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Valuing Executive Stock Options: A Simple Continuous-Time Model*

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1 Introduction

Executive and/or employee stock options (ESOs) have become increasingly popular and currently constitute a certain fraction of total compensation expense of many firms. ESOs are call options that give the option holder the right to buy their firm's stock for a fixed strike price during a specified period of time. Clearly, the exercise of an ESO triggers a dilution of the claims of the firm's existing shareholders, since the firm issues new stocks to the ESO holders. Hence, it is important to determine the fair value of ESOs from the view points both of accounting and of corporate finance; see Smith and Zimmerman (1976) for early research on valuing ESOs.

ESOs have features different from traded stock options (TSOs): While TSOs usually mature within one year, ESOs have the maturity over many years, typically, it is set equal to *ten* years. Also, ESOs are granted at-the-money to executives and/or employees, namely, its strike price is set equal to the current stock price. Usually, during the first portion of the option's life (*vesting period*), ESO holders cannot exercise their options and must *forfeit* the options on leaving the firm. Typically, the vesting date is *three* (*two*) years after the grant date in U.S. (Japan). After the vesting date, ESO holders can exercise the options at *any time* before the maturity date, *i.e.*, ESOs are of American-style. The most significant difference between ESOs and TSOs is that ESO holders cannot sell or otherwise transfer them. In the U.S., Section 16-c of the Securities Exchange Act prohibits insiders from selling their firm's stock short. An ESO holder leaving the firm is then forced to choose between forfeiting or exercising the options soon after his departure. Thus, the following features have the dominant effects in the fair valuation of ESOs (Rubinstein, 1995):

- Early Exercise
- Delayed Vesting
- Forfeiture
- Lack of Transferability
- Dilution

Among these features, Hull and White (2004) have pointed out that the dilution effects can be safely ignored in many situations. The lack of transferability implies that ESO holders cannot

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hedge their positions, and so that their personal valuations depend on their risk preferences and endowments. Thus, the non-transferability of ESOs may be realized in mathematical models by maximizing a utility function of ESO holders; see Lambert et al. (1991); Kulatilaka and Murcus (1994); Huddart (1994); Rubinstein (1995); Detemple and Sundaresan (1999); Hall and Murphy (2002); Agliardi and Andergassen (2005); Bettis et al. (2005); Rogers and Scheinkman (2007). Through an empirical analysis using data on ESO exercises from 40 firms, however, Carpenter (1998) showed that a simple American option pricing model performs well as an elaborate utility-maximizing model; see Aboody (1996); Huddart and Lang (1996); Murphy (1985) for further empirical analysis. Thus, the first three features above play principal roles in the ESO valuation.

In order to incorporate the early exercise feature into mathematical models, there are two different frameworks: European vs. American. No doubt, the American option model is a natural choice for realizing the early exercise feature. However, even a vanilla American call written on a dividend-paying stock has no closed-form valuation formula. Hence, ESO valuation with the American framework is inevitably based on a binomial-tree (*i.e.*, discrete-time) model (Ammann and Seiz, 2004). From the very nature of things, some artificial invention is required to realize the early exercise feature in the European framework. A very primitive idea is to put forward the maturity date, which was adopted in Statement of Financial Accounting Standards No. 123 (FAS 123). The Financial Accounting Standards Boards (FASB) published FAS 123 in 1995, which was revised in 2004. In FAS 123, the modified maturity is set to the ESO's expected life under the assumption that the ESO holder does not leave during the vesting period (Foster et al., 1991; Ammann and Seiz, 2004; Hull and White, 2004). Another idea is to model the early exercise behavior of ESO holders by assuming the exercise takes place whenever the stock price reaches a certain upper barrier. The barrier option models have been developed by Raupach (2003); Hull and White (2004). As an alternative idea, Carr and Linetsky (2000) introduced an intensity-based framework for realizing the early exercise feature, which has been used to model forfeiture due to voluntary or involuntary employment termination (Jennergren and Näslund, 1993, 1995; Hull and White, 2004).

In this paper, using a *quadratic approximation* originally developed for valuing vanilla American options (MacMillan, 1986; Barone-Adesi and Whaley, 1987), we will provide a simple continuous-time American option model in the Black-Scholes-Merton framework, which satisfies the (first three above) principal ESO features.

The rest of this paper is organized as follows: In Section 2, we briefly introduce the quadratic approximation for the value of an American call option, which approximates the target ESO value after vesting. By the principle of risk-neutral valuation, we obtain the values of the ESO with/without forfeiture at the grant date. In Section 3, we show that the valuation formula for the ESO with forfeiture gives exact values for two special cases either with no dividend or infinite maturity. Section 4 concludes with some remarks.

2 Approximate Valuation

2.1 A Quadratic Approximation

Suppose an economy with finite time period $[0, T]$, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. A Brownian motion process $W \equiv (W_t)_{t\in[0,T]}$ is defined on $(\Omega, \mathcal{F})$ and takes values in $\mathbb{R}$. The filtration is the natural filtration generated by W and $\mathcal{F}_T = \mathcal{F}$. Let $(S_t)_{t\in[0,T]}$ the price process of the underlying stock. For S_0 given, assume that $(S_t)_{t\in[0,T]}$ is a geometric Brownian motion process

$$\mathrm{d}S_t = (r - \delta)S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \quad t \in [0, T],$$

where the coefficients (r, δ, σ) are constant. Here r represents the risk-free rate of interest, δ the continuous dividend rate, and σ the volatility of the stock price. The stock price process $(S_t)_{t\in[0,T]}$ is represented under the equivalent martingale measure $\mathbb{P}$, which indicates that the stock has mean rate of return r, and the process W is a $\mathbb{P}$-Brownian motion.

Let T_1 $(\in (0, T))$ be the vesting date of ESO with strike price K. Also, let $C(S_u, u) \equiv C(S_u, u; T)$ and $c(S_u, u) \equiv c(S_u, u; T)$ be the values of the American and European call options with maturity date T at time $u \in [T_1, T]$, respectively. Then, the difference

$$e(S_u, u) = C(S_u, u) - c(S_u, u), \quad u \in [T_1, T] \tag{1}$$

represents the early exercise premium of the American option at time u. MacMillan (1986) and Barone-Adesi and Whaley (1987) have developed a simple approximation for the premium $e(S_u, u)$, which is given by

$$e(S_u, u) = \left\{1 - \mathrm{e}^{-\delta(T-u)}\Phi\big(d_+(\bar{S}_u, K, T-u)\big)\right\} \frac{\bar{S}_u}{\theta_u}\left(\frac{S_u}{\bar{S}_u}\right)^{\theta_u}, \tag{2}$$

for $S_u < \bar{S}_u$ $(u \in [T_1, T])$, where $\Phi(x)$ denotes the standard normal cumulative distribution function (cdf)

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}u^2}\mathrm{d}u, \quad x \in \mathbb{R},$$

$$d_\pm(x, y, \tau) = \frac{\log(x/y) + (r - \delta \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},$$

$\bar{S}_u$ $(> K)$ is a positive root of the equation

$$\bar{S}_u - K = c(\bar{S}_u, u) + \left\{1 - \mathrm{e}^{-\delta(T-u)}\Phi\big(d_+(\bar{S}_u, K, T-u)\big)\right\}\frac{\bar{S}_u}{\theta_u}, \quad u \in [T_1, T] \tag{3}$$

and $\theta_u > 1$ is a positive root of the quadratic equation

$$\tfrac{1}{2}\sigma^2\theta_u^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\theta_u - \nu(u) = 0, \tag{4}$$

namely,

$$\theta_u = \frac{2}{\sigma^2}\left\{-(r - \delta - \tfrac{1}{2}\sigma^2) + \sqrt{(r - \delta - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2\nu(u)}\right\}, \quad u \in [T_1, T],$$

for which $\nu(u)$ is defined by

$$\nu(u) = \frac{r}{1 - \mathrm{e}^{-r(T-u)}}, \quad u \in [T_1, T].$$

Thus, the American call value at the vesting date T_1 is approximately given by

$$C(S_{T_1}, T_1) = \begin{cases} S_{T_1} - K, & S_{T_1} \geq \bar{S}_{T_1} \\ c(S_{T_1}, T_1) + e(S_{T_1}, T_1), & S_{T_1} < \bar{S}_{T_1}, \end{cases} \tag{5}$$

which has been called a *quadratic approximation* (Hull, 2000, Appendix 16A). Clearly, the European call value $c(S_{T_1}, T_1; T)$ is given by the Black-Scholes formula

$$c(S_{T_1}, T_1; T) = S_{T_1}\mathrm{e}^{-\delta(T-T_1)}\Phi\big(d_+(S_{T_1}, K, T-T_1)\big) - K\mathrm{e}^{-r(T-T_1)}\Phi\big(d_-(S_{T_1}, K, T-T_1)\big). \tag{6}$$

Remark 1 Let $\mathcal{D}_0 = \{(S_u, u) \in \mathbb{R}_+ \times [t, T]\}$ denote the whole domain and let

$$\mathcal{D} = \{(S_u, u) \in \mathbb{R}_+ \times [T_1, T]\} \subset \mathcal{D}_0$$

denote the domain after the vesting date. Then, the curve $(\bar{S}_u)_{u \in [T_1, T]}$ in $\mathcal{D}$ is referred to as an *early exercise boundary*, splitting the subdomain $\mathcal{D}$ into two disjoint regions

$$\mathcal{C} = \left\{(S_u, u) \in [0, \bar{S}_u) \times [T_1, T]\right\} \quad \text{and} \quad \mathcal{E} = \left\{(S_u, u) \in [\bar{S}_u, \infty) \times [T_1, T]\right\},$$

which are called the *continuation region* and *exercise region*, respectively. An ESO is *optimally* exercised at a first-passage time τ_e defined by

$$\tau_e = \inf\{u \in [T_1, T] \mid (S_u, u) \in \mathcal{E}\}.$$

By *optimal* we mean that τ_e is a stopping time of the filtration $\mathbb{F}$, for which $C(S_u, u)$ ($u \in [T_1, T]$) is a solution of the *optimal stopping problem*

$$C(S_u, u) = \operatorname*{ess\,sup}_{\tau_e \in [u, T]} \mathbb{E}\left[\mathrm{e}^{-r(\tau_e - u)}(S_{\tau_e} - K)^+ \mid \mathcal{F}_u\right], \tag{7}$$

where the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$.

Remark 2 From (3), $\bar{S}_u$ for a given $u \in [T_1, T]$ can be obtained by solving a functional equation of the form $x = f(x)$, where f is an operator mapping defined by

$$f(x) \equiv K + c(x, u) + \left\{1 - \mathrm{e}^{-\delta(T-u)}\Phi\big(d_+(x, K, T-u)\big)\right\}\frac{x}{\theta_u}. \tag{8}$$

This equation can be solved by either Newton method or a recursive scheme such as

$$x_n = f(x_{n-1}), \quad n \geq 1,$$

for an appropriately selected initial value x_0, *e,g.* $x_0 = K$. Figure 1 illustrates some curves of early exercise boundaries $(\bar{S}_u)_{u \in [T_1, T]}$ of ESOs as functions of the elapsed time.

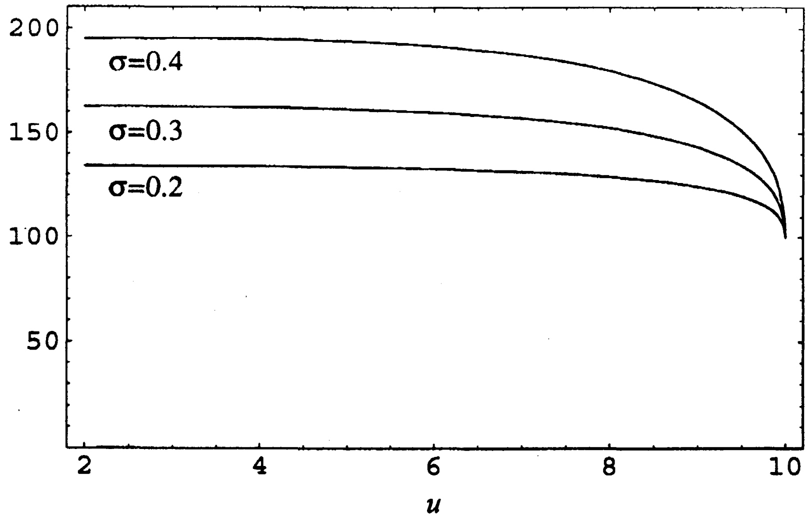

Figure 1: Quadratic approximations for the early exercise boundary $(\bar{S}_u)_{u\in[T_1,T]}$ ($t=0$, $T_1=2$, $T=10$, $K=100$, $r=0.03$, $\delta=0.05$, $\sigma=0.2,0.3,0.4$)

2.2 No Forfeiture

Let $V^{\circ}(S,t;T)$ be the ESO value at time $t<T_1$ with maturity date T and initial asset price $S\equiv S_t$, assuming no forfeiture before maturity. Then, by the principle of the risk-neutral valuation, we have

$$\begin{aligned} V^{\circ}(S,t;T) &= \mathrm{e}^{-r(T_1-t)}\mathbb{E}\left[C(S_{T_1},T_1;T)\mid\mathcal{F}_t\right] \\ &= \mathrm{e}^{-r(T_1-t)}\int_0^{\infty} C(S',T_1;T)G(S',T_1;S,t)\mathrm{d}S', \end{aligned} \tag{9}$$

where for $S'>0$

$$\begin{aligned} G(S',T_1;S,t) &= \frac{1}{\sqrt{2\pi(T_1-t)}\sigma S'}\exp\left[-\frac{\left\{\log(S'/S)-(r-\delta-\frac{1}{2}\sigma^2)(T_1-t)\right\}^2}{2\sigma^2(T_1-t)}\right] \\ &= \frac{1}{\sqrt{2\pi(T_1-t)}\sigma S'}\exp\left[-\tfrac{1}{2}\left\{d_-(S,S',T_1-t)\right\}^2\right] \end{aligned} \tag{10}$$

is the lognormal probability density function (pdf) of S_{T_1} starting from $S_t=S$, which is often referred to as *Green's function* of the Black-Scholes PDE.

Lemma 1 For $\bar{S}>0$,

$$\int_{\bar{S}}^{\infty} G(S',T_1;S,t)\mathrm{d}S' = \Phi\big(d_-(S,\bar{S},T_1-t)\big) \tag{11}$$

and

$$\int_{\bar{S}}^{\infty} S'G(S',T_1;S,t)\mathrm{d}S' = S\mathrm{e}^{(r-\delta)(T_1-t)}\Phi\big(d_+(S,\bar{S},T_1-t)\big). \tag{12}$$

Proof. The derivation of the two formulas (11) and (12) is straightforward. See Zhu et al. (2004, Chapter 2) for details. □

Lemma 2 For $\bar{S} > 0$ and $\theta \in \mathbb{R}$,

$$\begin{aligned}\int_0^{\bar{S}} &(S')^\theta G(S', T_1; S, t)\mathrm{d}S' \\ &= S^\theta \mathrm{e}^{\left(\frac{1}{2}\sigma^2\theta(\theta-1)+(r-\delta)\theta\right)(T_1-t)} \Phi\big(-d_-(S, \bar{S}, T_1 - t) - \theta\sigma\sqrt{T_1 - t}\big). \end{aligned} \tag{13}$$

Proof. See also Zhu et al. (2004, Chapter 2) for the derivation. □

Lemma 3 For $\bar{S} > 0$,

$$\begin{aligned}\mathrm{e}^{-r(T_1-t)} &\int_0^{\bar{S}} c(S', T_1; T) G(S', T_1; S, t)\mathrm{d}S' \\ = \;& S\mathrm{e}^{-\delta(T-t)}\Phi_2\big(-d_+(S, \bar{S}, T_1 - t), d_+(S, K, T - t); -\rho\big) \\ & - K\mathrm{e}^{-r(T-t)}\Phi_2\big(-d_-(S, \bar{S}, T_1 - t), d_-(S, K, T - t); -\rho\big), \end{aligned} \tag{14}$$

where $\Phi_2(x, y; \gamma)$ denote the bivariate standard normal cdf with the corelation coefficient γ given by

$$\Phi_2(x, y; \gamma) = \frac{1}{2\pi\sqrt{1-\gamma^2}} \int_{-\infty}^{x}\int_{-\infty}^{y} \mathrm{e}^{-\frac{1}{2}(u^2-2\gamma uv+v^2)/(1-\gamma^2)}\mathrm{d}v\mathrm{d}u, \quad (x, y) \in \mathbb{R}^2,$$

and

$$\rho = \sqrt{\frac{T_1 - t}{T - t}}.$$

Proof. The integral calculation similar to the left-hand-side of (14) can be found in the valuation of a European compound option (Geske, 1979; Geske and Johnson, 1984), *i.e.*, a European put on a European call. Following Zhu et al. (2004, pp. 184–190), we obtain the desired result. □

Theorem 1 For $t < T_1 < T$, let $V^\circ(S, t; T)$ be the value of ESO without forfeiture. Then, we have

$$\begin{aligned} V^\circ(S, t; T) = \;& S\mathrm{e}^{-\delta(T_1-t)}\Phi\big(d_+(S, \bar{S}_{T_1}, T_1 - t)\big) - K\mathrm{e}^{-r(T_1-t)}\Phi\big(d_-(S, \bar{S}_{T_1}, T_1 - t)\big) \\ & + S\mathrm{e}^{-\delta(T-t)}\Phi_2\big(-d_+(S, \bar{S}_{T_1}, T_1 - t), d_+(S, K, T - t);\, -\rho\big) \\ & - K\mathrm{e}^{-r(T-t)}\Phi_2\big(-d_-(S, \bar{S}_{T_1}, T_1 - t), d_-(S, K, T - t);\, -\rho\big) \\ & + e(S, T_1)\mathrm{e}^{-(r-\nu_1)(T_1-t)}\Phi\big(-d_-(S, \bar{S}_{T_1}, T_1 - t) - \theta_{T_1}\sigma\sqrt{T_1 - t}\big) \end{aligned} \tag{15}$$

where $\bar{S}_{T_1}$ is the positive root of the equation (3) for $u = T_1$,

$$e(S, T_1) = \Big\{1 - \mathrm{e}^{-\delta(T-T_1)}\Phi\big(d_+(\bar{S}_{T_1}, K, T - T_1)\big)\Big\} \frac{\bar{S}_{T_1}}{\theta_{T_1}} \left(\frac{S}{\bar{S}_{T_1}}\right)^{\theta_{T_1}}. \tag{16}$$

and

$$\nu_1 = \nu(T_1) = \frac{r}{1 - \mathrm{e}^{-r(T-T_1)}}. \tag{17}$$

Proof. For notational convenience, denote $S_{T_1} = S'$ and $\bar{S}_{T_1} = \bar{S}$. Then, from (5) and (9), we have

$$V^{\circ}(S,t;T) =$$
$$\mathrm{e}^{-r(T_1-t)}\left\{\int_0^{\bar{S}}(c(S',T_1)+e(S',T_1))G(S',T_1;S,t)\mathrm{d}S' + \int_{\bar{S}}^{\infty}(S'-K)G(S',T_1;S,t)\mathrm{d}S'\right\}.$$

Hence, the result immediately follows from Lemmas 1 through 3 and the relation

$$\frac{1}{2}\sigma^2\theta_{T_1}(\theta_{T_1}-1)+(r-\delta)\theta_{T_1}=\nu_1.$$

□

2.3 Independent Forfeiture

Following Jennergren and Näslund (1993), we assume that forfeiture occurs according to a Poisson process with an exogenous constant rate λ (>0); cf. Cuny and Jorion (1995); Carr and Linetsky (2000) for more general point processes with rate dependent on the stock price. In other words, there exists a stopping time τ_f, to be independent of the filtration $\mathcal{F}$. For $\tau_f \leq \tau_e$ (a.s.), the ESO is immediately paid off if exercisable, or forfeited otherwise. For $u \in [t,T]$, let

$$\pi(u;T)=\begin{cases}0, & t\leq u<T_1\\ (S_u-K)^+, & T_1\leq u\leq T,\end{cases} \tag{18}$$

and define $\pi(\tau_e\wedge\tau_f;T)$ to be the ESO's payoff with independent stopping. The payoff $\pi(\tau_e\wedge\tau_f;T)$ for a fixed τ_f is the same as the non-forfeited payoff $\pi(\tau_e;T\wedge\tau_f)$. Under the condition $\{\tau_f=\tau\}$ for a given $\tau>0$, we have

$$\begin{aligned}&\operatorname*{ess\,sup}_{\tau_e\in[t,T]}\mathbb{E}\left[\mathrm{e}^{-r(\tau_e\wedge\tau_f-t)}\pi(\tau_e\wedge\tau_f;T)\mid\tau_f=\tau,\mathcal{F}_t\right]\\ &\quad=\operatorname*{ess\,sup}_{\tau_e\in[t,T]}\mathbb{E}\left[\mathrm{e}^{-r(\tau_e\wedge\tau-t)}\pi(\tau_e;T\wedge\tau)\mid\mathcal{F}_t\right]=V(S,t;T\wedge\tau).\end{aligned} \tag{19}$$

Unfortunately, the arbitrage price of such a contingent claim is *not* defined uniquely, because there exists no replicating $\mathbb{P}$-admissible trading strategy (Musiela and Rutkowski, 1997, page 115). This is due to the fact that $\tau_e\wedge\tau_f$ is *not* a stopping time of $\mathcal{F}$. The same situation also can be found in the valuation of *Canadian options*, which are contingent claims with the exponentially random maturity (Carr, 1998). Raupach (2003) have recently showed that the contingent claim with payoff $\pi(\tau_e\wedge\tau_f;T)$ can be priced at its expected present value, just like a perfectly hedgeable contingent claim. Unconditioning the result (19), we obtain

Theorem 2 For $t<T_1<T$, let $V(S,t;T)$ be the value of ESO with forfeit rate λ. Then, we have

$$V(S,t;T)=\mathrm{e}^{-\lambda(T-t)}V^{\circ}(S,t;T)+\int_{T_1}^{T}\lambda\mathrm{e}^{-\lambda(u-t)}V^{\circ}(S,t;u)\mathrm{d}u. \tag{20}$$

Proof. For notational convenience, set $t = 0$ without loss of generality. With $\mathbb{P}\{\tau_f \geq u\} = \mathrm{e}^{-\lambda u}$ for $u \geq 0$, we have

$$\begin{aligned} V(S,0;T) &= \operatorname*{ess\,sup}_{\tau_e \in [0,T]} \mathbb{E}\left[\mathrm{e}^{-r(\tau_e \wedge \tau_f)}\pi(\tau_e; T \wedge \tau_f) \mid \mathcal{F}_0\right] \\ &= \operatorname*{ess\,sup}_{\tau_e \in [0,T]} \mathbb{E}\left[\mathrm{e}^{-r\tau_e}\pi(\tau_e; T)\mathbf{1}_{\{\tau_f \geq T\}} \mid \mathcal{F}_0\right] + \operatorname*{ess\,sup}_{\tau_e \in [0,T]} \mathbb{E}\left[\mathrm{e}^{-r(\tau_e \wedge \tau_f)}\pi(\tau_e; \tau_f)\mathbf{1}_{\{\tau_f < T\}} \mid \mathcal{F}_0\right] \\ &= \mathrm{e}^{-\lambda T} V^\circ(S,0;T) + \int_{T_1}^{T} \lambda \mathrm{e}^{-\lambda u} V^\circ(S,0;u)\mathrm{d}u, \end{aligned}$$

from which we obtain the desired result for $V(S,t;T)$. □

3 Exact Valuation for Two Special Cases

3.1 Non-Dividend Case

Consider the non-dividend case with $\delta = 0$, for which an American call option is always worth more alive than dead, *i.e.*, it is equivalent to the associated European call option with the same contractual features (Karatzas and Shreve, 1998, Theorem 6.1). This implies that $C(S_{T_1}, T_1; T) = c(S_{T_1}, T_1; T)$ instead of (5) and $\bar{S}_{T_1} = \infty$, and hence that $V^\circ(S,t;T) = c(S,t;T)$ for $t < T_1 < T$ (Huddart, 1994, Proposition 1), which also can be formally verified from Lemma 3 such that

$$\begin{aligned} V^\circ(S,t;T) &= \mathrm{e}^{-r(T_1 - t)} \int_0^\infty c(S', T_1; T) G(S', T_1; S, t)\mathrm{d}S' \\ &= \lim_{\bar{S} \to \infty} \big\{ S\Phi_2\big(-d_+(S,\bar{S},T_1 - t), d_+(S,K,T-t); -\rho\big) \\ &\qquad - K\mathrm{e}^{-r(T-t)}\Phi_2\big(-d_-(S,\bar{S},T_1 - t), d_-(S,K,T-t); -\rho\big)\big\} \\ &= S\Phi\big(d_+(S,K,T-t)\big) - K\mathrm{e}^{-r(T-t)}\Phi\big(d_-(S,K,T-t)\big) \\ &= c(S,t;T). \end{aligned}$$

Hence, we have

Theorem 3 For $t < T_1 < T$, let $\widetilde{V}(S,t;T)$ be the exact value of ESO with forfeit rate λ, being written on a non-dividend-paying asset. Then, we have

$$\widetilde{V}(S,t;T) = \mathrm{e}^{-\lambda(T-t)} c(S,t;T) + \int_{T_1}^{T} \lambda \mathrm{e}^{-\lambda(u-t)} c(S,t;u)\mathrm{d}u, \tag{21}$$

which can be used as an exact *lower* bound for the dividend-paying case, *i.e.*, $V(S,t;T) \geq \widetilde{V}(S,t;T)$ for $\delta > 0$.

Proof. The result (21) is a direct consequence of Theorem 2. Since $c(S,t;T) \leq C(S,t;T)$, the non-dividend value in (21) gives a lower bound of the dividend-paying case. □

3.2 Perpetual Case

Consider the perpetual case with $T = \infty$. Let $C_\infty(S')$ be the value of the perpetual American call option at the vesting date T_1 with strike price K and initial asset price $S_{T_1} \equiv S'$. Then, it has been well known that

$$C_\infty(S') = \begin{cases} \dfrac{\bar{S}}{\theta}\left(\dfrac{S'}{\bar{S}}\right)^\theta, & S' < \bar{S} \\ S' - K, & S' \geq \bar{S}, \end{cases} \tag{22}$$

where $\theta > 1$ is a positive root of the quadratic equation

$$\tfrac{1}{2}\sigma^2\theta^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\theta - r = 0, \tag{23}$$

namely,

$$\theta = \frac{2}{\sigma^2}\left\{-(r - \delta - \tfrac{1}{2}\sigma^2) + \sqrt{(r - \delta - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2 r}\right\},$$

and $\bar{S}$ ($> K$) is given by

$$\bar{S} = \frac{\theta}{\theta - 1}K. \tag{24}$$

See McKean (1965) and Karatzas and Shreve (1998, Theorem 6.7).

Theorem 4 For $t < T_1$, let $V_\infty(S,t)$ be the exact value of the perpetual ESO with forfeit rate λ. Then, we have

$$\begin{aligned} V_\infty(S,t) &= S\mathrm{e}^{-(\lambda+\delta)(T_1-t)}\Phi\big(d_+(S,\bar{S},T_1-t)\big) - K\mathrm{e}^{-(\lambda+r)(T_1-t)}\Phi\big(d_-(S,\bar{S},T_1-t)\big) \\ &\quad + \frac{\bar{S}}{\theta}\left(\frac{S}{\bar{S}}\right)^\theta \mathrm{e}^{-\lambda(T_1-t)}\Phi\big(-d_-(S,\bar{S},T_1-t) - \theta\sigma\sqrt{T_1-t}\big), \end{aligned} \tag{25}$$

which gives an exact *upper* bound of the finite-lived value, *i.e.*, $V(S,t;T) \leq V_\infty(S,t)$ for $T < \infty$.

Proof. Let $V_\infty^\circ(S,t)$ be the value of the perpetual ESO without forfeiture at time $t < T_1$. Then, by the principle of the risk-neutral valuation, we have

$$\begin{aligned} V_\infty^\circ(S,t) &= \mathrm{e}^{-r(T_1-t)}\mathbb{E}\left[C_\infty(S_{T_1}) \mid \mathcal{F}_t\right] \\ &= \mathrm{e}^{-r(T_1-t)}\left\{\int_0^{\bar{S}} \frac{\bar{S}}{\theta}\left(\frac{S'}{\bar{S}}\right)^\theta G(S',T_1;S,t)\mathrm{d}S' + \int_{\bar{S}}^\infty (S'-K)G(S',T_1;S,t)\mathrm{d}S'\right\} \\ &= S\mathrm{e}^{-\delta(T_1-t)}\Phi\big(d_+(S,\bar{S},T_1-t)\big) - K\mathrm{e}^{-r(T_1-t)}\Phi\big(d_-(S,\bar{S},T_1-t)\big) \\ &\quad + \frac{\bar{S}}{\theta}\left(\frac{S}{\bar{S}}\right)^\theta \Phi\big(-d_-(S,\bar{S},T_1-t) - \theta\sigma\sqrt{T_1-t}\big). \end{aligned}$$

Hence, from the relation

$$V_\infty(S,t) = \int_{T_1}^\infty \lambda\mathrm{e}^{-\lambda(u-t)}V_\infty^\circ(S,t)\mathrm{d}u = \mathrm{e}^{-\lambda(T_1-t)}V_\infty^\circ(S,t),$$

we obtain (25). Since $C(S_{T_1},T_1) \leq C_\infty(S_{T_1})$, the perpetual value in (25) gives an upper bound of the finite-lived value. □

Remark 3 From Figure 1, we can observe that the early exercise boundaries are almost flat during the initial and middle periods of the interval $[T_1, T]$. This is due to the ESO feature of long period up to maturity, which implies that the perpetual result can be used as an approximation for the finite-lived case. To examine this expectation, extensive numerical comparisons should be done.

4 Conclusion

In this paper, a simple continuous-time model for valuing ESOs with forfeiture is developed by using the quadratic approximation for the American vanilla call option. This model is a continuous-time version of a modified binomial model for the American call option adjusted for the forfeiture rate and the vesting period (Ammann and Seiz, 2004). The valuation formula of our model is consistent with the exact results for non-dividend and perpetual cases, which also give exact lower and upper bounds for the dividend-paying and finite-lived cases, respectively. To see the quality of the quadratic approximation, we need numerical experiments comparing the approximate value with a exact benchmark result computed, *e.g.* by the associated binomial model, being in progress.

The quadratic approximation method is so general that it can be applied to non-traditional ESOs such as reload options, performance-vested options, indexed options, repriceable options and so on (Brenner et al., 2000; Johnson and Tian, 2000a,b; Rogers and Scheinkman, 2007). Also, the quadratic approximation can be applied to the case that the underlying stock price process has jumps. These extensions remain as future studies.

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