On the cohomology of finite Chevalley groups and free loop spaces

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1 Introduction

Let p, ℓ be distinct primes and let q be a power of p. We denote by $\mathbb{F}_q$ the finite field with q-elements. Let G be a connected compact Lie group. There exists a reductive finite algebraic group $G(\mathbb{C})$ associated with G, called the complexification of G. One may consider $G(\mathbb{C})$ as $\mathbb{C}$-rational points of the reductive integral algebraic group scheme $G_{\mathbb{Z}}$. Taking the $\mathbb{F}_q$-rational points of $G_{\mathbb{Z}}$, we have the finite Chevalley group $G(\mathbb{F}_q)$. Denote by $\bar{\mathbb{F}}_q$ the algebraic closure of the finite field $\mathbb{F}_q$, so that

$$\bar{\mathbb{F}}_p = \bigcup_q \mathbb{F}_q.$$

We may consider the the finite Chevalley group $G(\mathbb{F}_q)$ as the fixed point set $G(\bar{\mathbb{F}}_p)^{\phi^q}$ where

$$\phi^q : G(\bar{\mathbb{F}}_p) \to G(\bar{\mathbb{F}}_p)$$

is the Frobenius map induced by the Frobenius homomorphism $\phi^q : \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$ sending x to x^q.

In[4], Quillen computed the mod ℓ cohomology of finite general linear group $GL_n(\mathbb{F}_q)$. The finite general linear group $GL_n(\mathbb{F}_q)$ is the finite Chevalley group associated with the unitary group $U(n)$. Quillen's computation could be explained by the the following Theorem 1.1 due to Friedlander [1, Theorem 12.2], Friedlander–Mislin [2, Theorem 1.4].

We fix a connected compact Lie group G. Let $BG^\wedge$ be the Bousfield–Kan $\mathbb{Z}/\ell$-completion of the classifying space BG of the connected compact Lie group G. We denote by $\mathrm{fib}(\alpha)$ the homotopy fibre of a map α. For the sake of notational simplicity, we write A, V for $H^*(BG; \mathbb{Z}/\ell)$, $H^*(G; \mathbb{Z}/\ell)$, respectively.

Theorem 1.1 There exist maps

$$D : BG(\bar{\mathbb{F}}_p) \to BG^\wedge$$

and

$$\phi^q : BG^\wedge \to BG^\wedge$$

satisfying the following three conditions:
(1) The induced homomorphism

$$D^* : H^*(BG^\wedge; \mathbb{Z}/\ell) \to H^*(BG(\bar{\mathbb{F}}_p); \mathbb{Z}/\ell)$$

is an isomorphism.
(2) $\phi^q \circ D \simeq D \circ \phi^q$ where

$$\phi^q : BG(\bar{\mathbb{F}}_p) \to BG(\bar{\mathbb{F}}_p)$$

is the Frobenius map induced by the Frobenius homomorphism $\phi^q : \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$.
(3) The induced map

$$\mathrm{fib}(D_q) \to \mathrm{fib}(\Delta)$$

induces an isomorphism

$$H^*(\mathrm{fib}(\Delta); \mathbb{Z}/\ell) \to H^*(\mathrm{fib}(D_q); \mathbb{Z}/\ell),$$

where the above map is induced by the following homotopy commutative diagram.

$$
\begin{array}{ccc}
BG(\mathbb{F}_q) & \longrightarrow & BG^\wedge \\
{\scriptstyle D_q} \downarrow & & \downarrow {\scriptstyle \Delta} \\
BG^\wedge & \xrightarrow{1 \times \phi^q \circ \Delta} & BG^\wedge \times BG^\wedge,
\end{array}
$$

where $D_q = D \circ i_q$, $i_q = BG(\mathbb{F}_q) \to BG(\bar{\mathbb{F}}_p)$ is the map induced by the inclusion of $\mathbb{F}_q$ into $\bar{\mathbb{F}}_p$ and $\Delta : BG^\wedge \to BG^\wedge \times BG^\wedge$ is the diagonal map.

On the one hand, there exists the Eilenberg-Moore spectral sequence converging to

$$gr\, H^*(BG(\mathbb{F}_q); \mathbb{Z}/\ell)$$

with the E_2-term

$$\mathrm{Tor}_{A \otimes A}(A, A).$$

On the other hand, for the free loop space

$$\mathcal{L}BG = \{\lambda : I = [0,1] \to BG \mid \lambda(1) = \lambda(0)\},$$

we have the following fibre square:

$$
\begin{array}{ccc}
\mathcal{L}BG & \longrightarrow & BG \\
\pi_0 \downarrow & & \downarrow \Delta \\
BG & \overset{\Delta}{\longrightarrow} & BG \times BG,
\end{array}
$$

where π_0 is the evaluation map at 0, so that $\pi_0(\lambda) = \lambda(0)$. There exists the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{A \otimes A}(A,A) \Rightarrow gr\, H^*(\mathcal{L}BG; \mathbb{Z}/\ell).$$

If $H_*(G; \mathbb{Z})$ has no ℓ-torsion and if $\ell | q - 1$, then $A = H^*(BG; \mathbb{Z}/\ell)$ is a polynomial algebra and the induced homomorphism

$$\phi^{q*} : A \to A$$

is the identity homomorphism. Hence, the above E_2-terms are the same polynomial tensor exterior algebra $A \otimes V$ and both spectral sequences collapse at the E_2-level. Thus, the mod ℓ cohomology of the free loop space of the classifying space BG is isomorphic to the mod ℓ cohomology of the finite Chevalley group $G(\mathbb{F}_q)$. Even if $H^*(G; \mathbb{Z})$ has ℓ-torsion, E_2-terms of the above spectral sequences are the same. Observing this phenomenon, Tezuka in [5] asked the following:

Conjecture 1.2 If $\ell | q - 1$ (resp. $4 | q - 1$) when ℓ is odd (resp. even), there exists an ring isomorphism between $H^*(BG(\mathbb{F}_q); \mathbb{Z}/\ell)$ and $H^*(\mathcal{L}BG; \mathbb{Z}/\ell)$.

In conjunction with this conjecture, in this paper, we give an outline of the proof of the following result:

Theorem 1.3 There exists an integer b such that, for $q = p^{ab}$ where a is an arbitrary positive integer, there exists an isomorphism between Leray-Serre spectral sequences associated with the map

$$D_q : BG(\mathbb{F}_q) \to BG^{\wedge}$$

and the diagonal map

$$\Delta : BG \to BG \times BG.$$

So, we have an isomorphism of graded $\mathbb{Z}/\ell$-algebras

$$gr\, H^*(BG(\mathbb{F}_q); \mathbb{Z}/\ell) = gr\, H^*(\mathcal{L}BG; \mathbb{Z}/\ell).$$

Remark 1.4 Although we give an example of the integer b in Theorem 1.3 in §2 as a function of $\dim G$ and $\dim V = H^*(G; \mathbb{Z}/\ell)$, it is not at all the best possible.

When we want to show that the cohomology of a space X is isomorphic to the cohomology of another space Y, we usually try to construct a chain of maps

$$X = X_0 \xleftarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longleftarrow \cdots \longleftarrow X_n \xrightarrow{f_n} X_{n+1} = Y$$

such that maps f_k's induce isomorphisms in mod ℓ cohomology. For example, Theorem 1.1 is proved by this method. However, when we try to prove Theorem 1.3 or Conjecture 1.2, we can not construct such a chain of maps. On the one hand, since the rational cohomology of finite group is trivial, we have

$$H^*(BG(\mathbb{F}_q); \mathbb{Q}) = \mathbb{Q}.$$

On the other hand, the rational cohomology of the free loop space is easy to compute and we have

$$
\begin{aligned}
H^*(\mathcal{L}BG; \mathbb{Q}) &= H^*(BG; \mathbb{Q}) \otimes H^*(G; \mathbb{Q}) \\
&= \mathbb{Q}[y_1, \ldots, y_n] \otimes \Lambda(x_1, \ldots, x_n),
\end{aligned}
$$

where n is the rank of the connected compact Lie group G. If there exists such a chain of maps, then they also induce isomorphisms of Bockstein spectral sequences. This contradicts the above observation on the rational (and integral) cohomology of $BG(\mathbb{F}_q)$ and $\mathcal{L}BG$. Thus, we construct maps which induce monomorphisms of Leray-Serre spectral sequences. By comparing the image of Leray-Serre spectral sequences, we construct an isomorphism between Leray-Serre spectral sequences. This isomorphism could not be realized by a chain of maps.

In §2, we define the integer b as a function of $\dim G$ and $\dim V = \dim H^*(G; \mathbb{Z}/\ell)$. In §3, we give a proof of Theorem 1.3 assuming Lemmas 3.2, 3.3 and Proposition 2.2.

2 The integer b

In this section, we define the integer b in Theorem 1.3. We define the integer b as

$$b = e_1{}^{\dim G}e_2$$

and we define e_1, e_2 in this section.

Recall that $V = H^*(G; \mathbb{Z}/\ell)$. We have isomorphisms

$$V = H^*(\mathrm{fib}(D_q); \mathbb{Z}/\ell) = H^*(\mathrm{fib}(\Delta); \mathbb{Z}/\ell) = H^*(\Omega BG^\wedge; \mathbb{Z}/\ell).$$

Denote by $GL(V)$ be the group of automorphisms of V.

Firstly, we define the integer e_1. Let q' be a power of q. The inclusion of $\mathbb{F}_q$ into $\mathbb{F}_{q'}$ induces maps

$$i : BG(\mathbb{F}_q) \to BG(\mathbb{F}_{q'})$$

and

$$j : \mathrm{fib}(D_q) \longrightarrow \mathrm{fib}(D_{q'}).$$

Suppose that $q' = q^e$ and $e = \ell m$ where m is the order of

$$\phi^{q*} : H^*(\Omega BG^\wedge; \mathbb{Z}/\ell) \to H^*(\Omega BG^\wedge; \mathbb{Z}/\ell)$$

as an element in $GL(V)$ (see the proof of Lemma 1.3 in [2]). Consider the induced homomorphism

$$j_* : \tilde{H}_*(\mathrm{fib}(D_q); \mathbb{Z}/\ell) \to \tilde{H}_*(\mathrm{fib}(D_{q'}); \mathbb{Z}/\ell),$$

where $\tilde{H}_*$ (resp. $\tilde{H}^*$) is the reduced homology (resp. reduced cohomology). Recall Lemma 1.3 in [2], we see that there hold the following:

(1) If $j_*(y) = 0$ for $\deg y < \deg x$ and if x is primitive, then $j_*(x) = 0$.

(2) If $j_*(y) = 0$ for $\deg y < \deg x$, then $j_*(x)$ is primitive.

Hence, if $q' = q^{e^2}$ and if $j_*(y) = 0$ for $\deg y < k$, then $j_*(x) = 0$ for $\deg x \leq k$. Let $e_1 = (\ell |GL(V)|)^{2 \dim G}$. Then, the induced homomorphism

$$\tilde{H}^*(\mathrm{fib}(D_{q^{e_1}}); \mathbb{Z}/\ell) \to \tilde{H}^*(\mathrm{fib}(D_q); \mathbb{Z}/\ell)$$

is zero. Therefore, we have the following lemma.

Lemma 2.1 The induced homomorphisms

$$\tilde{H}^*(\mathrm{fib}(D_{p^{e_1^{k+1}}}); \mathbb{Z}/\ell) \to \tilde{H}^*(\mathrm{fib}(D_{p^{e_1^k}}); \mathbb{Z}/\ell)$$

are zeros homomorphisms for $k = 0, \ldots, \dim G - 1$.

Secondly, we define the integer e_2. We consider the fibre square

$$
\begin{CD}
\mathrm{fib}(D_q) \times_{BG(\mathbb{F}_q)} \mathrm{fib}(D_q) @>>> \mathrm{fib}(D_q) \\
@VVV @VVV \\
\mathrm{fib}(D_q) @>>> BG(\mathbb{F}_q)
\end{CD}
$$

and the induced map

$$
1 \times \phi^q : \mathrm{fib}(D_q) \times_{BG(\mathbb{F}_q)} \mathrm{fib}(D_q) \to \mathrm{fib}(D_q) \times_{BG(\mathbb{F}_q)} \mathrm{fib}(D_q).
$$

We assume the following proposition.

Proposition 2.2 The local coefficients of $\mathrm{fib}(D_q) \times_{BG(\mathbb{F}_q)} \mathrm{fib}(D_q) \to \mathrm{fib}(D_q)$ are trivial and the E_2-term of the Leray-Serre spectral sequence for

$$
H^*(\mathrm{fib}(D_q) \times_{BG(\mathbb{F}_q)} \mathrm{fib}(D_q); \mathbb{Z}/\ell)
$$

is given by

$$
H^*(\mathrm{fib}(D_q); \mathbb{Z}/\ell) \otimes H^*(\Omega BG^\wedge; \mathbb{Z}/\ell) = V \otimes V.
$$

Hence, we have that

$$
\dim H^*(\mathrm{fib}(D_q) \times_{BG(\mathbb{F}_q)} \mathrm{fib}(D_q); \mathbb{Z}/\ell) \leq \dim(V \otimes V).
$$

Let $e_2 = |GL(V \otimes V)|$. Then, we have the following proposition.

Proposition 2.3 The induced homomorphism

$$
(1 \times \phi^q)^{ae_2*} : \tilde{H}^*(\mathrm{fib}(D_q) \times_{BG(\mathbb{F}_q)} \mathrm{fib}(D_q); \mathbb{Z}/\ell) \to \tilde{H}^*(\mathrm{fib}(D_q) \times_{BG(\mathbb{F}_q)} \mathrm{fib}(D_q); \mathbb{Z}/\ell)
$$

is the identity homomorphism for any positive integer a.

3 Proof of Theorem 1.3

Let X be a space and let $f : X \to X$ be a self-map of X with non-empty fixed point set X^f. Let A' be a subspace of the fixed point set X^f. We choose a base-point $*$ in A'. Let

$$
A = \{\lambda : I \to X \mid \lambda(1) \in A'\}.
$$

and let F be the homotopy fibre of the inclusion of A' into X, say

$$
F = \{\lambda : I \to X \mid \lambda(0) = *, \lambda(1) \in A'\}.
$$

We assume that X is simply connected and that F is connected. Let

$$\mathcal{L}_f X = \{\lambda : I \to X \mid \lambda(1) = f(\lambda(0))\}$$

and we call it the twisted loop space of f. When $f = 1$, the identity map, we denote $\mathcal{L}_1 X$ by $\mathcal{L}X$. This is the free loop space of X. There is an evaluation map

$$\pi_0 : \mathcal{L}_f X \to X$$

at 0, say $\pi_0(\lambda) = \lambda(0)$.

Firstly, we define a map

$$\varphi : \mathcal{L}_f X \times_X \mathcal{L}_f X \to \mathcal{L}X,$$

where

$$\mathcal{L}_f X \times_X \mathcal{L}_f X = \{(\lambda_1, \lambda_2) \in \mathcal{L}_f X \times \mathcal{L}_f X \mid \pi_0(\lambda_1) = \pi_0(\lambda_2)\}.$$

The map φ is defined by

$$\begin{cases} \varphi(\lambda_1, \lambda_2)(t) = \lambda_1(2t) & \text{for } 0 \leq t \leq \tfrac{1}{2}, \\ \varphi(\lambda_1, \lambda_2)(t) = \lambda_2(2 - 2t) & \text{for } \tfrac{1}{2} \leq t \leq 1. \end{cases}$$
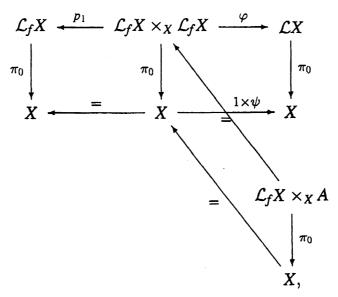
Since $\lambda_1(1) = f(\lambda_1(0))$, $\lambda_2(1) = f(\lambda_2(0))$ and $\lambda_1(0) = \lambda_2(0)$, this map is well-defined.

Next, we define a map from A to $\mathcal{L}_f X$, say $\psi : A \to \mathcal{L}_f X$ by

$$\begin{cases} \psi(\lambda)(t) = \lambda(2t) & \text{for } 0 \leq t \leq \tfrac{1}{2}, \\ \phi(\lambda)(t) = f(\lambda(2 - 2t)) & \text{for } \tfrac{1}{2} \leq t \leq 1. \end{cases}$$

Since $\lambda(1) = f(\lambda(1))$, this map is also well-defined.

Now, we consider the following diagram:

where

$$\mathcal{L}_f X \times_X A = \{(\lambda_1, \lambda_2) \in \mathcal{L}_f X \times A \mid \pi_0(\lambda_1) = \pi_0(\lambda_2)\},$$

and p_1 is the projection onto the first factor. Let us denote by $E_r(\xi)$ the Leray-Serre spectral sequence associated with a fibration $\xi \to X$. Then we have the following diagram of spectral sequences:

$$E_r(\mathcal{L}_f X) \xrightarrow{p_1^*} E_r(\mathcal{L}_f X \times_X \mathcal{L}_f X) \xleftarrow{\varphi^*} E_r(\mathcal{L}X)$$

$$\Big\downarrow 1 \times \psi^*$$

$$E_r(\mathcal{L}_f X \times_X A).$$

We denote by $\psi : F \to \Omega X$ the restriction of $\psi : \mathcal{L}_f X \to A$ to fibres. Let us consider a sufficient condition for the induced homomorphism

$$\psi^* : \tilde{H}^*(\Omega X; \mathbb{Z}/\ell) \to \tilde{H}^*(F; \mathbb{Z}/\ell)$$

to be zero. Let

$$F \times_{A'} F = \{(\lambda_1, \lambda_2) \in F \times F \mid \pi_1(\lambda_1) = \pi_1(\lambda_2)\},$$

where $\pi_1 : F \to A'$ is the evaluation map at $1 \in I$. We denote by $\varphi : \Omega X \times \Omega X \to \Omega X$ the restriction of $\varphi : \mathcal{L}_f X \times_X \mathcal{L}_f X \to \mathcal{L}_f X$ to fibres. The map $\psi : F \to X$ factors through

$$F \xrightarrow{\Delta} F \times_{A'} F \xrightarrow{1 \times f} F \times_{A'} F \xrightarrow{\varphi} \Omega X.$$

It is clear that the composition $\varphi \circ \Delta$ is null homotopic since an obvious null homotopy h_s is given by

$$\begin{cases} h_s(t) = \lambda(2st) & (0 \le t \le \tfrac{1}{2}), \\ h_s(t) = \lambda(2s - 2st) & (\tfrac{1}{2} \le t \le 1). \end{cases}$$

Thus, we have the following lemma.

Lemma 3.1 If the induced homomorphism

$$(1 \times f)^* : H^*(F \times_{A'} F; \mathbb{Z}/\ell) \to H^*(F \times_{A'} F; \mathbb{Z}/\ell)$$

is the identity homomorphisms, then the induced homomorphism

$$\psi^* : \tilde{H}^*(\Omega X; \mathbb{Z}/\ell) \to \tilde{H}^*(F; \mathbb{Z}/\ell)$$

is zero and

$$\mathrm{Im}\,(1 \times \psi)^* \circ p_1^* = \mathrm{Im}\,(1 \times \psi)^* \circ \varphi^* = H^*(X; \mathbb{Z}/\ell) \otimes H^*(\Omega X; \mathbb{Z}/\ell) \otimes \mathbb{Z}/\ell$$

$$\subset H^*(X; \mathbb{Z}/\ell) \otimes H^*(\Omega X; \mathbb{Z}/\ell) \otimes H^*(F; \mathbb{Z}/\ell) = E_2(\mathcal{L}_f X \times_X A)$$

in $E_2(\mathcal{L}_f X \times_X A)$.

We also need the following lemmas in the proof of Theorem 1.3.

Lemma 3.2 Let $\alpha : A' \to X$ be a map. If $H^i(\text{fib}(\alpha); \mathbb{Z}/\ell) = 0$ for $i > k$ and if there exists a sequence of maps

$$
\begin{array}{ccccccccc}
A_0' & \longrightarrow & A_1' & \longrightarrow & A_2' & \longrightarrow & \cdots & \longrightarrow & A_k' & \overset{=}{\longrightarrow} & A' \\
\downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\alpha_1} & & \downarrow & & & & \downarrow{\scriptstyle\alpha_k} & \swarrow{\scriptstyle\alpha} & \\
X & \overset{=}{\longrightarrow} & X & \overset{=}{\longrightarrow} & X & \overset{=}{\longrightarrow} & \cdots & \longrightarrow & X & &
\end{array}
$$

such that the induced map

$$\text{fib}(\alpha_i) \to \text{fib}(\alpha_{i+1})$$

induces a zero homomorphism

$$\tilde{H}^*(\text{fib}(\alpha_{i+1}); \mathbb{Z}/\ell) \to \tilde{H}^*(\text{fib}(\alpha_i); \mathbb{Z}/\ell)$$

for $i = 0, \ldots, k - 1$, then the map α induces a monomorphism $E_r(Y) \to E_r(Y \times_X A)$ of Leray-Serre spectral sequences for arbitrary fibration $Y \to X$.

Lemma 3.3 Let

$$E_r' \xrightarrow{\ \rho_r'\ } E_r \xleftarrow{\ \rho_r''\ } E_r''$$

be homomorphisms of spectral sequences. Suppose that

(1) $\operatorname{Im} \rho_2' = \operatorname{Im} \rho_2''$,

(2) ρ_r' is a monomorphism for $r \geq 2$.

Then, there exists an isomorphism of spectral sequences $\tau : E_r'' \to E_r'$.

Now, we prove Theorem 1.3 assuming Lemmas 3.2, 3.3 and Proposition 2.2.

Proof of Theorem 1.3 Let $q_0 = p^{e_1^{\dim G}}$, $q = q^{ae_2}$ ($a \geq 1$). Let X be the mapping cylinder of $D_{q_0} : BG(\mathbb{F}_{q_0}) \to BG^\wedge$, that is,

$$X = BG^\wedge \cup (BG(\mathbb{F}_{q_0}) \times I)/ \sim$$

where $D_{q_0}(x) \sim (x, 0)$. Choose a homotopy $H : BG(\mathbb{F}_q) \times I \to BG^\wedge$ between $\phi^q \circ D_{q_0}$ and D_{q_0}, so that

$$H(x', 0) = \phi^q \circ D_{q_0}(x'),$$
$$H(x', 1) = D_{q_0}(x').$$

Let f be the map defined by

$$\begin{aligned}
f(x) &= \phi^q(x) & &\text{for } x \in BG^\wedge, \\
f(x', t) &= H(x', 2t) & &(0 \leq t \leq \tfrac{1}{2}), \\
f(x', t) &= (x', 2t - 1) & &(\tfrac{1}{2} \leq t \leq 1).
\end{aligned}$$

Let $A' = BG(\mathbb{F}_{q_0}) \times \{1\} \subset X$. By definition, we have $A' \subset X^f$.

Let $A'_i = BG(\mathbb{F}_{p^{e_i}})$. Then, by Lemmas 2.1, 3.2, we have a monomorphism

$$E_r(\mathcal{L}_f X) \xrightarrow{(1 \times \psi)^* \circ p_1^*} E_r(\mathcal{L}_f X \times_X A).$$

By Proposition 2.3, the induced homomorphism $((1 \times \phi^{q_0})^*)^{ae_2}$ is the identity homomorphism. Hence, so is $1 \times f^*$. By Lemma 3.1, we obtain

$$\operatorname{Im}(1 \times \psi)^* \circ p_1^* = \operatorname{Im}(1 \times \psi)^* \circ \varphi^*$$

in $E_2(\mathcal{L}_f X \times_X A)$.

Therefore, using Lemma 3.3, we obtain an isomorphism between $E_r(\mathcal{L}_f X)$ and $E_r(\mathcal{L} X)$.

$\square$

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