem 2.2.

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