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A morphism of Green functors

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1 Introduction

This article is a survey of [Od07]. Bouc introduced the Dress construction for a *Green functor* ([Bo03a] Theorem 5.1): If A is a Green functor for G over a commutative ring $\mathcal{O}$, and Γ is a crossed G-monoid, then the Mackey functor A_Γ obtained by the Dress construction has a natural structure of a Green functor, and its evaluation $A_\Gamma(G)$ is an $\mathcal{O}$-algebra. Bouc's construction involves as special cases the construction of the crossed Burnside ring obtained from the Burnside ring Green functor, the Hochschild cohomology ring of G obtained from the group cohomology Green functor, and the Grothendieck ring of the Drinfel'd double of G obtained from the Grothendieck ring Green functor for a group algebra. We also point out that Bouc's construction is discussed in [Wi04]. In this paper, we obtain an induction theorem for the Drinfel'd double for G by using a formula for the primitive idempotents of the crossed Burnside ring [OY01], Bouc's construction, and some properties of Witherspoon's Green functor $R(D_G(*))$. The theorem implies Artin's induction theorem for a group algebra over $\mathbb{C}$. This is a new proof of Artin induction theorem.

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2 Crossed G-sets

(2.1) Notation. Let G be a finite group. If H is a subgroup of G, and $g \in G$, the conjugate subgroup gHg^{-1} of G is denoted by ${}^gH$. The normalizer of H in G is denoted by $N_G(H)$. The centralizer of H (resp. $g \in G$) in G is denoted by $C_G(H)$ (resp. $C_G(g)$). A set of representatives in G of G/H is denoted by $[G/H]$. If X is a G-set, the stabilizer in G of element x of X is denoted by G_x. If X and Y are G-sets, the intersection $G_x \cap G_y$ of stabilizers in G of element (x, y) of $X \times Y$ is denoted by $G_{x,y}$. The set of orbits of H on X is denoted by $H\backslash X$, and $[H\backslash X]$ denotes a set of representatives in X of $H\backslash X$.

(2.2) Crossed Burnside rings. Let G be a finite group. In [Bo03a], Bouc defined a crossed G-monoid as follows. A *crossed G-monoid* (Γ, φ) is a pair consisting of a finite monoid Γ with a left action of G by monoid automorphisms (denoted by $(g, \gamma) \mapsto g\gamma$ or $(g, \gamma) \mapsto {}^g\gamma$, for $g \in G$ and $\gamma \in \Gamma$), and a map of G-monoids φ from Γ to the G-set

G^c with G-action defined by conjugation (i.e. a map φ which is both a map of monoids and a map of G-sets). In this paper, since we use only the trivial crossed G-monoid $(\Gamma, \varphi) = (G^c, id_{G^c})$, we denote by Γ or G^c a crossed G-monoid. A *crossed G-set* (X, α) over a crossed G-monoid Γ, is a pair consisting of a finite G-set X, together with a map α of G-sets from X to Γ. A *morphism* of crossed G-sets from (X, α) to (Y, β) is a G-map f from X to Y such that $\beta \circ f = \alpha$. Crossed G-sets over Γ and crossed G-maps make a category G-**xset**$/\Gamma$. The *tensor product* of crossed G-sets (X, α) and (Y, β) is defined by $(X \times Y, \alpha.\beta)$, where $X \times Y$ is the direct product of X and Y, with diagonal G-action, and $\alpha.\beta$ is the map from $X \times Y$ to G^c defined by $\alpha.\beta(x, y) = \alpha(x)\beta(y)$. We denote by $X\Omega(G, \Gamma)$ the Grothendieck ring of the category G-**xset**$/\Gamma$ with respect to disjoint union and tensor product. We call it the *crossed Burnside ring*. The crossed Burnside ring G-**xset**$/1^c$ over the crossed 1^c-monoid is the ordinary Burnside ring $B(G)$. Since any crossed G-set is a disjoint union of *transitive* crossed G-sets (see 2.12 of [OY01]), G-**xset**$/\Gamma$ has the following free $\mathbb{Z}$-basis as an abelian group:

$$\{(G/D)_s \mid D \in [G \setminus S(G)],\ s \in [G \setminus C_\Gamma(D)]\}.$$

If Γ is a normal subgroup of G or an abelian group, then a formula for the primitive idempotents of $\mathbb{K}X\Omega(G, \Gamma)$ over a splitting field $\mathbb{K}$ of characteristic 0 has been given by Oda and Yoshida (see Lemma (5.5) of [OY01]).

(2.3) Theorem. [OY01] *Let $\mathbb{K}$ be a field of characteristic* 0 *which is a splitting field for all subgroups of G.*

(1) *For $H \leq G$ and an irreducible $\mathbb{K}$-character θ of $C_\Gamma(H)$, we put*

$$e_{H,\theta} = \frac{\theta(1)}{|N_G(H)||C_\Gamma(H)|} \sum_{D \leq H} \sum_{s \in C_\Gamma(H)} |D| \mu(D, H) \widetilde{\theta}(s^{-1}) (G/D)_s,$$

where $\widetilde{\theta}$ is the sum of all distinct $N_G(H)$-conjugates of θ. Then

$$\{e_{H,\theta} \mid H \in [G \setminus S(G)],\ \theta \in [N_G(H) \setminus \mathrm{Irr}_{\mathbb{K}}(C_\Gamma(H))]\}$$

is a set of orthogonal idempotents of the crossed Burnside ring $\mathbb{K}X\Omega(G, \Gamma)$ over $\mathbb{K}$ such that

$$(G/G)_{1_G} = 1_{\mathbb{K}X\Omega(G,\Gamma)} = \sum_{H,\theta} e_{H,\theta}.$$

Moreover, the idempotents $e_{H,\theta}$ are all primitive and conversely any primitive idempotent of $\mathbb{K}X\Omega(G, \Gamma)$ has this form.

A formula for the primitive idempotents of $\mathcal{O}X\Omega(G, G^c)$ over a p-local ring $\mathcal{O}$ has been given by Bouc [Bo03b].

3 Bouc's constructions of Green functors

(3.1) Burnside Green functors. We recall the crossed Burnside ring Green functor $X\Omega(*, G^c)$ in terms of subgroups of G (see 4.1 of [OY04]). Let $S(H)$ be the family of all subgroups of $H \leq G$ and $C_G(D)$ the centralizer of $D \leq H$. Then the assignment

$$H(\leq G) \longmapsto X\Omega(H, G^c) = \langle (H/D)_s \mid D \in [H \setminus S(H)]\ s \in [H \setminus C_G(D)] \rangle_{\mathbb{Z}}$$

gives a Green functor for G over $\mathbb{Z}$ equipped with

$$\begin{array}{rlll}
\mathrm{ind}_L^H & : X\Omega(L,G^c) \longrightarrow X\Omega(H,G^c) & : (L/D)_s \longmapsto (H/D)_s, \\
\mathrm{res}_L^H & : X\Omega(H,G^c) \longrightarrow X\Omega(L,G^c) & : (H/D)_s \longmapsto \displaystyle\sum_{g\in[L\backslash H/D]} (L/L\cap {}^gD)_{{}^gs}, \\
\mathrm{con}_{H,g} & : X\Omega(H,G^c) \longrightarrow X\Omega({}^gH,G^c) & : (H/D)_s \longmapsto ({}^gH/{}^gD)_{{}^gs},
\end{array}$$

where $D \leq L \leq H \leq G$ and $g \in G$. In order to note the Green functor structure of $X\Omega(*,G^c)$, we shall discuss briefly an equivalence between the category $G\text{-}\mathbf{set}{\downarrow}_{(G/H\times G^c)}$ of finite G-sets over the G-set $G/H \times G^c$ (see 2.4 of [Bo97]) and the category $H\text{-}\mathbf{set}{\downarrow}_{G^c}$ of finite H-sets over the H-set G^c with the H-action defined by conjugation. Let Ω be the Burnside Green functor for G over $\mathbb{Z}$ in terms of G-sets. By Proposition 2.4.2 of [Bo97], $\Omega_{G^c}(G/H) = \Omega((G/H)\times G^c)$ is isomorphic to the Grothendieck group of $G\text{-}\mathbf{set}{\downarrow}_{(G/H\times G^c)}$, with relations given by decomposition into disjoint union. It is easy to see that the G-sets

$$[K,s] : G/K \to G/H \times G^c : gK \mapsto (gH, {}^gs)$$

over $G/H \times G^c$, for $K \in [H\backslash S(H)]$ and $s \in [H\backslash C_G(K)]$, form a basis of $\Omega(G/H \times G^c)$ over $\mathbb{Z}$. We denote by $(G/K,[K,s])$ an element of the basis of $\Omega(G/H \times G^c)$. Theorem 5.1 of [Bo03a] shows that Ω_{G^c} is a Green functor. If $(G/K,[K,s])$ and $(G/L,[L,t])$ are elements of the basis of $\Omega(G/H \times G^c)$, then we have the following commutative diagram

$$\begin{array}{ccccc}
\displaystyle\bigsqcup_{x\in K\backslash H/L} G/K\cap {}^xL & \xrightarrow{\sqcup_x \pi^K_{K\cap^x L}\times(\pi^L_{K^x\cap L}c_{K\cap^x L,x})} & G/K\times G/L & \xrightarrow{[K,s]\times[L,t]} & (G/H\times G^c)\times(G/H\times G^c) \\
\Big\downarrow {\scriptstyle \sqcup_x[K\cap^x L,s\cdot^x t]} & P.B. & \Big\downarrow & & \Big\downarrow {\scriptstyle f} \\
G/H\times G^c & \xrightarrow{\delta_{G/H}\times Id_{G^c}} & G/H\times G/H\times G^c & \xrightarrow{id} & G/H\times G/H\times G^c,
\end{array}$$

where the map f from $G/H \times G^c \times G/H \times G^c$ to $G/H \times G/H \times G^c$ maps $(xK,\gamma_1,yL,\gamma_2)$ to $(xK,yL,\gamma_1\gamma_2)$ (see section 5 of [Bo03a]). The left square is a pullback square. Theorem 5.1 of [Bo03a] shows that the product of $(G/K,[K,s])$ and $(G/L,[L,t])$ on $\Omega(G/H \times G^c)$ is given by

$$(G/K,[K,s])\cdot(G/L,[L,t]) = \sum_{x\in[K\backslash H/L]} (G/K\cap {}^xL,[K\cap {}^xL, s\cdot {}^xt]). \qquad (3.1.1)$$

We have a functor F mapping $G\text{-}\mathbf{set}{\downarrow}_{(G/H\times G^c)}$ to $H\text{-}\mathbf{set}{\downarrow}_{G^c}$ defined for a transitive G-set $[K,s] : G/K \to G/H \times G^c$ over $G/H \times G^c$ by

$$F : (G/K,[K,s]) \mapsto \langle K,s\rangle : [K,s]^{-1}(\{H\}\times G^c) \to G^c$$

as in Lemma 2.4.1 of [Bo97]. We also denote by $[K,s]$ the H-map $H/K \to G^c$ defined by $gK \mapsto {}^gs$. It is clear that the H-sets

$$[K,s] : H/K \to G^c : gK \mapsto {}^gs$$

over the H-set G^c, for $K \in [H\backslash S(H)]$ and $s \in [H\backslash C_G(K)]$, form a basis of $\Omega{\downarrow}_H^G(G^c)$ over $\mathbb{Z}$, where $\Omega{\downarrow}_H^G$ is a Green functor for H given by the restricion to H of G. We denote by $(H/K,[K,s])$ this element of the basis of $\Omega{\downarrow}_H^G(G^c)$. It is easy to see that F gives an equivalence of categories from $G\text{-}\mathbf{set}{\downarrow}_{(G/H\times G^c)}$ to $H\text{-}\mathbf{set}{\downarrow}_{G^c}$ for any subgroup H of G. The inverse equivalence is given by the induction functor from H-$\mathbf{set}\downarrow_{G^c}$ to G-$\mathbf{set}\downarrow_{(G/H\times G^c)}$. The images of (3.1.1) under F are

$$(H/K,[K,s])\cdot(H/L,[L,t]) = \sum_{x\in[K\backslash H/L]} (H/K\cap {}^xL,[K\cap {}^xL, s\cdot {}^xt]).$$

in H-$\mathbf{set}\downarrow_{G^c}$. The Grothendieck group of H-$\mathbf{set}\downarrow_{G^c}$ is isomorphic to $X\Omega(H,G^c)$. We can define a product

$$(H/K)_s\cdot(H/L)_t = \sum_{x\in[K\backslash H/L]} (H/K\cap {}^xL)_{s\cdot {}^xt}$$

for any two elements $(H/K)_s$ and $(H/L)_t$ of the basis of $X\Omega(H,G^c)$. It is clear that the element $(H/H)_{1_G}$ for the identity element 1_G of G is the identity element of $X\Omega(H,G^c)$. This gives a unitary ring structure to $X\Omega(H,G^c)$ for a subgroup H of G.

(3.2) Witherspoon's Green functor. Witherspoon introduced a Green functor $R_{\mathbb{C}}(D_G(*))$ for G over $\mathbb{Z}$ (see [Wi96] Section 5). For each subgroup H of G, there is a subalgebra

$$D_G(H) = \sum_{g\in G,\, h\in H} \mathbb{C}\phi_g h$$

of the Drinfel'd (quantum) double $D(G)$ of $\mathbb{C}G$ [Dr86], where ϕ_g is an element of the basis $\{\phi_g\}_{g\in G}$ of the dual space $(\mathbb{C}G)^* = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}G,\mathbb{C})$. Note that $D_G(G) = D(G)$ and $R(D(G))$ is the representation ring of $D(G)$ or equivalently the Grothendieck ring of Hopf bimodules for the Hopf algebra $\mathbb{C}G$ ([Ro95], [Bo03a], [OY04]). Let $R_{\mathbb{C}}(D_G(H))$ be the Grothendieck (representation) ring of $D_G(H)$ for subgroup H of G. Then the assignment

$$H \longmapsto R_{\mathbb{C}}(D_G(H))$$

where $H\le G$ gives a Green functor for G over $\mathbb{Z}$ with operations given by

$$\begin{array}{lllllll}
\mathrm{Dres}^H_L & : & R_{\mathbb{C}}(D_G(H)) & \longrightarrow & R_{\mathbb{C}}(D_G(L)) & : & U \longmapsto U\downarrow_{D_G(L)},\\
\mathrm{Dind}^H_L & : & R_{\mathbb{C}}(D_G(L)) & \longrightarrow & R_{\mathbb{C}}(D_G(H)) & : & V \longmapsto D_G(H)\otimes_{D_G(L)} V,\\
\mathrm{Dconj}_{H,g} & : & R_{\mathbb{C}}(D_G(H)) & \longrightarrow & R_{\mathbb{C}}(D_G({}^gH)) & : & U \longmapsto {}^gU = gD_G(H)\otimes_{D_G(H)} U,
\end{array}$$

where $U\downarrow_{D_G(L)}$ is a $D_G(L)$-module by restriction of the action from $D_G(H)$ to $D_G(L)$, $L\le H\le G$ and $g\in G$. We use the equivalence of the category of H-vector bundles on G^c with the category of $D_G(H)$-modules (see [Wi96] Section 2).

(3.3) A morphism of Green functors. Let Ω be the Burnside ring Green functor for G over $\mathbb{Z}$ (see [Bo97] 2.4.2):

- If X is a finite G-set, then $\Omega(X)$ is the Grothendieck ring of the category of finite G-sets over X, where the relations are given by decomposition into disjoint union and product of G-sets.
- If $X\to X'$ is a G-map, then $\Omega_*(f):\Omega(X)\to\Omega(X')$ is defined by $\Omega_*(f)((Y,\phi)) = (Y,f\phi)$ for any G-set $(Y,\phi) = Y\xrightarrow{\phi} X$ over X.

- If $X' \to X$ is a G-map, then $\Omega^*(f) : \Omega(X) \to \Omega(X')$ is defined by $\Omega^*(f)((Y,\phi)) = (Y',\phi')$, where (Y',ϕ') is the pull-back of (Y,ϕ) along f, obtained by filling the cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{a} & Y \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi} \\ X' & \xrightarrow[f]{} & X. \end{array}$$

Suppose that $R_{\mathbb{C}}$ is the $\mathbb{C}$-representation (character ring) Green functor for G over $\mathbb{Z}$, defined on subgroups of G. Then setting $R_{\mathbb{C}}(G/H) = R_{\mathbb{C}}(H)$ leads by linearity to a definition of G-equivariant $\mathbb{C}$-vector bundles $R_{\mathbb{C}}(X)$ on a G-set X (see [Wi96] Section 2) by using Remark 2.3 of [Bo03a]:

- If X is a finite G-set, then $R_{\mathbb{C}}(X)$ is the Grothendieck ring of the category of G-equivariant $\mathbb{C}$-vector bundles on the G-set X, for relations given by decomposition into direct sum of vector bundles, the ring structure being induced by the tensor product of vector bundles: one can set

$$R_{\mathbb{C}}(X) = \left(\bigoplus_{x\in X} R_{\mathbb{C}}(G_x)\right)^G$$

  where the exponent denotes fixed points under the natural action of G on $\oplus_{x\in X} R_{\mathbb{C}}(G_x)$ by permutation of the components, and G_x is the stabilizer of x in G.

- If $f : X \to X'$ is a G-map, then $R_{\mathbb{C}*}(f) : R_{\mathbb{C}}(X) \to R_{\mathbb{C}}(X')$ is defined by

$$R_{\mathbb{C}*}(f)(u)_y = \sum_{x\in[G_y\backslash f^{-1}(y)]} t_{G_x}^{G_y}(u_x),$$

  where $t_{G_x}^{G_y}$ is the induction map from $R_{\mathbb{C}}(G_x)$ to $R_{\mathbb{C}}(G_y)$, $u \in R_{\mathbb{C}}(X)$, and $y \in X'$.

- If $f : X' \to X$ is a G-map, then $R_{\mathbb{C}}^*(f) : R_{\mathbb{C}}(X) \to R_{\mathbb{C}}(X')$ is defined by

$$R_{\mathbb{C}}^*(f)(v)_x = r_{G_{f(x)}}^{G_x}(v_{f(x)}),$$

  where $r_{G_{f(x)}}^{G_x}$ is the restriction map from $R_{\mathbb{C}}(G_x)$ to $R_{\mathbb{C}}(G_{f(x)})$, $v \in R_{\mathbb{C}}(X)$, and $x \in X'$.

- The product of the elements $a \in R_{\mathbb{C}}(X)$ and $b \in R_{\mathbb{C}}(Y)$ is defined by

$$(a \times b)_{x,y} = r_{G_{(x,y)}}^{G_x}(a_x) \cdot r_{G_{(x,y)}}^{G_y}(b_y).$$

If X is a finite G-set, denote the natural morphism

$$\theta : \Omega \to R_{\mathbb{C}}$$

of Green functors defined by the maps $\theta(X) : \Omega(X) \to R_{\mathbb{C}}(X)$ by

$$(Y, \varphi) = (\varphi : Y \to X) \longmapsto \{\mathbb{C}[\varphi^{-1}(x)]\}_{x \in X},$$

where $\mathbb{C}[\varphi^{-1}(x)]$ is the permutation module associated to the G_x-set $\varphi^{-1}(x)$.

The following theorem is essential in the proof of Theorem 4.1 of this paper.

(3.4) Theorem (Bouc [Bo03a] 5.1). *Let A be a Green functor for G over a commutative ring $\mathcal{O}$, Γ a crossed G-monoid, and ε_A an element of $A(\bullet)$ such that for any G-set X and for any $a \in A(X)$*

$$A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a)$$

denoting by p_x (resp. q_x) the bijective projection from $X \times \bullet$ (resp. from $\bullet \times X$) to X (see 1.2.1 of [Bo03a]). Then the functor A_Γ is a Green functor for G over $\mathcal{O}$, with unit ε_{A_Γ}, where ε_{A_Γ} is the element $A_*\left(\begin{smallmatrix}\bullet\\ \downarrow\\ 1_G\end{smallmatrix}\right)(\varepsilon_A)$ of $A(\Gamma) = A_\Gamma(\bullet)$. Moreover the correspondence $A \mapsto A_\Gamma$ is an endo-functor of the category of Green functors for G over $\mathcal{O}$.

(3.5) Lemma. *Let Ω be the Burnside ring Green functor and G^c the crossed G-monoid. Then there is an isomorphism of Green functors*

$$X\Omega(*, G^c) \cong \Omega_{G^c}.$$

We will denote by $\mathbb{C}[X]$ the $\mathbb{C}$-permutation module associated to a set X. The endo-functor of the category of Green functors of Theorem (3.4) applied to the morphism θ from Ω to $R_{\mathbb{C}}$ leads to the following lemma.

(3.6) Lemma. *Let $\theta : \Omega \to R_{\mathbb{C}}$ be the natural morphism from the Burnside Green functor to the Grothendieck ring Green functor. Then the morphism $\theta_{G^c} : \Omega_{G^c} \to R_{\mathbb{C}G^c}$ given by the Bouc's construction is a morphism of Green functors.*

(3.7) Lemma. *There is a morphism*

$$\theta_{G^c} : X\Omega(*, G^c) \to R_{\mathbb{C}}(D_G(*))$$

of Green functors.

Let $(H/L)_g$ be an element of the basis of $X\Omega(H, G^c)$. Then the previous lemma shows that $\theta_{G^c}((H/L)_g)$ is an H-vector bundle on G^c. We denote by $[H/L]_g$ this H-vector bundle. Lemma (3.5) shows the following lemma.

(3.8) Lemma. *The H-vector bundle $[H/L]_g$ is the $\mathbb{C}C_H({}^xg)$-module $\mathbb{C}[[{}^xL, {}^xg]^{-1}({}^xg)]$ in the ${}^xg$-component, for $x \in [H/C_H(g)]$, and 0 in all other components.*

We recall the maps $\mathrm{Incl}_{J,h} : R_{\mathbb{C}}(J) \to R_{\mathbb{C}}(D_G(J))$, where J is a subgroup of G and $h \in C_G(J)$, introduced in Section 2 of [Wi96]: Given a $\mathbb{C}J$-module V, $\mathrm{Incl}_{J,h}(V)$ is the $D_G(J)$-module which is V in the h-component and 0 elsewhere.

(3.9) Lemma. *Let θ_{G^c} be the ring homomorphism $\theta_{(G/G)\times G^c}$ from the crossed Burnside ring $X\Omega(G, G^c)$ to the Grothendieck ring $R_{\mathbb{C}}(D_G(G))$ given by the previous lemma. Then the $D(G)$-module corresponding to the G-vector bundle $\theta_{G^c}((G/L)_g)$ is the induced module*

$$D(G) \otimes_{D_G(L)} \mathrm{Incl}_{L,g}(\mathbb{C}[L/L]).$$

(3.10) Sub-Green functors. There is a sub-Green functor $X\Omega(*, G^c)_1$ which assigns to each subgroup H of G the subring $X\Omega(H, G^c)_1$ of $X\Omega(H, G^c)$ generated by the elements $(H/L)_{1_G}$. There is also a sub-Green functor $R_{\mathbb{C}}(D_G(*)_1)$ which assigns to each subgroup H of G the subring $R_{\mathbb{C}}(D_G(H)_1)$ of $R_{\mathbb{C}}(D_G(H))$ generated by $\mathrm{Incl}_{H,1_G}(V)'s$, where $\mathrm{Incl}_{H,1_G}$ is a functor embedding the category of $\mathbb{C}H$-modules as a full subcategory of the category of $D_G(H)$-modules (see, [Wi96] Section 1) and V is a $\mathbb{C}H$-module. It is easy to see that $X\Omega(H, G^c)_1$ is isomorphic to the Burnside ring $\Omega(H)$ and $R_{\mathbb{C}}(D_G(H)_1)$ is isomorphic to the ordinary character ring $R_{\mathbb{C}}(H)$. The homomorphism $\theta_{G^c}\downarrow_{X\Omega(H,G^c)_1}$ is the natural ring homomorphism from $\Omega(H)$ to $R_{\mathbb{C}}(H)$.

(3.11) Characters. Witherspoon pointed out the character of a $\mathbb{C}D(G)$-module in [Wi96], that appeared in [Lu87]. For $g \in G$ and an irreducible character ρ of $C_G(g)$, a character $\chi_{g,\rho}$ of a $\mathbb{C}D(G)$-module $U = \{U_h\}_{h\in G^c}$ is given by the formula

$$\chi_{g,\rho}(U) = \frac{1}{\deg\rho} \sum_{h\in C_G(g)} \mathrm{Tr}(g, U_h)\rho(h).$$

The characters of the crossed Burnside ring have been considered by Oda and Yoshida ([OY01], Section 5). For a subgroup H of G and an irreducible character θ of $C_G(H)$, the linear map $\omega_{H,\theta}$ of $X\Omega(G, G^c)$ to $\mathbb{C}$ is the composite of Burnside homomorphism φ_H and a central character $\tilde{\omega}_{H,\theta}$: given a crossed G-set X over G^c, $H \le G$, and an irreducible character ρ of the group algebra $\mathbb{C}C_G(H)$, $\omega_{H,\rho}(X) = \tilde{\omega}_{H,\rho} \circ \varphi_H(X)$.

For each $h \in G^c$ the h-component of the crossed G-set (X, α) is defined by

$$X[h] = \{x \in X | \alpha(x) = h\}.$$

(3.12) Lemma. *Let g be an element of G, ρ an irreducible character of $\mathbb{C}C_G(g)$, and θ_{G^c} the homomorphism from $X\Omega(G, G^c)$ to $R_{\mathbb{C}}(D(G))$. Then $\chi_{g,\rho}\theta_{G^c} = \omega_{\langle g\rangle,\rho}$, where $\langle g\rangle$ is the cyclic subgroup generated by g.*

4 Induction theorems

The proof of the following theorem is similar to the proof of Theorem 3.5.2 in [Bo00].

(4.1) Theorem. *Let G be a finite group. Then*

$$\mathbb{C}R_{\mathbb{C}}(D(G)) = \sum_{H\in C(G)} \mathrm{Dind}_H^G \mathbb{C}R_{\mathbb{C}}(D_G(H)),$$

where $C(G)$ is the family of cyclic subgroups of G. In other words, any complex character of $D(G)$ is a linear combination with rational coefficients of characters induced from cyclic groups of G.

The previous theorem and (3.10) show the following corollary.

(4.2) Corollary (Artin). *Let G be a finite group. Then*

$$\mathbb{Q}R_{\mathbb{C}}(G) = \sum_{H \in C(G)} \mathrm{Ind}_H^G \mathbb{Q}R_{\mathbb{C}}(H).$$

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