THE DRINFELD CENTER OF THE CATEGORY OF MACKEY FUNCTORS (COHOMOLOGY THEORY OF FINITE GROUPS AND RELATED TOPICS)

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THE DRINFELD CENTER OF THE
CATEGORY OF MACKEY FUNCTORS

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We determine the center of the tensor category of Mackey functors for a finite group. Details are in [5].

1. The center of a tensor category

The center of a tensor category was defined by Drinfeld, Joyal and Street ([3]), and Magid ([4]). We review the definition. Let $\mathcal{A}$ be a tensor category over a field. The tensor product of objects $A, B \in \mathcal{A}$ is denoted by $A \otimes B$, and the unit object of $\mathcal{A}$ is denoted by $I$.

The center $\mathbb{Z}(\mathcal{A})$ is a category defined as follows. An object of $\mathbb{Z}(\mathcal{A})$ is a pair $(A, \theta)$, where $A \in \mathcal{A}$ and $\theta$ is a family of isomorphisms $\theta_B: B \otimes A \to A \otimes B$ for all $B \in \mathcal{A}$ satisfying the conditions

$$
\theta_{B \otimes B'} = (\theta_B \otimes 1) \circ (1 \otimes \theta_{B'}) \quad \text{for all } B, B' \in \mathcal{A},
$$

$$
\theta_I = 1.
$$

A morphism $(A, \theta) \to (A', \theta')$ of $\mathbb{Z}(\mathcal{A})$ is a morphism $f: A \to A'$ of $\mathcal{A}$ satisfying

$$
(f \otimes 1) \circ \theta_B = \theta'_{B} \circ (1 \otimes f) \quad \text{for all } B \in \mathcal{A}.
$$

2. Mackey functors

We review the definition of a Mackey functor ([1], [2]). Let $G$ be a finite group. Denote by $S$ the category of finite $G$-sets. For $X, Y \in S$, we write the direct product $X \times Y$ as $XY$, and the disjoint sum of $X$ and $Y$ as $X + Y$. Let $k$ be a field. Denote by $\mathcal{V}$ the category of vector spaces over $k$.

A Mackey functor $M$ on $S$ consists of $k$-vector spaces $M(X)$ for all $G$-sets $X$ and linear maps $f_*: M(X) \to M(Y)$ and $f^*: M(Y) \to M(X)$ for all $G$-maps $f: X \to Y$ satisfying the following conditions:

(i) $M(X)$ and $f_*$ form a functor $S \to \mathcal{V}$.

(ii) $M(X)$ and $f^*$ form a functor $S^{op} \to \mathcal{V}$. 
For a pullback diagram
\[
\begin{array}{ccc}
X & \xrightarrow{p} & X' \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{q} & Y'
\end{array}
\]
in \( S \), the diagram
\[
\begin{array}{ccc}
M(X) & \xleftarrow{p^*} & M(X') \\
\downarrow f_* & & \downarrow f'_* \\
M(Y) & \xleftarrow{q^*} & M(Y')
\end{array}
\]
is commutative.

(iv) Let \( i_1: X_1 \rightarrow X_1 + X_2 \) and \( i_2: X_2 \rightarrow X_1 + X_2 \) be the inclusion maps in \( S \). Then the maps
\[
(i_{1*}, i_{2*}): M(X_1) \oplus M(X_2) \rightarrow M(X_1 + X_2),
\]
\[
(i_{1}^{*}, i_{2}^{*}): M(X_1 + X_2) \rightarrow M(X_1) \oplus M(X_2)
\]
are inverse to each other.

(v) \( M(\emptyset) = 0 \).

The category of Mackey functors on \( S \) is denoted by \( \text{M}(S) \).

We use the following fact later. If \( M \) is a Mackey functor and \( i: Y \rightarrow X \) is a monomorphism in \( S \), then the composite
\[
M(Y) \xrightarrow{i_*} M(X) \xrightarrow{i^*} M(Y)
\]
is the identity. So the composite
\[
M(X) \xrightarrow{i^*} M(Y) \xrightarrow{i_*} M(X)
\]
is an idempotent endomorphism.

The category \( \text{M}(S) \) is a tensor category. Its tensor product is defined as follows. Let \( M, M', M'' \in \text{M}(S) \). A bilinear morphism \( \phi: (M, M') \rightarrow M'' \) is a family of linear maps \( \phi_{X,Y}: M(X) \otimes M'(Y) \rightarrow M''(XY) \) which commute with \( f_* \) and \( f^* \) for the both variables \( X, Y \). Given \( M, M' \in \text{M}(S) \), there exists a bilinear morphism \( (M, M') \rightarrow M_0 \) which is universal among all bilinear morphisms \( (M, M') \rightarrow M'' \). We define \( M_0 = M \otimes M' \).

If \( C \) is a category with pullbacks and sums, Mackey functors on \( C \) are similarly defined. We denote by \( \text{M}(C) \) the category of Mackey functors on \( C \).
3. The main result

We define a category $\mathcal{T}_{c*}$. An object of $\mathcal{T}_{c*}$ is a pair $(X, a)$ of $X \in S$ and an automorphism $a : X \to X$ of $S$ such that $a$ leaves all $G$-orbits in $X$ stable. A morphism $(X, a) \to (X', a')$ of $\mathcal{T}_{c*}$ is a morphism $f : X \to X'$ of $S$ such that $f \circ a = a' \circ f$.

The category $\mathcal{T}_{c*}$ has pullbacks and sums, so the category $M(\mathcal{T}_{c*})$ is defined.

A construction of pullback in $\mathcal{T}_{c*}$ is as follows. Given a diagram

$$(Y, b)$$

$$(X, a) \longrightarrow (Z, c)$$

in $\mathcal{T}_{c*}$, form a pullback

$$W \longrightarrow Y$$

$$\downarrow \downarrow$$

$$X \longrightarrow Z$$

in $S$. The maps $a : X \to X$ and $b : Y \to Y$ induce $d : W \to W$. Put

$$V = \bigcup \{U \mid U \text{ is a } G\text{-orbit in } W, d(U) = U\}$$

and $e = d|V$. Then

$$(V, e) \longrightarrow (Y, b)$$

$$(X, a) \longrightarrow (Z, c)$$

is a pullback in $\mathcal{T}_{c*}$.

Our result is

**Theorem.** An equivalence of categories $\mathbb{Z}(M(S)) \simeq M(\mathcal{T}_{c*})$.

By definition an object of $\mathbb{Z}(M(S))$ is a pair $(M, \theta)$ of $M \in M(S)$ and a family $\theta$ of isomorphisms $\theta_{M'} : M' \otimes M \to M \otimes M'$ for all $M' \in M(S)$ satisfying certain conditions. We may also regard an object of $\mathbb{Z}(M(S))$ as a pair $(M, \omega)$ of $M \in M(S)$ and a family $\omega$ of isomorphisms $\omega_{X,Y} : M(XY) \to M(YX)$ for all $X, Y \in S$ satisfying (i)–(iii):

(i) $\omega_{X,Y}$ is natural in $X, Y$.

(ii) The diagram

$$\begin{array}{ccc}
M(XYZ) & \xrightarrow{\omega_{X,Y,Z}} & M(YZX) \\
\downarrow \omega_{XY,Z} & & \downarrow \omega_{Y,ZX} \\
N(ZXY) & \xrightarrow{\omega_{ZXY}} & N(ZXY)
\end{array}$$

is a pullback in $\mathcal{T}_{c*}$.
commutes for all $X, Y, Z \in S$.

(iii) $\omega_{1, X} = 1$ for a one-element $G$-set 1.

The equivalence of the theorem is given as follows. Let $(M, \omega) \in Z(M(S))$. For an object $(X, a) \in T_{c*}$, define $L(X, a)$ as the pullback

$$
\begin{array}{c}
L(X, a) \rightarrow M(X) \\
\downarrow \quad \downarrow (a, 1)* \\
M(X) \rightarrow M(XX)
\end{array}
$$

where $(a, 1): X \rightarrow XX$ is the map $x \mapsto (a(x), x)$, and $(1, 1): X \rightarrow XX$ is the diagonal map. Then the assignment $(X, a) \mapsto L(X, a)$ becomes a Mackey functor $L$ on $T_{c*}$. The functor $(M, \omega) \mapsto L$ gives the equivalence $Z(M(S)) \simeq M(T_{c*})$.

4. Outline of the proof

The equivalence of the theorem is obtained as the composite of equivalences

$$
Z(M(S)) \simeq sM(S,S) \simeq M_0(\mathcal{W}') \simeq M(\mathcal{W}_{ic*}) \simeq M(T_{c*})
$$

We will sketch each equivalence in order.

(1) $Z(M(S)) \simeq sM(S,S)$.

A bi-Mackey functor $N$ on $S$ consists of vector spaces $N(X, Y)$ for all $G$-sets $X$ and $Y$, and linear maps

$$
\begin{align*}
(f, g)_*: N(X, Y) &\rightarrow N(X', Y'), \\
(f, g)^*: N(X', Y') &\rightarrow N(X, Y)
\end{align*}
$$

for all $G$-maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ satisfying (i)–(ix):

(i) The collection of $N(X, Y)$ and $(f, g)_*$ forms a functor $S \times S \rightarrow \mathcal{V}$.

(ii) The collection of $N(X, Y)$ and $(f, g)^*$ forms a functor $S^{op} \times S^{op} \rightarrow \mathcal{V}$.

(iii) For $G$-maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$, the diagrams

are commutative.
(iv) If

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & X'_1 \\
\downarrow p & & \downarrow p' \\
X_2 & \xrightarrow{f_2} & X'_2
\end{array}
\]

is a pullback diagram, then

\[
\begin{array}{ccc}
N(X_1, Y) & \xrightarrow{(f_1, 1)_*} & N(X'_1, Y) \\
\langle p, 1 \rangle^* & \uparrow & \uparrow \langle p', 1 \rangle^* \\
N(X_2, Y) & \xrightarrow{(f_2, 1)_*} & N(X'_2, Y)
\end{array}
\]

is commutative.

(v) The analogue of (iv) for the second variable.

(vi) Let \(i_1: X_1 \to X_1 + X_2\), \(i_2: X_2 \to X_1 + X_2\) denote the inclusion maps. Then the maps

\[
((i_1, 1)_*, (i_2, 1)_*): N(X_1, Y) \oplus N(X_2, Y) \to N(X_1 + X_2, Y),
\]

\[
((i_1, 1)^*, (i_2, 1)^*): N(X_1 + X_2, Y) \to N(X_1, Y) \oplus N(X_2, Y)
\]

are inverse to each other.

(vii) The analogue of (vi) for the second variable.

(viii) \(N(\emptyset, Y) = 0\).

(ix) \(N(X, \emptyset) = 0\).

A **bi-Mackey functor on \(S\) with two-sided action** is a bi-Mackey functor \(N\) on \(S\) equipped with maps

\[
\begin{array}{ccc}
Z!: N(X, Y) & \to & N(ZX, ZY) \\
!Z: N(X, Y) & \to & N(XZ, YZ)
\end{array}
\]

for \(X, Y, Z \in S\) satisfying (i)-(ix):

(i) For \(G\)-maps \(f: X \to X'\) and \(g: Y \to Y'\), the diagrams

\[
\begin{array}{ccc}
N(X, Y) & \xrightarrow{(f, g)_*} & N(X', Y') \\
\downarrow !Z & & \downarrow !Z \\
N(XZ, YZ) & \xrightarrow{(1f, 1g)_*} & N(X'Z, Y'Z)
\end{array}
\]
and
\[
\begin{array}{ccc}
N(X, Y) & \xleftarrow{(f, g)^*} & N(X', Y') \\
!Z & & \downarrow !Z \\
N(XZ, YZ) & \xleftarrow{(1f, 1g)^*} & N(X'Z, Y'Z)
\end{array}
\]
are commutative.

(ii) For \(G\)-map \(h: Z \rightarrow Z'\), the diagrams
\[
\begin{array}{ccc}
N(X, Y) & \xrightarrow{!Z} & N(XZ, YZ) \\
!Z' & & \downarrow (1,1h)^* \\
N(XZ', YZ') & \xrightarrow{(1h, 1)^*} & N(XZ, YZ')
\end{array}
\]
and
\[
\begin{array}{ccc}
N(X, Y) & \xrightarrow{!Z} & N(XZ, YZ) \\
!Z' & & \downarrow (1h, 1)^* \\
N(XZ', YZ') & \xrightarrow{(1,1h)^*} & N(XZ', YZ)
\end{array}
\]
are commutative.

(iii) The diagram
\[
\begin{array}{ccc}
N(X, Y) & \xrightarrow{!Z} & N(XZ, YZ) \\
!(ZZ') & & \downarrow !Z' \\
N(XZZ', YZZ') & & N(XZ', YZ')
\end{array}
\]
is commutative.

(iv) For a one-element \(G\)-set \(1\),
\[
!1: N(X, Y) \rightarrow N(X1, Y1)
\]
is the identity.

(v)–(viii) The analogue of (i)–(iv) for \(Z!\).

(ix) The diagram
\[
\begin{array}{ccc}
N(X, Y) & \xrightarrow{Z!} & N(ZX, ZY) \\
!W & & \downarrow !W \\
N(XW, YW) & \xrightarrow{Z!} & N(ZXW, ZYW)
\end{array}
\]
is commutative.

The category of bi-Mackey functors on \(S\) with two-sided action is denoted by \(sM(S, S)_{S}\).
Proposition. We have an equivalence $\mathbb{Z}(M(S)) \simeq sM(S, S)_S$.

This equivalence takes an object $(M, \omega) \in \mathbb{Z}(M(S))$ to an object $N \in sM(S, S)_S$ defined as follows. For $X, Y \in S$

$$N(X, Y) = M(XY).$$

The operation

$$!Z : N(X, Y) \to N(XZ, YZ)$$

is the composite

$$M(XY) \xrightarrow{(1\ell)^*} M(XYZ) \xrightarrow{(1\Delta)} M(XYZZ) \xrightarrow{(1\tau)} M(XZZ) \xrightarrow{\Delta} M(XZZZ) \xrightarrow{(1p)} M(XZYZ),$$

where $\Delta : Z \to ZZ$ is the diagonal map and $\tau : YZ \to ZY$ is the transposition.

The operation

$$Z! : N(X, Y) \to N(ZX, ZY)$$

is the composite

$$M(XY) \xrightarrow{(11\ell)^*} M(XYY) \xrightarrow{(11\Delta)} M(XZXY) \xrightarrow{\omega_{X, Y}} M(ZXY) \xrightarrow{(1\tau)} M(ZXXY).$$

(2) $sM(S, S)_S \simeq \mathcal{M}_0(\mathcal{W}')$.

Let $\mathcal{W}'$ be the category whose objects are diagrams

$$\begin{array}{ccc}
U & \xrightarrow{V} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{V} & Z
\end{array}$$

of $G$-sets such that the induced maps $U \to XY$, $V \to XY$ are injective, and morphisms are natural ones. This has pullbacks and sums, so one has the category $\mathcal{M}(\mathcal{W}')$ of Mackey functors on $\mathcal{W}'$.

Suppose that $(M, \omega) \in \mathbb{Z}(M(S))$ corresponds to $N \in sM(S, S)_S$ under the equivalence (1). Let

$$X = \begin{pmatrix}
\begin{array}{c}
U \\
X \\
V
\end{array}
\end{pmatrix} \in \mathcal{W}'.
$$

As noted after the definition of a Mackey functor, the injection $V \to XY$ determines an idempotent endomorphism on $M(XY)$. As $M(XY) = N(X, Y)$, this is an idempotent endomorphism on $N(X, Y)$, which we denote by

$$e^R(X \leftarrow V \to Y).$$

Similarly the injection $U \to YX$ determines an idempotent endomorphism on $M(YX)$. Through the isomorphism $\omega_{X, Y} : M(XY) \to M(YX)$ and $M(XY) = N(X, Y)$, this yields an idempotent endomorphism on $N(X, Y)$, which we denote by

$$e^L(X \leftarrow U \to Y).$$
**Lemma.** The idempotent endomorphisms $e^L(X \leftarrow U \rightarrow Y)$ and $e^R(X \leftarrow V \rightarrow Y)$ on $N(X, Y)$ commute with each other.

We set

$$H(X) = \text{Im} e^L(X \leftarrow U \rightarrow Y) \cap \text{Im} e^R(X \leftarrow V \rightarrow Y)$$

Then the assignment $X \mapsto H(X)$ becomes a Mackey functor $H$ on $\mathcal{W}'$. We thus obtain a functor

$$sm\mathcal{M}(S, S) \rightarrow \mathcal{M}(\mathcal{W}')$$

$$N \mapsto H.$$

This is fully faithfull. To describe its image, we define a full subcategory $\mathcal{M}_0(\mathcal{W}')$ of $\mathcal{M}(\mathcal{W}')$.

An object of $\mathcal{M}_0(\mathcal{W}')$ is an object $H$ of $\mathcal{M}(\mathcal{W}')$ which satisfies (i)–(viii):

(i) Suppose that

$$X = \begin{pmatrix} U_1 + U_2 \\ X \\ Y \end{pmatrix}$$

is an object of $\mathcal{W}'$. Put

$$X_1 = \begin{pmatrix} U_1 \\ X \\ Y \end{pmatrix}, \quad X_2 = \begin{pmatrix} U_2 \\ X \\ Y \end{pmatrix}$$

and let $i_1: X_1 \rightarrow X$, $i_2: X_2 \rightarrow X$ be the natural injections. Then the maps

$$(i_{1*}, i_{2*}): H(X_1) \oplus H(X_2) \rightarrow H(X),$$

$$(i_{1'}^*, i_{2'}^*): H(X) \rightarrow H(X_1) \oplus H(X_2)$$

are inverse to each other.

(ii)

$$H \begin{pmatrix} \emptyset \\ X \\ Y \end{pmatrix} = 0.$$

(iii) The analogue of (i) for the $V$-component.

(iv) The analogue of (ii) for the $V$-component.

(v) Let

$$X_1 = \begin{pmatrix} U_1 \\ X_1 \\ V_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} U_2 \\ X_2 \\ V_2 \end{pmatrix}$$
be objects of $\mathcal{W}'$. Put

$$X = \left( \begin{array}{c} U_1 + U_2 \\ X_1 + X_2 \\ Y \\ V_1 + V_2 \end{array} \right)$$

and let $j_1 : X_1 \to X$, $j_2 : X_2 \to X$ be the natural injections. Then the maps

$$(j_1^*, j_2^*): H(X_1) \oplus H(X_2) \to H(X),$$

$$(j_1^*, j_2^*): H(X) \to H(X_1) \oplus H(X_2)$$

are inverse to each other.

(vi) The analogue of (v) for the $Y$-component.

(vii) Let

$$X = \left( \begin{array}{c} U \\ X_1 \\ Y \\ V \end{array} \right)$$

be an object of $\mathcal{W}'$. Let

$$V_1 \xrightarrow{(c_1, d_1)} UU$$

$e \downarrow \quad \downarrow ab$

$$V \xrightarrow{(c, d)} XY$$

be a pullback. Put

$$U = \left( \begin{array}{c} U \\ U_1 \\ V_1 \end{array} \right)$$

and

$$a = \left( \begin{array}{c} 1 \\ e \\ b \end{array} \right) : U \to X.$$

Then the maps

$$a_* : H(U) \to H(X),$$

$$a^* : H(X) \to H(U)$$

are inverse to each other.

(viii) The analogue of (vii) for the $V$-component.

The functor $sM(S, S)_S \to M(\mathcal{W}')$ constructed before has the image $M_0(\mathcal{W}')$, and yields
Proposition. An equivalence $\mathcal{M}(S, S) \simeq \mathcal{M}_0(W')$.

(3) $\mathcal{M}_0(W') \simeq \mathcal{M}(W_{ic*})$.
Let $W_{ic*}$ be the full subcategory of $W'$ consisting of finite sums of diagrams

\[
\begin{array}{c}
U \\
\downarrow \\
X \\
\downarrow \\
V \\
\downarrow \\
Y \\
\uparrow \\
\end{array}
\]

in which $X, Y, U, V$ are transitive $G$-sets and the four arrows are isomorphisms.

Lemma. The inclusion functor $W_{ic*} \to W'$ has a right adjoint.

Denote the inclusion $W_{ic*} \to W'$ by $i$ and a right adjoint by $R$.

Proposition. We have an equivalence $\mathcal{M}_0(W') \simeq \mathcal{M}(W_{ic*})$.

Under the equivalence objects $H \in \mathcal{M}_0(W')$ and $K \in \mathcal{M}(W_{ic*})$ correspond if

$H \cong K \circ R, \quad K \cong H \circ i$.

(4) $\mathcal{M}(W_{ic*}) \simeq \mathcal{M}(T_{c*})$.
An object of the category $T_{c*}$ is a pair $(X, a)$ of $X \in S$ and an automorphism $a: X \to X$ such that $a$ leaves all $G$-orbits stable. The functor

\[(X, a) \mapsto \begin{pmatrix}
  X \\
  \downarrow \\
  X \\
  \downarrow \\
  1 \\
  \end{pmatrix}
\]

gives an equivalence $T_{c*} \simeq W_{ic*}$. This yields

Proposition. An equivalence $\mathcal{M}(W_{ic*}) \simeq \mathcal{M}(T_{c*})$.

Combining (1)–(4), we obtain $\mathbb{Z}(\mathcal{M}(S)) \simeq \mathcal{M}(T_{c*})$.

References
5. D. Tambara, The Drinfeld center of the category of Mackey functors, in submission.