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β-FILTRATION OF COHOMOLOGY OF GROUPS

NOBUAKI YAGITA
柳田伸顕 (茨城大学教育学部)

1. INTRODUCTION

Let G be a compact Lie group, (e.g., finite group). The structure of $H^*(BG;\mathbb{Z}/p)$ seems very complicated, in general. Most cases are difficult to write down it, even if it can be computable. So it may be reasonable to consider an adequate filtration and the associated graded algebra $grH^*(BG;\mathbb{Z}/p)$.

In this paper, we study natural filtrations such that

$$P^j(F_i) \subset F_i \quad and \quad \beta(F_i) \subset F_{i-1}$$

where P^j is the reduced power operation and β is the Bockstein operation. (We say such filtrations to be a β-filtration. For the precise definition, see §2 bellow.) The β-filtration seems not unique. However, we see it is determined uniquely for the cases $G=(\mathbb{Z}/p)^n, O_3, PGL_3, p_+^{1+2}$. (However the ring structures of $grH^*(BG;\mathbb{Z}/p)$ do not become more simple.)

An example of β-filtrations is given in §6 by using the motivic cohomology $H^{*,*'}(BG;\mathbb{Z}/p)$ (over $k=\mathbb{C}$). In fact we define the (motivic) filtration such that $x\in F_i$ if

$$x \in \oplus_{i=2*'-*} Im(t_{\mathbb{C}} : H^{*,*'}(BG;\mathbb{Z}/p) \to H^*(BG;\mathbb{Z}/p))$$

where $t_{\mathbb{C}}$ is the realization map. Moreover, we see that this filtration coincides the coniveau filtration defined by Grothendieck [Bl-Og] for cases $(\mathbb{Z}/p)^n, O_n, PGL_p$.

2. β-FILTRATION

Let p be a prime number. Let G be a compact Lie group (e.g., finite group) and BG be their classifying spaces.

Definition. We say that a filtration

$$(0)=F_{-1}\subset F_0\subset F_1\subset ... \subset F_\infty = H^*(BG;\mathbb{Z}/p)$$

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is a β-filtration if it satisfies the following (1) – (5) ;

(1) It is natural for the induced map and the transfer. That is, for $f : G \to G'$ and an injection $g : G \to G'$ of finite cokernel, let F' be the filtration of $H^*(BG';\mathbb{Z}/p)$. Then

$$f^*(F_i') \subset F_i \quad and \quad g_*(F_i) \subset F_i',$$

(2) $H^i(BG;\mathbb{Z}/p) \subset F_i$,

(3) $F_i \cdot F_j \subset F_{i+j}$,

(4) $Ch(G) \subset F_0$ where $Ch(G)$ is the Chern subring.

(5) For the reduced and the (usual and higher) Bockstein operations P^j and β,

$$P^j(F_i) \subset F_i \quad and \quad \beta(F_i) \subset F_{i-1}.$$

Let us write the associated graded algebra

$$grH^*(BG;\mathbb{Z}/p) = \oplus_i F_i/F_{i-1} \quad and \quad gr^iH^*(BG;\mathbb{Z}/p) = F_i/F_{i-1}.$$

For each element $x \in H^*(BG;\mathbb{Z}/p)$, define the (weight) degree

$$w(x) = i \quad when\ 0 \neq x \in gr^iH^*(BG;\mathbb{Z}/p).$$

First note that the β-filtration of a finite group is decided from that of its Sylow p-subgroup.

Lemma 2.1. *Let G be a finite group, S its Sylow p-subgroup and $i : S \subset G$ the inclusion. Let $F_j(G)$ and $F_j(S)$ be its β-filtration of G and S. Then $i_*F_j(S) = F_j(G)$.*

Proof. From the property (1),

$$i_*i^*F_j(G) \subset i_*F_j(S) \subset F_j(G).$$

Let $[G;S] = m$ the number prime to p. Then $i_*i^*F_j(G) = mF_j(G) = F_j(G)$. Hence we see $i_*F_j(S) = F_j(G)$. □

Quillen showed that the following restriction map r is an F-isomorphism

$$r : H^*(BG;\mathbb{Z}/p) \to lim_{\leftarrow A}H^*(BA;\mathbb{Z}/p)$$

where A runs over a set of conjugacy classes of elementary abelian p-subgroups of G. Here an F-isomorphism means that its kernel is generated by nilpotent elements and for each $x \in H^*(BG;\mathbb{Z}/p)$ there is $s \geq 0$ such that $x^{p^s} \in Im(r)$. Let $\bar{C}h(G)$ be the Mackey closure of $Ch(G)$, which is recursively defined by transfers of Chern classes from subgroups. From (1) and (4), we see $\bar{C}h(G) \subset F_0$. It is known from Green-Leary [Gr-Le] that the inclusion $\bar{C}h(G) \subset H^*(BG;\mathbb{Z}/p)$ is F-epic . Hence we see ;

Lemma 2.2. *Each element in $gr^iH^*(BG;\mathbb{Z}/p), i > 0$ is nilpotent in $grH^*(BG;\mathbb{Z}/p)$.*

Next we recall the Milnor operation Q_i which is inductively defined as $Q_0 = \beta$ and $Q_{i+1} = [Q_i, P^{p^i}]$. It is known that $Q_iQ_j = -Q_jQ_i$ and $Q_i^2 = 0$. Let us write the exterior algebra

$$Q(n) = \Lambda(Q_0, ..., Q_n).$$

From the property (5), we see $Q_jF_i \subset F_{j-1}$. Moreover $Q_{i_1}...Q_{i_s}F_j \subset F_{j-s}$. Thus we have

Lemma 2.3. *Let $H^*(BG;\mathbb{Z}/p) \cong \oplus_i Q(n)G_n$ with $G_n \subset F_{n+1}$ where $Q(n)G_n$ means the free $Q(n)$-module generated by G_n. Then*

$$gr^iH^*(BG;\mathbb{Z}/p) = \oplus_{i=n-s}Q_{i_1}...Q_{i_s}G_n.$$

That means if $0 \neq x \in gr^iH^*(BG;\mathbb{Z}/p)$ if and only if $Q_0..Q_{i-1}x \neq 0$ and $Q_0...Q_i(x) = 0$.

Proof. Let $x' = Q_{i_1}...Q_{i_s}x$ for some $0 \neq x \in G_n$. Then $x' \in F_{n-s}$. Moreover

$$0 \neq Q_0...Q_nx = (\pm)Q_{j_1}...Q_{j_{n-s}}x'$$

with $(j_1, ..., j_{n-s}, i_1, ..., i_s) = (0, ..., n)$. This implies $x \notin F_{n-s-1}$. Thus $0 \neq x' \in gr^{n-s}H^*(BG;\mathbb{Z}/p)$. □

Remark. If the condition (4) is weaken so that $\beta(F_i) \subset F_i$ for only the *usual* Bockstein operation, then letting $w(x) = n$ as the largest number n such that $Q_{i_1}...Q_{i_s}(x) \neq 0$, gives an example of such filtrations.

3. EXAMPLES

We do not see that β-filtrations are unique. However we give here the cases where the β-filtration (so $grH^*(BG;\mathbb{Z}/p)$) is determined uniquely.

At first consider the case $G = \mathbb{Z}/2$. Of course $H^*(BZ/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]$ with $deg(x) = 1$. Since x^2 is represented as the first Chern class c_1 of the canonical bundle, $x^2 \in F_0$. Let us write x^2 by y. Since $\beta(x) = y$, we show $w(x) \geq 1$. But $x \in F_1$ from (2). So $w(x) = 1$ exactly. Thus we have the isomorphism

$$grH^*(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[y] \otimes \Lambda(x) \qquad w(y) = 0,\ w(x) = 1$$

as the case for odd prime. Indeed, we have

$$grH^*((BZ/p)^n;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, ..., y_n] \otimes \Lambda(x_1, ..., x_n)$$

with $\beta(x_i) = y_i$ for all primes p. Of course for $i_1 < ... < i_s$,

$$w(yx_{i_1}...x_{i_s}) = s, \quad with\ 0 \neq y \in \mathbb{Z}/p[y_1, ..., y_n].$$

Note that $Q_i(x) = y^{p^i}$ and each Q_i is a derivation, and hence

$$Q_0...Q_{s-1}(yx_1...x_s) = y\sum sgn(j_1, ..., j_n)y_1^{p^{j_1}}y_2^{p^{j_2}}...y_s^{p^{j_s}} \neq 0$$

where $(j_1, ..., j_s)$ are permutations of $(0, ..., s-1)$.

For the n-th unitary group U_n, it is immediate from (4),

$$F_0 = H^*(BU_n; \mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, ..., c_n].$$

The mod 2 cohomology of the classifying space BO_n of the n-th orthogonal group is

$$H^*(BO_n; \mathbb{Z}/2) \cong H^*((B\mathbb{Z}/2)^n; \mathbb{Z}/2)^{S_n} \cong \mathbb{Z}/2[w_1, ..., w_n]$$

where S_n is the n-th symmetry group, w_i is the Stiefel-Whiteney class which restricts the elementary symmetric polynomial in $\mathbb{Z}/2[x_1, ..., x_n]$. Each element w_i^2 is represented by Chern class c_i of the induced representation $O(n) \subset U(n)$. Let us write w_i^2 by c_i.

Since $Q_{i-1}...Q_0(w_i) \neq 0$, we see each $w(w_i) = i$. However even the module structure of $gr^*H^*(BO_n; \mathbb{Z}/2)$ seems complicated. W.S.Wilson ([Wi],[Ko-Ya]) found a good $Q(i) = \Lambda(Q_0, ..., Q_i)$-module decomposition for $X = BO_n$, namely,

$$H^*(X; \mathbb{Z}/2) = \oplus_{-1}Q(i)G_i \quad with\ Q_0...Q_iG_i \in \mathbb{Z}/2[c_1, ..., c_n].$$

Here G_{k-1} is quite complicated , namely, it is generated by symmetric functions

$$\Sigma x_1^{2i_1+1}...x_k^{2i_k+1}x_{k+1}^{2j_1}...x_{k+q}^{2j_q}, \quad k + q \leq n,$$

with $0 \leq i_1 \leq ... \leq i_k$ and $0 \leq j_1 \leq ... \leq j_q$; and if the number of j equal to j_u is odd, then there is some $s \leq k$ such that $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$.

From Lemma 2.3 in the preceding section, we have

Proposition 3.1. *If $w(G_i) = i + 1$, then we have the bidegree module isomorphism*

$$grH^*(BO_n; \mathbb{Z}/2) \cong (grH^*((B\mathbb{Z}/2)^n; \mathbb{Z}/2))^{S_n} \cong (\oplus_i Q(i)G_i).$$

We can prove the assumption $w(G_i) = i+1$ for the motivic filtration, which is an example of β-filtrations.

Since the direct decomposition of BO_3 is complicated to write, we only write here that of SO_3 since $O_3 \cong SO_3 \times \mathbb{Z}/2$.

$$H^*(BSO_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, w_3]/(w_1) \cong \mathbb{Z}/2[w_2, w_3]$$

$$\cong \mathbb{Z}/2[c_2, c_3]\{1, w_2, w_3 = Q_0w_2, w_2w_3 = Q_1w_2\}$$

$$\cong \mathbb{Z}/2[c_2,c_3]\{w_2, Q_0w_2, Q_1w_2, c_3 = Q_0Q_1w_2\} \oplus \mathbb{Z}/2[c_2]$$
$$\cong \mathbb{Z}/2[c_2,c_3]Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_2].$$

Of course, this case $w(w_2) = 2$ and the assumption in the above proposition is satisfied.

From here we consider the case $p = odd$. One of the easiest examples is the case $G = PGL_3$ and $p = 3$. The mod 3 cohomology is given by ([Ko-Ya],[Ve])

$$H^*(BPGL_3; \mathbb{Z}/3) \cong (\mathbb{Z}/3[y_2]\{y_2\} \oplus \mathbb{Z}/3[y_8]\{1, y_2, y_3, y_7\}) \otimes \mathbb{Z}/3[y_{12}]$$

where the suffice i of y_i means its degree. It is known that y_2^2, y_2^3, y_8^2 and y_{12} are represented by Chern classes. The cohomology operations are given

$$y_2 \xrightarrow{\beta} y_3 \xrightarrow{P^1} y_7 \xrightarrow{\beta} y_8$$

Hence $Q_1Q_0(y_2) = y_8$. (The element y_8 is not represented by a Chern class.) Thus we see

$$F_0 = (\mathbb{Z}/3[y_2]\{y_2^2\} \oplus \mathbb{Z}/3[y_8]) \otimes \mathbb{Z}/3[y_{12}].$$

Theorem 3.2. *Let $w(y_2) = 2$. Then the bidegree module $grH^*(BPGL_3; \mathbb{Z}/3)$ is isomorphic to*

$$(\mathbb{Z}/3[y_2]\{y^2\} \oplus \mathbb{Z}/3\{1\} \oplus \mathbb{Z}/3[y_8] \otimes Q(1)\{y_2\}) \otimes \mathbb{Z}/3[y_{12}].$$

In §10, we will study the motivic filtration of $BPGL_p$ for each odd prime.

Next consider the extraspecial p-group $E = p_+^{1+2}$ for odd prime. (The similar argument also holds for the dihedral group $D_8 = 2_+^{1+2}$.) This group is a noncommutative group of exponent p such that there is the central extension

$$0 \to \mathbb{Z}/p \to p_+^{1+2} \xrightarrow{\pi} \mathbb{Z}/p \oplus \mathbb{Z}/p \to 0.$$

The ordinary cohomology is known by Lewis (see also Leary [Lew], [Le], [Te-Ya]), namely,

$$H^{even}(BE)/p \cong (\mathbb{Z}/p[y_1,y_2]/(y_1^py_2 - y_1y_2^p) \oplus \mathbb{Z}/p\{c_2, ..., c_{p-1}\}) \otimes \mathbb{Z}/p[c_p].$$

$$H^{odd}(BE) \cong \mathbb{Z}/p[y_1,y_2,c_p]\{a_1,a_2\}/(y_1a_2 - y_2a_1, y_1^pa_2 - y_2^pa_1).$$

Here y_1,y_2 are the 1-st Chern classes of 1-dimensional representation induced from the map π, c_j is the j-th Chern class of the inductive p-dimensional representation from the maximal abelian subgroup ($\cong \mathbb{Z}/p \oplus \mathbb{Z}/p$). The 3-dimensional elements a_i satisfies $Q_1(a_i) = y_ic_p$.

Theorem 3.3. *Let $\partial_p : H^*(BE;\mathbb{Z}/p) \to H^*(BE)/p$ be the (higher) Bockstein map. Then*

$$grH^*(BE;\mathbb{Z}/p) \cong \{1, \partial_p^{-1}\}(H^*(BE)/p) - \{\partial_p^{-1}1\}$$

where $w(H^{even}(BE)/p) = 0, w(H^{odd}(BE)) = 1$ and ∂_p^{-1} ascents the weight one.

Proof. Since all elements in $H^{even}(BE)$ are generated by Chern classes, we have the isomorphism $F_0 = H^{even}(BE)/p$. We know $H^{odd}(BE)/p$ is generated as an $(H^{even}(BE)/p)$-module by two elements a_1, a_2 such that $Q_1(a_i) = y_i c_p$ [Te-Ya].

The mod p-cohomology is written additively

$$H^*(BE;\mathbb{Z}/p) \cong \{1, \partial_p^{-1}\}H^*(BE)/p.$$

All elements in $H^{odd}(BE)$ are just p-torsion and we can take $a_i' \in H^2(BE;\mathbb{Z}/p)$ such that $\beta(a_i') = a_i$. Since $|a_i'| = 2$, $a_i' \in F_2$ from (2). Hence we take $w(a_i') = 2$ and so $w(a_i) = 1$.

Next consider elements $x = \partial_p^{-1}(y)$, $y \in H^{even}(BE)/p$. If $y \in (Ideal(y_1, y_2))$, then $\partial_p^{-1}(y) = \sum x_i b_i$ for $b_i \in H^{even}(BE)/p$, and hence we can take $w(\partial_p^{-1}(y)) = 1$. For other elements $y = c_i c$ with $c \in \mathbb{Z}/p[c_p]$, we can prove ([Ly]) that those elements are represented by transfer from a subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Therefore we can also prove that $w(\partial_p^{-1}(y)) = 1$. Thus we complete the proof. □

4. MOTIVIC FILTRATION

The motivic cohomology of the classifying space is defined as follows. Let G be a linear algebraic group over k. Let V be a representation of G such that G acts freely on $V - S$ for some closed subset S. Then $(V - S)/G$ exists as a quasi-projective variety over k. According to Totaro ([To1]) and V.Voevodsky, define

$$H^{*,*}(BG;\mathbb{Z}/p) = \lim_{dim(V), codim(S) \to \infty} H^{*,*}((V-S)/G;\mathbb{Z}/p).$$

The topological space $BG(\mathbb{C}) = \lim((V - S)/G)(\mathbb{C})$ is homotopic to the usual classifying space BG. Hence we write the $\mathbb{C}$-value points $BG(\mathbb{C})$ simply by BG.

We still know the motivic cohomologies of $B\mathbb{G}_m$ and $B\mathbb{Z}/p$. Since BGL_n is cellular, we have (Hu-Kriz [Hu-Kr])

$$H^{*,*'}(BGL_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, ..., c_n] \otimes H^{*,*'}(pt;\mathbb{Z}/p)$$

where the Chern class c_i with $deg(c_i) = (2i, i)$ are identified with the elementary symmetric polynomial in $H^{*,*}((\mathbb{P}^\infty)^n; \mathbb{Z}/p)$. So we can define the Chern class $\rho^*(c_i) \in H^{2*,*}(BG; \mathbb{Z}/p)$ for each representation $\rho : G \to GL_n$.

Hereafter we assume $k = \mathbb{C}$.

Definition. We define the motivic filtration of $H^*(BG; \mathbb{Z}/p)$ by,

$$F_i = Im(t_{\mathbb{C}}^{2*-i,*}) = \oplus_n t_{\mathbb{C}}(H^{2n-i,n}(BG; \mathbb{Z}/p)).$$

Note $t_{\mathbb{C}} = cl : H^{*,*'}(X; \mathbb{Z}/p) \to H^*_{et}(X; \mathbb{Z}/p)$ identifying $H^*_{et}(X; \mathbb{Z}/p) \cong H^*(X(\mathbb{C}); \mathbb{Z}/p)$ when $k = \mathbb{C}$. Moreover the cycle map is identified with

$$\times\tau^{*-*'} : H^{*,*'}(X; \mathbb{Z}/p) \to H^{*,*}(X; \mathbb{Z}/p),$$

from the $BL(*, p)$ condition. So we know $F_i \subset F_{i+1}$.

Theorem 4.1. *The motivic filtration is a β-filtration.*

Proof. By the dimensional condition, we know

$$H^{2*-i,*}(BG; \mathbb{Z}/p) = 0 \quad for\ i < 0.$$

This implies $F_{-1} = 0$. By the $BL(i, p)$ condition, we see $F_i = H^i(BG; \mathbb{Z}/p)$, which is the condition (2) of the β-filtration. Hence of course $F_\infty = H^*(BG; \mathbb{Z}/p)$. Thus the motivic filtration is indeed filtration of $H^*(BG; \mathbb{Z}/p)$.

The condition (1) is satisfied, indeed $H^{*,*'}(X; \mathbb{Z}/p)$ has the Gysin exact sequence and transfers. Of course the sum of the bidegrees are

$$(2*-n, *) + (2*'-n', *') = (2(* + *') - (n + n'), * + *'),$$

which shows the condition (3). Since we can also define the Chern class $c_i \in H^{2i,i}(X; \mathbb{Z}/p)$ from the above argument, we get (4), namely $Ch(G) \subset F_0$. The existence of cohomology operations implies the condition (5). $\square$

Remark. The motivic filtration can be extend for all smooth X and $k \subset \mathbb{C}$ but not only BG and $k = \mathbb{C}$ with changing the realization map to the cycle map, and $H^*(X(\mathbb{C}); \mathbb{Z}/p)$ to $H^*_{et}(X; \mathbb{Z}/p)$. Moreover the condition (1) should be extended for each projective map g and the condition (3) be changed $cl(CH^*(X)) = F_0$.

From the above theorem, we can know some information of the motivic cohomology without using any theory of algebraic geometry (just arguments in §2 or §3). In [Ya2], we define

$$h^{*,*'}(X; \mathbb{Z}/) = \oplus_{m,n} H^{m,n}(X; \mathbb{Z}/p)/(Ker(t_{\mathbb{C}}^{m,n}),$$

and compute them, for example, the cases $X = BG$ stated in §3. In fact, it is immediate that

$$h^{*,*'}(X; \mathbb{Z}/p) \cong gr^{2*'-*} H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau].$$

5. CONIVEAU FILTRATION

The motivic cohomology is known to be a cohomology of a complex of some Zarisky sheaves. Recall that $\mathbb{Z}/p(n)$ ([Vo1],[Vo4]) is the complex of sheaves in Zarisky topology such that $H^{m,n}(X;\mathbb{Z}/p) \cong H^m_{Zar}(X;\mathbb{Z}/p(n))$. Let α be the obvious map of sites from Zarisky topology to etale topology so that

$$H^m_{et}(X;\mathbb{Z}/p) \cong H^m_{et}(X;\mu_p^{\otimes n}) \cong H^m_{Zar}(X, R\alpha_*\alpha^*\mathbb{Z}/p(n)).$$

For $k \leq n$, let $\tau_{\leq k+1}R\alpha_*\alpha^*\mathbb{Z}/p(n)$ be the canonical truncation of $R\alpha_*\alpha^*\mathbb{Z}/p(n)$ of level $k+1$. Then we have the short exact sequence of sheaves

$$0 \to \tau_{\leq n}R\alpha_*\alpha^*\mathbb{Z}/p(n) \to \tau_{\leq n+1}R\alpha_*\alpha^*\mathbb{Z}/p(n) \to H^n_{\mathbb{Z}/p} \to 0$$

where $H^n_{\mathbb{Z}/p}$ the Zarisky sheaf induced from the presheaf $H^n_{et}(V;\mathbb{Z}/p)$ for open subset V of X. The Beilinson and Lichtenbaum conjecture (hence $BL(n,p)$condition) (see [Vo4],[Vo5]) implies

$$\mathbb{Z}/p(k) \cong \tau_{\leq k+1}R\alpha_*\alpha^*\mathbb{Z}/p(n) \quad \textit{quasi equivalence.}$$

Thus we get the long exact sequence

$$\to H^{m,n-1}(X;\mathbb{Z}/p) \xrightarrow{\times\tau} H^{m,n}(X;\mathbb{Z}/p)$$
$$\to H^{m-n}_{Zar}(X;H^n_{\mathbb{Z}/p}) \to H^{m+1,n-1}(X;\mathbb{Z}/p) \xrightarrow{\times\tau}.$$

Hence we have the isomorphism ;

Lemma 5.1. *([Or-Vi-Vo], [Ya4])*

$$H^{*-*'}_{Zar}(X;H^{*'}_{\mathbb{Z}/p}) \cong H^{*,*'}(X;\mathbb{Z}/p)/(\tau) \oplus Ker(\tau|H^{*+1,*'-1}(X;\mathbb{Z}/p).$$

The filtration *coniveau* is given by

$$N^cH^m_{et}(X;\mathbb{Z}/p) = \cup_Z Ker\{H^m_{et}(X;\mathbb{Z}/p) \to H^m_{et}(X-Z;\mathbb{Z}/p)\}$$

where Z runs in the set of closed subschemes of X of $codim = c$.

Grothendieck wrote down the E_1-term of the spectral sequence induced from the above *coniveau* filtration.

$$E(c)_1^{c,m-c} \cong \Pi_{x\in X^{(c)}}H^{m-c}_{et}(k(x);\mathbb{Z}/p) \Longrightarrow grH^m_{et}(X;\mathbb{Z}/p)$$

where $X^{(c)}$ is the set of primes of codimension c and $k(x)$ is the function field of x. We can regard $i_xH^{m-c}_{et}(k(x);\mathbb{Z}/p)$ as a constant sheaf $H^{m-c}_{et}(k(x);\mathbb{Z}/p)$ on $\{\bar{x}\}$ and extend it by zero to X. Then the differentials of the spectral sequence give us a complex on sheaves on X

$$0 \to H^q_{\mathbb{Z}/p} \to \Pi_{x\in X^{(0)}}i_xH^q_{et}(k(x);\mathbb{Z}/p) \to \Pi_{x\in X^{(1)}}i_xH^{q-1}_{et}(k(x);\mathbb{Z}/p)$$
$$\to \dots \to \Pi_{x\in X^{(q)}}i_xH^0_{et}(k(x);\mathbb{Z}/p) \to 0.$$

Bloch-Ogus [Bl-Og] proved that the above sequence of sheaves is exact and the E_2-term is given by

$$E(c)_2^{c,m-c} \cong H^c_{Zar}(X, H^{m-c}_{\mathbb{Z}/p}).$$

Theorem 5.2. *([Ya4]) For each bidegree $(*,*')$, let $H^{*,*'}(X;\mathbb{Z}/p)$ be a finite group and the cycle map $cl^{*,*'}$ injective. Then the coniveau spectral sequence collapses from the E_2-term, namely,*

$$E_\infty^{*-*',*'} \cong E_2^{*-*',*'} \cong H^{*,*'}(X;\mathbb{Z}/p)/(\tau).$$

Thus the motivic filtration is the coniveau filtration (changing degree)

$$F_i(H^*_{et}(X;\mathbb{Z}/p)) = \oplus_{*-2c=i} N^c H^*_{et}(X;\mathbb{Z}/p).$$

Proof. Let $cl^{*,*'}$ be injective for each $(*.*')$. Then $Ker(\tau)|H^{*,*'}(X;\mathbb{Z}/p) = 0$. From the preceding lemma, we have the isomorphism for $E_2^{*,*'}$-term in the theorem.

From the injectivity of cl also, we see

$$\oplus_s^{s=*} H^{*,s}(X;\mathbb{Z}/p)/(\tau) \cong H^*_{et}(X;\mathbb{Z}/p).$$

Suppose $d_r(x) \neq 0$ for some $x \in E_2^{*,*'}$ and $r \geq 2$. Then

$$\oplus_s rank_p E_\infty^{*-s,s} < \oplus_s rank_p E_2^{*-s,s} = rank_p H^*_{et}(X;\mathbb{Z}/p).$$

(Here note that ranks are finite since all cohomology are finite.)

On the other hand, $E_\infty^{*,*'} \cong \tilde{gr} H_{et}(x;\mathbb{Z}/p)$ where $\tilde{gr}$ means the graded algebra associated with the coniveau filtration. This is a contradiction. Thus $d_r(x) = 0$ for all x and $r \geq 2$. □

For example, when $X = (B\mathbb{Z}/p)^n$, the motivic filtration is the coniveau one. We give here some geometric explanation as following.

Let $x \in N^c H^m_{et}(X;\mathbb{Z}/p)$. Take a closed subscheme K of X with $codim = c$ such that $i^*x = 0$ for $i^* : H^*_{et}(X;\mathbb{Z}/p) \to H^*_{et}(X - Z;\mathbb{Z}/p)$. Consider the exact sequence

$$(*) \quad \to H^*_{et}(X/X-K;\mathbb{Z}/p) \xrightarrow{q^*} H^*_{et}(X;\mathbb{Z}/p) \xrightarrow{i^*} H^*_{et}(X-K;\mathbb{Z}/p) \to .$$

Here we assume that the embedding $i : K \subset X$ is regular. Since there is the Thom isomorphism

$$Th : H^{*-2c}_{et}(K;\mathbb{Z}/p) \cong H^*_{et}(X/X-K;\mathbb{Z}/p),$$

we can take $x' \in H^{*-2c}_{et}(K;\mathbb{Z}/p)$ such that $q^*(Th(x')) = x$. By $BL(n,p)$-condition, we have $x'' \in H^{*-2c,*-2c}(K;\mathbb{Z}/p)$ in the motivic cohomology with $cl^{*,*}(x'') = x'$. We consider the Thom map in the motivic cohomology

$$H^{*-2c,*-2c}(K;\mathbb{Z}/p) \xrightarrow{Th} H^{*,*-c}(Th(K);\mathbb{Z}/p) \cong H^{*,*-c}(X/X-K;\mathbb{Z}/p).$$

So we get the element $\tilde{x} = q^*Th(x'') \in H^{*,*-c}(X;\mathbb{Z}/p)$ with $cl^{*,*-c}(\tilde{x}) = x$. Our case $* = m$ and this means $x \in F_{m-2c}$.

Let $L_p^N \cong (\mathbb{C}^N - \{0\})/\mathbb{Z}/p$ the $2N$-dimensional lens space so that $H^*(L_p^N;\mathbb{Z}/p) \cong (\mathbb{Z}/p[y] \otimes \Lambda(x))/(y^{N+1}, y^N x)$. Let us write by j the (regular) embedding

$$j : Z = L_p^{N-i_1} \times \ldots \times L_l^{N-i_n} \subset X = (L_p^N)^{\times n}.$$

Recall the Gysin map j_* is defined by q^*Th in the exact sequence $(*)$. Since $j_*(1) = y_1^{i_1}\ldots y_n^{i_n}$, we see that

$$x = y_1^{i_1}\ldots y_n^{i_n} x_{j_1}\ldots x_{j_s} \in Im(j_*)$$

$$\subset Ker(i^*) \subset N^{i_1+\ldots+i_n} H^*((L_p^N)^{\times n};\mathbb{Z}/p).$$

This means that $x \in F_s^m$ implies $x \in N^{(m-s)/2}$. When $i_1 + \ldots + i_n = c$, the above elements x make a $\mathbb{Z}/p$-basis of N^c and the embeddings are regular. Thus we have $F_{m-2c} = N^c$ as desired.

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DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, IBARAKI UNIVERSITY, MITO, IBARAKI, JAPAN

E-mail address: yagita@mx.ibaraki.ac.jp