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Kyoto University
A morphism of Green functors

富山工業高等専門学校・一般科目 小田 文仁 (Fumihito Oda)

Department of Liberal Arts, Toyama National College of Technology
E-mail: oda@toyama-nct.ac.jp

1 Introduction

This article is a survey of [Od07]. Bouc introduced the Dress construction for a Green functor ([Bo03a] Theorem 5.1): If $A$ is a Green functor for $G$ over a commutative ring $O$, and $\Gamma$ is a crossed $G$-monoid, then the Mackey functor $A_{\Gamma}$ obtained by the Dress construction has a natural structure of a Green functor, and its evaluation $A_{\Gamma}(G)$ is an $O$-algebra. Bouc's construction involves as special cases the construction of the crossed Burnside ring obtained from the Burnside ring Green functor, the Hochschild cohomology ring of $G$ obtained from the group cohomology Green functor, and the Grothendieck ring of the Drinfel'd double of $G$ obtained from the Grothendieck ring Green functor for a group algebra. We also point out that Bouc's construction is discussed in [Wi04]. In this paper, we obtain an induction theorem for the Drinfel'd double for $G$ by using a formula for the primitive idempotents of the crossed Burnside ring [OY01], Bouc's construction, and some properties of Witherspoon's Green functor $R(D_{G}(\ast))$. The theorem implies Artin's induction theorem for a group algebra over $\mathbb{C}$. This is a new proof of Artin induction theorem.

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2 Crossed $G$-sets

(2.1) Notation. Let $G$ be a finite group. If $H$ is a subgroup of $G$, and $g \in G$, the conjugate subgroup $gHg^{-1}$ of $G$ is denoted by $^{g}H$. The normalizer of $H$ in $G$ is denoted by $N_{G}(H)$. The centralizer of $H$ (resp. $g \in G$) in $G$ is denoted by $C_{G}(H)$ (resp. $C_{G}(g)$). A set of representatives in $G$ of $G/H$ is denoted by $[G/H]$. If $X$ is a $G$-set, the stabilizer in $G$ of element $x$ of $X$ is denoted by $G_{x}$. If $X$ and $Y$ are $G$-sets, the intersection $G_{x} \cap G_{y}$ of stabilizers in $G$ of element $(x, y)$ of $X \times Y$ is denoted by $G_{x,y}$. The set of orbits of $H$ on $X$ is denoted by $H\backslash X$, and $[H\backslash X]$ denotes a set of representatives in $X$ of $H\backslash X$.

(2.2) Crossed Burnside rings. Let $G$ be a finite group. In [Bo03a], Bouc defined a crossed $G$-monoid as follows. A crossed $G$-monoid $(\Gamma, \varphi)$ is a pair consisting of a finite monoid $\Gamma$ with a left action of $G$ by monoid automorphisms (denoted by $(g, \gamma) \mapsto g\gamma$ or $(g, \gamma) \mapsto {}^{g}\gamma$, for $g \in G$ and $\gamma \in \Gamma$), and a map of $G$-monoids $\varphi$ from $\Gamma$ to the $G$-set
$G^c$ with $G$-action defined by conjugation (i.e. a map $\varphi$ which is both a map of monoids and a map of $G$-sets). In this paper, since we use only the trivial crossed $G$-monoid $(\Gamma, \varphi) = (G^c, id_{G^c})$, we denote by $\Gamma$ or $G^c$ a crossed $G$-monoid. A crossed $G$-set $(X, \alpha)$ over a crossed $G$-monoid $\Gamma$, is a pair consisting of a finite $G$-set $X$, together with a map $\alpha$ of $G$-sets from $X$ to $\Gamma$. A morphism of crossed $G$-sets from $(X, \alpha)$ to $(Y, \beta)$ is a $G$-map $f$ from $X$ to $Y$ such that $\beta \circ f = \alpha$. Crossed $G$-sets over $\Gamma$ and crossed $G$-maps make a category $G$-$xset/\Gamma$. The tensor product of crossed $G$-sets $(X, \alpha)$ and $(Y, \beta)$ is defined by $(X \times Y, \alpha \cdot \beta)$, where $X \times Y$ is the direct product of $X$ and $Y$, with diagonal $G$-action, and $\alpha \cdot \beta$ is the map from $X \times Y$ to $G^c$ defined by $\alpha \cdot \beta(x, y) = \alpha(x) \beta(y)$. We denote by $X \Omega(G, \Gamma)$ the Grothendieck ring of the category $G$-$xset/\Gamma$ with respect to disjoint union and tensor product. We call it the crossed Burnside ring. The crossed Burnside ring $G$-$xset/1^c$ over the crossed $1^c$-monoid is the ordinary Burnside ring $B(G)$. Since any crossed $G$-set is a disjoint union of transitive crossed $G$-sets (see 2.12 of [OY01]), $G$-$xset/\Gamma$ has the following free $\mathbb{Z}$-basis as an abelian group:

$$\{(G/D)_s | D \in [G \setminus S(G)], s \in [G \setminus C_{\Gamma}(D)]\}.$$  

If $\Gamma$ is a normal subgroup of $G$ or an abelian group, then a formula for the primitive idempotents of $KX \Omega(G, \Gamma)$ over a splitting field $K$ of characteristic 0 has been given by Oda and Yoshida (see Lemma (5.5) of [OY01]).

(2.3) Theorem, [OY01] Let $K$ be a field of characteristic 0 which is a splitting field for all subgroups of $G$.

(1) For $H \leq G$ and an irreducible $K$-character $\theta$ of $C_{\Gamma}(H)$, we put

$$e_{H, \theta} = \frac{\theta(1)}{|N_G(H)||C_T(H)|} \sum_{D \leq H} \sum_{s \in C_{\Gamma}(H)} |D| \mu(D, H) \bar{\theta}(s^{-1}) (G/D)_s,$$

where $\bar{\theta}$ is the sum of all distinct $N_G(H)$-conjugates of $\theta$. Then

$$\{e_{H, \theta} | H \in [G \setminus S(G)], \theta \in [N_G(H) \setminus Irr_K(C_{\Gamma}(H))]\}$$

is a set of orthogonal idempotents of the crossed Burnside ring $KX \Omega(G, \Gamma)$ over $K$ such that

$$(G/G)_{1_G} = 1_{KX \Omega(G, \Gamma)} = \sum_{H, \theta} e_{H, \theta}.$$

Moreover, the idempotents $e_{H, \theta}$ are all primitive and conversely any primitive idempotent of $KX \Omega(G, \Gamma)$ has this form.

A formula for the primitive idempotents of $O \times \Omega(G, G^c)$ over a $p$-local ring $O$ has been given by Bouc [Bo03b].

3 Bouc's constructions of Green functors

(3.1) Burnside Green functors. We recall the crossed Burnside ring Green functor $X \Omega(\ast, G^c)$ in terms of subgroups of $G$ (see 4.1 of [OY04]). Let $S(H)$ be the family of all subgroups of $H \leq G$ and $C_G(D)$ the centralizer of $D \leq H$. Then the assignment

$$H(\leq G) \mapsto X \Omega(H, G^c) = ((H/D)_s | D \in [H \setminus S(H)] \ s \in [H \setminus C_G(D)] \}_{\mathbb{Z}}$$

gives a Green functor for $G$ over $\mathbb{Z}$ equipped with

\[
\text{ind}^H_L : X\Omega(L, G^c) \rightarrow X\Omega(H, G^c) : (L/D) \mapsto (H/D),
\]
\[
\text{res}^H_L : X\Omega(H, G^c) \rightarrow X\Omega(L, G^c) : (H/D) \mapsto \sum_{g \in [L/H/D]} (L/L \cap gD)_{gs},
\]
\[
\text{con}_{H,g} : X\Omega(H, G^c) \rightarrow X\Omega(gH, G^c) : (H/D) \mapsto (gH/gD)_{gs},
\]

where $D \leq L \leq H \leq G$ and $g \in G$. In order to note the Green functor structure of $X\Omega(\ast, G^c)$, we shall discuss briefly an equivalence between the category $G$-set$_{(G/H \times G^c)}$ of finite $G$-sets over the $G$-set $G/H \times G^c$ (see 2.4 of [Bo97]) and the category $H$-set$_{(G/H \times G^c)}$ of finite $H$-sets over the $H$-set $G^c$ with the $H$-action defined by conjugation. Let $\Omega$ be the Burnside Green functor for $G$ over $\mathbb{Z}$ in terms of $G$-sets. By Proposition 2.4.2 of [Bo97], $\Omega_{G^c}(G/H) = \Omega((G/H) \times G^c)$ is isomorphic to the Grothendieck group of $G$-set$_{(G/H \times G^c)}$, with relations given by decomposition into disjoint union. It is easy to see that the $G$-sets

\[
[K, s] : G/K \rightarrow G/H \times G^c : gK \mapsto (gH, ^g s)
\]

over $G/H \times G^c$, for $K \in [H \backslash S(H)]$ and $s \in [H \backslash C_G(K)]$, form a basis of $\Omega(G/H \times G^c)$ over $\mathbb{Z}$. We denote by $(G/K, [K, s])$ an element of the basis of $\Omega(G/H \times G^c)$. Theorem 5.1 of [Bo03a] shows that $\Omega_{G^c}$ is a Green functor. If $(G/K, [K, s])$ and $(G/L, [L, t])$ are elements of the basis of $\Omega(G/H \times G^c)$, then we have the following commutative diagram

\[
\begin{array}{ccc}
G/K \cap \mathbb{Z}L & \xrightarrow{\mathbb{Z}\cap\mathbb{Z}L} & G/K \times G/L \\
\downarrow \mathbb{Z}\in\mathbb{Z}L & & \downarrow \mathbb{Z}\in\mathbb{Z}L \\
G/H \times G^c & \xrightarrow{\delta_{G/H} \times \text{Id}_{G^c}} & G/H \times G/H \times G^c \\
\downarrow \delta_{G/H} \times \text{Id}_{G^c} & & \downarrow \text{Id} \\
G/H \times G/H \times G^c & \xrightarrow{\text{Id}} & G/H \times G/H \times G^c
\end{array}
\]

where the map $f$ from $G/H \times G^c \times G/H \times G^c$ to $G/H \times G/H \times G^c$ maps $(xK, \gamma_1, yL, \gamma_2)$ to $(xK, yL, \gamma_1 \gamma_2)$ (see section 5 of [Bo03a]). The left square is a pullback square. Theorem 5.1 of [Bo03a] shows that the product of $(G/K, [K, s])$ and $(G/L, [L, t])$ on $\Omega(G/H \times G^c)$ is given by

\[
(G/K, [K, s]) \cdot (G/L, [L, t]) = \sum_{x \in [K \cap H/L]} (G/K \cap \mathbb{Z}L, [K \cap \mathbb{Z}L, s \cap \mathbb{Z}t]).
\]  

(3.1.1)

We have a functor $F$ mapping $G$-set$_{(G/H \times G^c)}$ to $H$-set$_{(G^c)}$ defined for a transitive $G$-set $[K, s] : G/K \rightarrow G/H \times G^c$ over $G/H \times G^c$ by

\[
F : (G/K, [K, s]) \mapsto [K, s]^{-1}(\{H\} \times G^c) \rightarrow G^c
\]

as in Lemma 2.4.1 of [Bo97]. We also denote by $[K, s]$ the $H$-map $H/K \rightarrow G^c$ defined by $gK \mapsto ^g s$. It is clear that the $H$-sets

\[
[K, s] : H/K \rightarrow G^c : gK \mapsto ^g s
\]

over the $H$-set $G^c$, for $K \in [H \backslash S(H)]$ and $s \in [H \backslash C_G(K)]$, form a basis of $\Omega_H(G^c)$ over $\mathbb{Z}$, where $\Omega_H(G^c)$ is a Green functor for $H$ given by the restriction to $H$ of $G$. We denote by $(H/K, [K, s])$ this element of the basis of $\Omega_H(G^c)$. It is easy to see that $F$ gives an equivalence of categories from $G$-set$_{(G/H \times G^c)}$ to $H$-set$_{(G^c)}$ for any subgroup $H$ of $G$. The
inverse equivalence is given by the induction functor from $H$-$\text{set}\downarrow_{G^c}$ to $G$-$\text{set}\downarrow_{(G/H \times G^c)}$. The images of (3.1.1) under $F$ are

$$(H/K, [K, s]) \cdot (H/L, [L, t]) = \sum_{z \in [K \setminus H/L]} (H/K \cap zL, [K \cap zL, s \cdot zt]).$$

in $H$-$\text{set}\downarrow_{G^c}$. The Grothendieck group of $H$-$\text{set}\downarrow_{G^c}$ is isomorphic to $X\Omega(H, G^c)$. We can define a product

$$(H/K)_s \cdot (H/L)_t = \sum_{z \in [K \setminus H/L]} (H/K \cap zL)_{s \cdot zt}$$

for any two elements $(H/K)_s$ and $(H/L)_t$ of the basis of $X\Omega(H, G^c)$. It is clear that the element $(H/H)_1^g$ for the identity element $1_G$ of $G$ is the identity element of $X\Omega(H, G^c)$. This gives a unitary ring structure to $X\Omega(H, G^c)$ for a subgroup $H$ of $G$.

(3.2) Witherspoon's Green functor. Witherspoon introduced a Green functor $R_C(D_G(\ast))$ for $G$ over $\mathbb{Z}$ (see [Wi96] Section 5). For each subgroup $H$ of $G$, there is a subalgebra

$$D_G(H) = \sum_{g \in G, h \in H} C\phi_g h$$

of the Drinfel'd (quantum) double $D(G)$ of $\mathbb{C}G$ [Dr86], where $\phi_g$ is an element of the basis $\{ \phi_g \}_{g \in G}$ of the dual space $(\mathbb{C}G)^* = \text{Hom}_{\mathbb{C}}(\mathbb{C}G, \mathbb{C})$. Note that $D_G(G) = D(G)$ and $R(D(G))$ is the representation ring of $D(G)$ or equivalently the Grothendieck ring of Hopf bimodules for the Hopf algebra $\mathbb{C}G$ ([Ro95], [Bo03a], [OY04]). Let $R_C(D_G(H))$ be the Grothendieck (representation) ring of $D_G(H)$ for subgroup $H$ of $G$. Then the assignment

$$H \mapsto R_C(D_G(H))$$

where $H \leq G$ gives a Green functor for $G$ over $\mathbb{Z}$ with operations given by

- $\text{Dres}^H_L$: $R_C(D_G(H)) \rightarrow R_C(D_G(L)) : U \mapsto U \downarrow_{D_G(L)}$,
- $\text{Dind}_L^H$: $R_C(D_G(L)) \rightarrow R_C(D_G(H)) : V \mapsto D_G(H) \otimes_{D_G(L)} V$,
- $\text{Dconj}_{H,g}$: $R_C(D_G(H)) \rightarrow R_C(D_G(gH)) : U \mapsto gU = gD_G(H) \otimes_{D_G(H)} U$,

where $U \downarrow_{D_G(L)}$ is a $D_G(L)$-module by restriction of the action from $D_G(H)$ to $D_G(L)$, $L \leq H \leq G$ and $g \in G$. We use the equivalence of the category of $H$-vector bundles on $G^c$ with the category of $D_G(H)$-modules (see [Wi96] Section 2).

(3.3) A morphism of Green functors. Let $\Omega$ be the Burnside ring Green functor for $G$ over $\mathbb{Z}$ (see [Bo97] 2.4.2):

- If $X$ is a finite $G$-set, then $\Omega(X)$ is the Grothendieck ring of the category of finite $G$-sets over $X$, where the relations are given by decomposition into disjoint union and product of $G$-sets.
- If $X \rightarrow X'$ is a $G$-map, then $\Omega_*(f) : \Omega(X) \rightarrow \Omega(X')$ is defined by $\Omega_*(f)((Y, \phi)) = (Y, f\phi)$ for any $G$-set $(Y, \phi) = Y \rightarrow X$ over $X$. 

• If $X' \to X$ is a $G$-map, then $\Omega^*(f) : \Omega(X) \to \Omega(X')$ is defined by $\Omega^*(f)((Y, \phi)) = (Y', \phi')$, where $(Y', \phi')$ is the pull-back of $(Y, \phi)$ along $f$, obtained by filling the cartesian square

\[
\begin{array}{ccc}
Y' & \to & Y \\
\downarrow \phi' & & \downarrow \phi \\
X' & \overset{f}{\to} & X.
\end{array}
\]

Suppose that $R_C$ is the $\mathbb{C}$-representation (character ring) Green functor for $G$ over $\mathbb{Z}$, defined on subgroups of $G$. Then setting $R_C(G/H) = R_C(H)$ leads by linearity to a definition of $G$-equivariant $\mathbb{C}$-vector bundles $R_C(X)$ on a $G$-set $X$ (see [Wi96] Section 2) by using Remark 2.3 of [Bo03a]:

• If $X$ is a finite $G$-set, then $R_C(X)$ is the Grothendieck ring of the category of $G$-equivariant $\mathbb{C}$-vector bundles on the $G$-set $X$, for relations given by decomposition into direct sum of vector bundles, the ring structure being induced by the tensor product of vector bundles: one can set

\[
R_C(X) = \left( \bigoplus_{x \in X} R_C(G_x) \right)^G
\]

where the exponent denotes fixed points under the natural action of $G$ on $\bigoplus_{x \in X} R_C(G_x)$ by permutation of the components, and $G_x$ is the stabilizer of $x$ in $G$.

• If $f : X \to X'$ is a $G$-map, then $R_C*(f) : R_C(X) \to R_C(X')$ is defined by

\[
R_C*(f)(u)_{y} = \sum_{x \in [G_y \backslash f^{-1}(y)]} t_{G_x}^{G_y}(u_x),
\]

where $t_{G_x}^{G_y}$ is the induction map from $R_C(G_x)$ to $R_C(G_y)$, $u \in R_C(X)$, and $y \in X'$.

• If $f : X' \to X$ is a $G$-map, then $R_C^*(f) : R_C(X) \to R_C(X')$ is defined by

\[
R_C^*(f)(v)_x = r_{G_{f(x)}^{x}}^{G}(v_{f(x)}),
\]

where $r_{G_{f(x)}^{x}}^{G}$ is the restriction map from $R_C(G_x)$ to $R_C(G_{f(x)})$, $v \in R_C(X)$, and $x \in X'$.

• The product of the elements $a \in R_C(X)$ and $b \in R_C(Y)$ is defined by

\[
(a \times b)_{x,y} = r_{G_{(x,y)}^{x}}^{G}(a_x) \cdot r_{G_{(x,y)}^{y}}^{y}(b_y).
\]

If $X$ is a finite $G$-set, denote the natural morphism

\[
\theta : \Omega \to R_C
\]
of Green functors defined by the maps \( \theta(X) : \Omega(X) \rightarrow R_C(X) \) by

\[
(Y, \varphi) = (\varphi : Y \rightarrow X) \mapsto \{C[\varphi^{-1}(x)]\}_{x \in X},
\]

where \( C[\varphi^{-1}(x)] \) is the permutation module associated to the \( G_x \)-set \( \varphi^{-1}(x) \).

The following theorem is essential in the proof of Theorem 4.1 of this paper.

(3.4) **Theorem** (Bouc [Bo03a] 5.1). Let \( A \) be a Green functor for \( G \) over a commutative ring \( \mathcal{O} \), \( \Gamma \) a crossed \( G \)-monoid, and \( \varepsilon_A \) an element of \( A(\bullet) \) such that for any \( G \)-set \( X \) and for any \( a \in A(X) \)

\[
A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a)
\]
denoting by \( p_x \) (resp. \( q_x \)) the bijective projection from \( X \times \bullet \) (resp. from \( \bullet \times X \)) to \( X \) (see 1.2.1 of [Bo03a]). Then the functor \( A_{\Gamma} \) is a Green functor for \( G \) over \( \mathcal{O} \), with unit \( \varepsilon_{A_{\Gamma}} \), where \( \varepsilon_{A_{\Gamma}} \) is the element \( A_*(\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon_A \end{pmatrix}) \) of \( A(\Gamma) = A_{\Gamma}(\bullet) \). Moreover the correspondence \( A \mapsto A_{\Gamma} \) is an endo-functor of the category of Green functors for \( G \) over \( \mathcal{O} \).

(3.5) **Lemma.** Let \( \Omega \) be the Burnside ring Green functor and \( G^c \) the crossed \( G \)-monoid. Then there is an isomorphism of Green functors

\[
X\Omega(\bullet, G^c) \cong \Omega_{G^c}.
\]

We will denote by \( C[X] \) the \( \mathcal{C} \)-permutation module associated to a set \( X \). The endo-functor of the category of Green functors of Theorem (3.4) applied to the morphism \( \theta \) from \( \Omega \) to \( R_C \) leads to the following lemma.

(3.6) **Lemma.** Let \( \theta : \Omega \rightarrow R_C \) be the natural morphism from the Burnside Green functor to the Grothendieck ring Green functor. Then the morphism \( \theta_{G^c} : \Omega_{G^c} \rightarrow R_{\mathcal{C}G^c} \) given by the Bouc's construction is a morphism of Green functors.

(3.7) **Lemma.** There is a morphism

\[
\theta_{G^c} : X\Omega(\bullet, G^c) \rightarrow R_C(D_G(\bullet))
\]

of Green functors.

Let \((H/L)_g\) be an element of the basis of \( X\Omega(H, G^c) \). Then the previous lemma shows that \( \theta_{G^c}((H/L)_g) \) is an \( H \)-vector bundle on \( G^c \). We denote by \([H/L]_g\) this \( H \)-vector bundle. Lemma (3.5) shows the following lemma.

(3.8) **Lemma.** The \( H \)-vector bundle \([H/L]_g\) is the \( \mathcal{C}C_H(\star g) \)-module \( \mathcal{C}[[xL, xg]^{-1}(\star g)] \) in the \( \star g \)-component, for \( x \in [H/C_H(g)] \), and 0 in all other components.

We recall the maps \( \text{Incl}_{J,h} : R_C(J) \rightarrow R_C(D_G(J)) \), where \( J \) is a subgroup of \( G \) and \( h \in C_G(J) \), introduced in Section 2 of [Wi96]: Given a \( \mathcal{C}J \)-module \( V \), \( \text{Incl}_{J,h}(V) \) is the \( D_G(J) \)-module which is \( V \) in the \( h \)-component and 0 elsewhere.
(3.9) Lemma. Let $\theta_{G'}$ be the ring homomorphism $\theta_{(G/G)\times G'}$ from the crossed Burnside ring $X\Omega(G,G')$ to the Grothendieck ring $R_{C}(D_{G}(G))$ given by the previous lemma. Then the $D(G)$-module corresponding to the $G$-vector bundle $\theta_{G'}((G/L)_{g})$ is the induced module

$$D(G) \otimes_{D_{G}(L)} \text{Incl}_{L,g}(C[L/L]).$$

(3.10) Sub-Green functors. There is a sub-Green functor $X\Omega(\ast,G')_{1}$ which assigns to each subgroup $H$ of $G$ the subring $X\Omega(H,G')_{1}$ of $X\Omega(H,G')$ generated by the elements $(H/L)_{1G}$. There is also a sub-Green functor $R_{C}(D_{G}(\ast))_{1}$ which assigns to each subgroup $H$ of $G$ the subring $R_{C}(D_{G}(\ast))_{1}$ of $R_{C}(D_{G}(H))$ generated by $\text{Incl}_{H,1G}(V)'s$, where $\text{Incl}_{H,1G}$ is a functor embedding the category of $CH$-modules as a full subcategory of the category of $D_{G}(H)$-modules (see, [Wi96] Section 1) and $V$ is a $CH$-module. It is easy to see that $X\Omega(H,G')_{1}$ is isomorphic to the Burnside ring $\Omega(H)$ and $R_{C}(D_{G}(H))_{1}$ is isomorphic to the ordinary character ring $R_{C}(H)$. The homomorphism $\theta_{G'} \downarrow_{X\Omega(H,G')_{1}}$ is the natural ring homomorphism from $\Omega(H)$ to $R_{C}(H)$.

(3.11) Characters. Witherspoon pointed out the character of a $C\mathbb{D}(G)$-module in [Wi96], that appeared in [Lu87]. For $g \in G$ and an irreducible character $\rho$ of $C_{G}(g)$, a character $\chi_{g,\rho}$ of a $C\mathbb{D}(G)$-module $U = \{U_{h}\}_{h \in G'}$ is given by the formula

$$\chi_{g,\rho}(U) = \frac{1}{\deg \rho} \sum_{h \in C_{G}(g)} \text{Tr}(g, U_{h}) \rho(h).$$

The characters of the crossed Burnside ring have been considered by Oda and Yoshida ([OY01], Section 5). For a subgroup $H$ of $G$ and an irreducible character $\theta$ of $C_{G}(H)$, the linear map $\omega_{H,\theta}$ of $X\Omega(G,G')$ to $\mathbb{C}$ is the composite of Burnside homomorphism $\varphi_{H}$ and a central character $\tilde{\omega}_{H,\theta}$: given a crossed $G$-set $X$ over $G'$, $H \leq G$, and an irreducible character $\rho$ of the group algebra $\mathbb{C}C_{G}(H)$, $\omega_{H,\rho}(X) = \tilde{\omega}_{H,\rho} \circ \varphi_{H}(X)$.

For each $h \in G'$ the $h$-component of the crossed $G$-set $(X, \alpha)$ is defined by

$$X[h] = \{x \in X | \alpha(x) = h\}.$$

(3.12) Lemma. Let $g$ be an element of $G$, $\rho$ an irreducible character of $\mathbb{C}C_{G}(g)$, and $\theta_{G'}$ the homomorphism from $X\Omega(G,G')$ to $R_{C}(D(G))$. Then $\chi_{g,\rho}(g) = \omega_{(g),\rho}$, where $(g)$ is the cyclic subgroup generated by $g$.

4 Induction theorems

The proof of the following theorem is similar to the proof of Theorem 3.5.2 in [Bo00].

(4.1) Theorem. Let $G$ be a finite group. Then

$$\mathbb{C}R_{C}(D(G)) = \sum_{H \in \mathcal{C}(G)} \text{Dind}_{H}^{G} \mathbb{C}R_{C}(D_{G}(H)),$$

where $\mathcal{C}(G)$ is the family of cyclic subgroups of $G$. In other words, any complex character of $D(G)$ is a linear combination with rational coefficients of characters induced from cyclic groups of $G$.

The previous theorem and (3.10) show the following corollary.
Corollary (Artin). Let $G$ be a finite group. Then

$$QR_{C}(G) = \sum_{H \in \mathcal{C}(G)} \text{Ind}_H^G QR_{C}(H).$$

References


