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On eigenvalues of Cartan matrices

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1 Introduction

Let $G$ be a finite group and let $(O, K, F)$ be a $p$-modular system which is large enough for $G$. Let $B$ be a block of $FG$ with defect group $D$. We study the Cartan matrix $C$ of $B$, especially the relations between eigenvalues and elementary divisors of $C$. First we recall the definition of Cartan matrix of $B$. Let $S_1, \ldots, S_l$ ($l = l(B)$) be the set of simple $B$-modules and $P_i$ be the projective cover of $S_i$. The integers $c_{ij} = \dim_F \text{Hom}_{FG}(P_i, P_j)$ are called Cartan invariants and the $l$ by $l$ matrix $C = (c_{ij})$ is the Cartan matrix of $B$. The following facts on the Cartan matrix $C$ are well-known.

(Fact 1) The determinant of $C$, $\det C$, is a power of $p$.
(Fact 2) $C$ has the unique maximal elementary divisor, which is equal to $|D|$, and the other elementary divisors are less than $|D|$.
(Fact 3) All eigenvalues of $C$ are positive real numbers, and the maximal eigenvalue is a simple root. It is called the Frobenius eigenvalue of $C$, denoted by $\rho(C)$.

In [K-M-W], we posed the following two conjectures on eigenvalues of $C$.

(Conjecture 1) If $\rho(C) = |D|$ holds, then is it true that the eigenvalues of $C$ coincides with the elementary divisors of $C$?

(Conjecture 2) If $\rho(C)$ is an integer, then is it true that $\rho(C) = |D|$?

In [K-M-W], we showed that Conjecture 1 is affirmative under one of the following three assumptions:
(a) $G$ is $p$-solvable,
(b) $D \trianglelefteq G$,
(c) $B$ is finite type or tame type, i.e. $D$ is cyclic, dihedral, semi-dihedral or quaternion.
Conjecture 2 is also proved under the condition (b) or (c). I can not prove it.
under the condition (a).

In [W], Wada considered the following.

(Conjecture 3) Let $f_C(x)$ be the characteristic polynomial of $C$. Let

$$f_C(x) = f_1(x) \cdots f_t(x)$$

be the decomposition of $f_C(x)$ into monic irreducible polynomials in $\mathbb{Z}[x]$. Suppose $\rho(C)$ is a root of $f_1(x)$. Then, do we have a decomposition of the elementary divisors of $C$ into $t$ subsets $E_1, \ldots, E_t$ with the following properties?

1. $\deg f_i = |E_i| \quad (i = 1, \ldots, t)$,
2. $f_i(0) = \pm \prod_{e \in E_i} e \quad (i = 1, \ldots, t)$,
3. $|D| \in E_1$.

Note that Conjecture 3 is a generalization of Conjecture 2. Wada proved in [W] that Conjecture 3 holds when $B$ is finite type with $l(B) \leq 5$ or tame type. If Conjecture 3 is true, then many interesting properties of the Cartan matrix are derived from it. For example, Conjecture 3 implies that if $C$ has an integer eigenvalue $\lambda$, then $\lambda$ is an elementary divisor of $C$. It also implies that if $C$ has $k$ eigenvalues which are units in the ring of algebraic integers, then first $k$ elementary divisors of $C$ are all 1. The last statement on unit eigenvalues is proved without Conjecture 3.

2 Results

**Proposition 1** (Nomura-Kiyota) Let $C$ be the Cartan matrix of a block $B$. If $C$ has $k$ eigenvalues which are units in the ring of algebraic integers, then first $k$ elementary divisors of $C$ are all 1.

For the proof, we use the following lemma.

**Lemma 2** $\text{rank}(\bar{C}) = \text{the number of multiplicity of 1 among the elementary divisors of } C$, where $\bar{C}$ is the matrix over $\text{GF}(p)$ defined by $C \pmod{p}$.

For $p$-solvable groups $G$, we have the following.

**Proposition 3** Let $C$ be the Cartan matrix of a block in $p$-solvable group. Let $\lambda$ be an eigenvalue of $C$. If $\lambda$ is a unit in the ring of algebraic integers, then we have $\lambda = 1$. 
Proposition 3 comes from the following.

**Proposition 4** Let $C$ be the Cartan matrix of a block $B$. Suppose that every simple $B$-module is liftable. If $\lambda$ is a unit in the ring of algebraic integers, then we have $\lambda = 1$.

### 3 Problems

Recall that $(K, O, F)$ is a $p$-modular system. Let $v$ be the corresponding valuation on $K$. We assume all eigenvalues of $C$ are in $O$. We consider the following two conditions of the Cartan matrix $C$.

(*) There exists a 1-1 correspondence between the eigenvalues of $C$ and the elementary divisors of $C$ preserving the valuation $v$. i.e. the correspondants have the same valuations.

(**) There exists $R$ in $GL_4(O)$ such that $R^{-1}CR$ is a diagonal matrix.

We remark that (***) implies (*) and that (*) implies Conjecture 3 (except (3)). But (*) does not hold in general, as the example $G = SL(2,5), p=5$ shows. So we should study the following.

(Problem 1) What is the condition under which (*) holds?

We can prove the following.

**Proposition 5** If $G$ is $p$-solvable and $l(B) = 2$, then (***) holds.

So natural question arises.

(Problem 2) If $G$ is $p$-solvable, then is it true that (***) holds?
References

