<table>
<thead>
<tr>
<th>Title</th>
<th>SURFACE SYMMETRIES, HOMOLOGY REPRESENTATIONS, AND GROUP COHOMOLOGY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>AKITA, TOSHIYUKI</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2008(1581): 103-108</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/81425">http://hdl.handle.net/2433/81425</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
SURFACE SYMMETRIES, HOMOLOGY REPRESENTATIONS, AND GROUP COHOMOLOGY

TOSHIYUKI AKITA

Given a finite group $G$ of automorphisms of a compact Riemann surface, we discuss a relation between Mumford-Morita-Miller classes of odd indices and the homology representation of $G$. Since most participants were group theorists rather than topologists, I separate the algebraic and the topological ingredients and explain the former in detail.

1. SURFACE SYMMETRIES

1.1. The Grieder group of a finite group. Let $G$ be a finite group and $\gamma$ the conjugacy class of $\gamma \in G$. We denote by $\langle \gamma_1, \ldots, \gamma_q \rangle$ an unordered $q$-tuple ($q \geq 0$) of conjugacy classes of nontrivial elements of $G$ satisfying $\gamma_1 \gamma_2 \cdots \gamma_q \in [G, G]$, and $M_G$ the set of all such $q$-tuples. We can define an abelian monoid structure on $M_G$ by

$$\langle \gamma_1, \ldots, \gamma_q \rangle + \langle \gamma_{q+1}, \ldots, \gamma_r \rangle = \langle \gamma_1, \ldots, \gamma_q, \gamma_{q+1}, \ldots, \gamma_r \rangle.$$ 

The identity element is the empty tuple $\langle \rangle$. We call $M_G$ the Grieder monoid of $G$. Now let $M'_G$ be the submonoid generated by $\langle \gamma, \gamma^{-1} \rangle$ ($\gamma \in G$) and set $A_G := M_G / M'_G$. The quotient $A_G$ is an abelian group. The inverse element is given by

$$-\langle \gamma_1, \ldots, \gamma_q \rangle = \langle \gamma_1^{-1}, \ldots, \gamma_q^{-1} \rangle \text{ in } A_G.$$ 

We call $A_G$ the Grieder group of $G$. As the names suggest, $M_G$ and $A_G$ were introduced and studied by Grieder [5, 6] to study surface symmetries. First of all, $A_G$ is finitely generated:

Proposition 1 ([5]). $A_G \cong \mathbb{Z}^m \oplus \mathbb{Z}_2^n$ for some $m, n \geq 0$.

A homomorphism $f : H \to G$ of groups induces a homomorphism $f_* : A_H \to A_G$ of abelian groups by $f_* \langle \gamma_1, \ldots, \gamma_q \rangle = \langle f(\gamma_1), \ldots, f(\gamma_q) \rangle$ so that the assignment $G \mapsto A_G$ is a covariant functor. In addition, for an inclusion $i : H \hookrightarrow G$, one can also define the restriction $i^* : A_G \to A_H$ via surface symmetries. Grieder [5] verified the double coset formula and hence proved the following proposition:
Proposition 2. The assignment $G \mapsto \mathcal{M}_G$ is a Mackey functor.

1.2. Ramification data. By a surface symmetry we mean a pair $(G, C)$, where $C$ is a compact Riemann surface of genus $g \geq 2$, and $G$ is a finite group of automorphisms of $C$. For each $x \in C$, let $G_x$ be the isotropy subgroup at $x$. Note that $G_x$ is necessary cyclic. Set $\mathcal{S} = \{x \in C \mid G_x \neq 1\}$, and let $\mathcal{S}/G = \{x_1, x_2, \ldots, x_q\}$ be a set of representatives of $G$-orbits of elements of $\mathcal{S}$. For each $x_i \in \mathcal{S}/G$, choose a generator $\gamma_i$ of $G_{x_i}$ such that $\gamma_i$ acts on the holomorphic tangent space $T_{x_i}C$ by $z \mapsto \exp(2\pi \sqrt{-1}/|G_{x_i}|)z$ with respect to a suitable local coordinate $z$ at $x_i$. The ramification data of $(G, C)$, abbreviated by $\delta(G, C)$, is the unordered $q$-tuple $\langle \gamma_1, \gamma_2, \ldots, \gamma_q \rangle$. It satisfies $\gamma_1 \gamma_2 \cdots \gamma_q \in [G, G]$, and hence $\delta(G, C)$ is an element of the Grieder monoid $\mathcal{M}_G$. Conversely, we have the following proposition.

Proposition 3 (see [5]). For any element $\mu \in \mathcal{M}_G$, there exists a surface symmetry $(G, C)$ whose ramification data coincides with $\mu$.

2. GROUP COHOMOLOGY

2.1. The first Chern class. Let $\langle \gamma \rangle$ be a cyclic group of order $m$ generated by $\gamma$ and $\rho_\gamma : \langle \gamma \rangle \to \mathbb{C}^\times$ a linear character defined by $\gamma \mapsto \exp(2\pi i/m)$. For any finite group $G$, we have natural isomorphisms

$$\text{Hom}(G, \mathbb{C}^\times) \cong H^1(G, \mathbb{C}^\times) \cong H^2(G, \mathbb{Z}).$$

Here, the latter isomorphism is the connecting homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$. Define $c(\gamma) \in H^2(\langle \gamma \rangle, \mathbb{Z})$ to be the image of $\rho_\gamma$ under the isomorphism $\text{Hom}(\langle \gamma \rangle, \mathbb{C}^\times) \cong H^2(\langle \gamma \rangle, \mathbb{Z})$. The cohomology class $c(\gamma)$ is sometimes called the first Chern class of $\rho_\gamma$.

2.2. MMM classes (algebra). For each element $\mu = \langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle$ of $\mathcal{M}_G$, define a series of cohomology classes $e_k(\mu) \in H^{2k}(G, \mathbb{Z})$ ($k \geq 1$) by

$$e_k(\mu) := \sum_{i=1}^q \text{Tr}_{\langle \gamma_i \rangle}^G(c(\gamma_i)^k) \in H^{2k}(G, \mathbb{Z}),$$

where $\text{Tr}_{\langle \gamma \rangle}^G : H^*(\langle \gamma \rangle, \mathbb{Z}) \to H^*(G, \mathbb{Z})$ is the transfer. We call $e_k(\mu)$ the $k$-th Mumford-Morita-Miller class of $\mu$ (MMM class in short). The definition of $e_k(\mu)$ is motivated by topology, as will be explained in the next subsection. Observe that the assignment $\mu \mapsto e_k(\mu)$ defines a well-defined homomorphism $\mathcal{M}_G \to H^{2k}(G, \mathbb{Z})$ of abelian monoids. For $k$ odd, it induces a well-defined homomorphism $\mathcal{A}_G \to H^{2k}(G, \mathbb{Z})$ of abelian groups, for we have $c(\gamma^{-1}) = -c(\gamma)$. In addition, we can prove the following proposition:
Proposition 4. For odd $k \geq 1$, the homomorphism $\mathcal{A}_G \to H^{2k}(G, \mathbb{Z})$ is a natural transformation of Mackey functors.

2.3. MMM classes (topology). The definition of $e_k(\mu)$ is inspired by a result of Kawazumi and Uemura [8] concerning of characteristic classes of oriented surface bundles. Let $\Sigma_g$ be the closed oriented surface of genus $g \geq 2$. Let $\pi : E \to B$ an oriented $\Sigma_g$-bundle, $T^vE$ the tangent bundle along the fiber of $\pi$, and $e \in H^2(E; \mathbb{Z})$ the Euler class of $T^vE$. Define $e_k^{\text{top}}(\pi) \in H^{2k}(B; \mathbb{Z})$ by $e_k^{\text{top}}(\pi) := \pi_!(e^{k+1})$ where $\pi_! : H^*(E; \mathbb{Z}) \to H^{*-2}(B; \mathbb{Z})$ is the Gysin homomorphism (the superscript "top" stands for "topology"). $e_k^{\text{top}}(\pi)$ is called the $k$-th Mumford-Morita-Miller class of $\pi$, as it was introduced in [11, 10, 9].

Now let $(G, C)$ be a surface symmetry as in Section 1.2. Associated with $(G, C)$, there is an oriented surface bundle $\pi : EG \times_G C \to BG$ called the Borel construction, where $EG \to BG$ is the universal $G$-bundle. We denote by $e_k^{\text{top}}(G, C) \in H^{2k}(G, \mathbb{Z})$ the $k$-th MMM class of the Borel construction $\pi$. A result of Kawazumi and Uemura [8] implies the following result:

Theorem 5. We have $e_k^{\text{top}}(G, C) = e_k(\delta(G, C))$ where $\delta(G, C)$ is the ramification data of $(G, C)$.

3. Homology representations

3.1. Algebra. In what follows, we denote by $R(G)$ the complex representation ring (or the character ring) of a finite group $G$. Let $\langle \gamma \rangle$ be a cyclic group of order $m$ generated by $\gamma$ and $\rho_\gamma : \langle \gamma \rangle \to \mathbb{C}^\times$ a linear character as in Section 2.1. Define $\Delta_\gamma \in R(\langle \gamma \rangle) \otimes \mathbb{Q}$ by

$$\Delta_\gamma := 2 \sum_{k=1}^{m-1} \left( \frac{k}{m} - \frac{1}{2} \right) \rho_\gamma^\otimes k = \frac{2}{m} \sum_{k=1}^{m-1} k \rho_\gamma^\otimes k - r_\langle \gamma \rangle + 1_\langle \gamma \rangle,$$

where $r_\langle \gamma \rangle$ is the regular representation and $1_\langle \gamma \rangle$ is the trivial 1-dimensional representation of $\langle \gamma \rangle$. Now, for each element $\mu = \langle \gamma_1, \ldots, \gamma_q \rangle$ of $\mathcal{M}_G$, define the $G$-signature $\sigma(\mu)$ of $\mu$ by

$$\sigma(\mu) := \sum_{k=1}^{q} \text{Ind}_{\langle \gamma_k \rangle}^G(\Delta_\gamma) \in R(G) \otimes \mathbb{Q}.$$

Proposition 6. $\sigma(\mu) \in R(G)$ for every $G$ and $\mu \in \mathcal{M}_G$.

See the next section for the proof. Note that, in case $\mu \in \mathcal{M}_G$ consists of a single conjugacy class ($\mu = \langle \gamma \rangle$ for $\gamma \in [G, G]$), Proposition 6 was proved by T. Yoshida [13].
The assignment \( \mu \mapsto \sigma(\mu) \) yields a homomorphism \( \mathcal{M}_G \to R(G) \) of monoids, which induces a well-defined homomorphism \( \mathcal{A}_G \to R(G) \) of abelian groups. In addition, we can prove the following proposition:

**Proposition 7.** \( \mathcal{A}_G \to R(G) \) is a natural transformation of Mackey functors.

### 3.2. Topology.

Let \((G, C)\) be a surface symmetry, and \(H_C\) the space of holomorphic 1-forms on \(C\). Note that \(\dim_C H_C = g\) where \(g\) is the genus of the Riemann surface \(C\). Then \(G\) acts on \(H_C\) and hence \(H_C\) is a complex representation of \(G\). A virtual representation \(\sigma^{\text{top}}(G, C) := H_C - \overline{H}_C \in R(G)\) is called the \(G\)-signature of \((G, C)\), where \(\overline{H}_C\) is the complex conjugate.

**Proposition 8.** We have \(\sigma^{\text{top}}(G, C) = \sigma(\delta(G, C))\) where \(\delta(G, C)\) is the ramification data of \((G, C)\).

The character of \(\sigma^{\text{top}}(G, C)\) is given by the Eichler trace formula (see [4] for instance). The proposition can be verified by comparing characters of \(\sigma^{\text{top}}(G, C)\) and \(\sigma(\delta(G, C))\). An alternative proof was given by N. Kawazumi (unpublished manuscript). Since every \(\mu \in \mathcal{M}_G\) can be realized as a ramification data of a surface symmetry, Proposition 6 follows from the last proposition. The following fact is an easy consequence of Proposition 8.

**Corollary 9.** If all the complex characters of \(G\) are \(\mathbb{R}\)-valued, then \(\sigma(\mu) = 0\) for all \(\mu \in \mathcal{M}_G\).

**Proof.** Choose a surface symmetry \((G, C)\) with \(\delta(G, C) = \mu\). Then we have \(\sigma(\mu) = \sigma^{\text{top}}(G, C) = H_C - \overline{H}_C = 0\) since \(H_C = \overline{H}_C\) by the assumption. \(\square\)

### 4. A RELATION OF \(e_k(\mu)\) AND \(\sigma(\mu)\)

**Theorem 10.** Let \(G\) be a finite group and \(\mu, \nu \in \mathcal{M}_G\).

1. If \(\sigma(\mu) = \sigma(\nu)\) then \(e_k(\mu) = e_k(\nu)\) for all odd \(k \geq 1\).
2. If \(\sigma(\mu) = 0\) then \(e_k(\mu) = 0\) for all odd \(k \geq 1\).

Since \(R(G)\) is free as an abelian group, the homomorphism \(\mathcal{A}_G \to R(G)\) in Section 3.1 induces \(\phi_1 : \mathcal{A}_G/\text{Tor}(\mathcal{A}_G) \to R(G)\), where \(\text{Tor}(\mathcal{A}_G)\) is the torsion subgroup of \(\mathcal{A}_G\). For odd \(k \geq 1\), let \(\phi_2 : \text{Tor}(\mathcal{A}_G) \to \mathbb{H}^{2k}(G, \mathbb{Z})\) be the restriction of the homomorphism \(\mathcal{A}_G \to \mathbb{H}^{2k}(G, \mathbb{Z})\) in Section 2.2. The proof of Theorem 10 is based on the following two facts:

**Theorem 11.** For any finite group \(G\) and any odd \(k \geq 1\),

1. The homomorphism \(\phi_1 : \mathcal{A}_G/\text{Tor}(\mathcal{A}_G) \to R(G)\) is injective.
2. The homomorphism \(\phi_2 : \text{Tor}(\mathcal{A}_G) \to \mathbb{H}^{2k}(G, \mathbb{Z})\) is trivial.
The first statement is proved by using a result of Edmonds and Ewing [3], while the second statement is proved by considering the cohomology of metacyclic 2-groups. The detail will appear elsewhere. Theorem 10 and Corollary 9 imply the following corollary:

**Corollary 12.** If all the complex characters of $G$ are $\mathbb{R}$-valued, then $e_k(\mu) = 0$ for all $\mu \in M_G$ and odd $k \geq 1$.

Define $\mathcal{R}_G$ to be the image of $\phi_1 : \mathcal{A}_G / \text{Tor}(\mathcal{A}_G) \to R(G)$. In view of Theorem 11, there exists a series of homomorphisms $\Phi_k : \mathcal{R}_G \to H^{2k}(G, \mathbb{Z})$ ($k$ odd) which assigns $e_k(\mu)$ to $\sigma(\mu)$. Let $c : \text{Hom}(G, \mathbb{C}^\times) \to H^2(G, \mathbb{Z})$ be the natural isomorphism as in Section 2.1 and $\text{det} : R(G) \to \text{Hom}(G, \mathbb{C}^\times)$ the determinant homomorphism (see [13] for precise). Then the homomorphism $\Phi_1$ is determined by the following proposition:

**Proposition 13.** $e_1(\mu) = 6 \cdot c(\det(\sigma(\mu)))$ for all $\mu \in M_G$.

The proposition follows from the Grothendieck-Riemann-Roch theorem and a result of Harer [7] (see also [1, Proposition 6]). Proposition 13 can be generalized to larger $k$, provided $G$ is cyclic. Recall that, for every finite group $G$, there is a series of homomorphisms $s_k : R(G) \to H^{2k}(G, \mathbb{Z})$ ($k \geq 0$) of abelian groups, which satisfies the following properties:

1. $s_1(p) = c(\det p)$ for all $p \in R(G)$.
2. If $p$ is a linear character, then $s_k(p) = c(p)^k$.

$s_k(p)$ is called the $k$-th Newton class of $p \in R(G)$. See [12] for further details. Let $B_{2k}$ be the $2k$-th Bernoulli number and $N_{2k}, D_{2k}$ coprime integers satisfying $B_{2k}/k = N_k/D_k$. Then a result of the author and Kawazumi [2] implies the following result:

**Theorem 14.** If $G$ is cyclic, then $N_{2k} \cdot e_{2k-1}(\mu) = D_{2k} \cdot s_{2k-1}(\sigma(\mu))$ holds for all $\mu \in M_G$ and $k \geq 1$.

Now let $G$ be a cyclic group of order $m$, and suppose that $N_{2k}$ is prime to $m$. Choose an integer $N_{2k}^*$ satisfying $N_{2k} \cdot N_{2k}^* \equiv 1 \pmod{m}$. Under these assumptions, we have

$$e_{2k-1}(\mu) = N_{2k}^* D_{2k} \cdot s_{2k-1}(\sigma(\mu))$$

for all $\mu \in M_G$, and hence determining $\Phi_{2k-1}$ for these cases. In particular, we have $e_1(\mu) = 6 \cdot s_1(\sigma(\mu))$, $e_3(\mu) = -60 \cdot s_3(\sigma(\mu))$, $e_5(\mu) = 126 \cdot s_5(\sigma(\mu))$, $e_7(\mu) = -120 \cdot s_7(\sigma(\mu))$ for any cyclic group $G$ and $\mu \in M_G$, since $N_{2k} = 1$ for $1 \leq k \leq 4$. 


REFERENCES


DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO, 060-0810 JAPAN

E-mail address: akita@math.sci.hokudai.ac.jp