<table>
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<th>On eigenvalues of Cartan matrices (Cohomology Theory of Finite Groups and Related Topics)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2008, 1581: 41-44</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/81433">http://hdl.handle.net/2433/81433</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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On eigenvalues of Cartan matrices

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1 Introduction

Let \( G \) be a finite group and let \((O, K, F)\) be a \( p \)-modular system which is large enough for \( G \). Let \( B \) be a block of \( FG \) with defect group \( D \). We study the Cartan matrix \( C \) of \( B \), especially the relations between eigenvalues and elementary divisors of \( C \). First we recall the definition of Cartan matrix of \( B \). Let \( S_1, \ldots, S_l (l = l(B)) \) be the set of simple \( B \)-modules and \( P_i \) be the projective cover of \( S_i \). The integers \( c_{ij} = \text{dim}_F \text{Hom}_{FG}(P_i, P_j) \) are called Cartan invariants and the \( l \) by \( l \) matrix \( C = (c_{ij}) \) is the Cartan matrix of \( B \). The following facts on the Cartan matrix \( C \) are well-known.

(Fact 1) The determinant of \( C \), \( \det C \), is a power of \( p \).
(Fact 2) \( C \) has the unique maximal elementary divisor, which is equal to \( |D| \), and the other elementary divisors are less than \( |D| \).
(Fact 3) All eigenvalues of \( C \) are positive real numbers, and the maximal eigenvalue is a simple root. It is called the Frobenius eigenvalue of \( C \), denoted by \( \rho(C) \).

In [K-M-W], we posed the following two conjectures on eigenvalues of \( C \).

(Conjecture 1) If \( \rho(C) = |D| \) holds, then is it true that the eigenvalues of \( C \) coincides with the elementary divisors of \( C \)?

(Conjecture 2) If \( \rho(C) \) is an integer, then is it true that \( \rho(C) = |D| \)?

In [K-M-W], we showed that Conjecture 1 is affirmative under one of the following three assumptions:
(a) \( G \) is \( p \)-solvable,
(b) \( D \triangleleft G \),
(c) \( B \) is finite type or tame type, i.e. \( D \) is cyclic, dihedral, semi-dihedral or quaternion.
Conjecture 2 is also proved under the condition (b) or (c). I can not prove it
under the condition (a).

In [W], Wada considered the following.

(Conjecture 3) Let $f_C(x)$ be the characteristic polynomial of $C$. Let

$$f_C(x) = f_1(x) \cdots f_t(x)$$

be the decomposition of $f_C(x)$ into monic irreducible polynomials in $\mathbb{Z}[x]$. Suppose $\rho(C)$ is a root of $f_1(x)$. Then, do we have a decomposition of the elementary divisors of $C$ into $t$ subsets $E_1, \ldots, E_t$ with the following properties?

1. $\deg f_i = |E_i|$ \hspace{1cm} ($i = 1, \ldots, t$),
2. $f_i(0) = \pm \prod_{e \in E_i} e \hspace{1cm} (i = 1, \ldots, t)$,
3. $|D| \in E_i$.

Note that Conjecture 3 is a generalization of Conjecture 2. Wada proved in [W] that Conjecture 3 holds when $B$ is finite type with $l(B) \leq 5$ or tame type. If Conjecture 3 is true, then many interesting properties of the Cartan matrix are derived from it. For example, Conjecture 3 implies that if $C$ has an integer eigenvalue $\lambda$, then $\lambda$ is an elementary divisor of $C$. It also implies that if $C$ has $k$ eigenvalues which are units in the ring of algebraic integers, then first $k$ elementary divisors of $C$ are all 1. The last statement on unit eigenvalues is proved without Conjecture 3.

2 Results

Proposition 1 (Nomura-Kiyota) Let $C$ be the Cartan matrix of a block $B$. If $C$ has $k$ eigenvalues which are units in the ring of algebraic integers, then first $k$ elementary divisors of $C$ are all 1.

For the proof, we use the following lemma.

Lemma 2 rank($\overline{C}$) = the number of multiplicity of 1 among the elementary divisors of $C$, where $\overline{C}$ is the matrix over GF(p) defined by $C \pmod{p}$.

For $p$-solvable groups $G$, we have the following.

Proposition 3 Let $C$ be the Cartan matrix of a block in $p$-solvable group. Let $\lambda$ be an eigenvalue of $C$. If $\lambda$ is a unit in the ring of algebraic integers, then we have $\lambda = 1$. 
Proposition 3 comes from the following.

**Proposition 4** Let $C$ be the Cartan matrix of a block $B$. Suppose that every simple $B$-module is liftable. If $\lambda$ is a unit in the ring of algebraic integers, then we have $\lambda = 1$.

### 3 Problems

Recall that $(K,O,F)$ is a $p$-modular system. Let $v$ be the corresponding valuation on $K$. We assume all eigenvalues of $C$ are in $O$. We consider the following two conditions of the Cartan matrix $C$.

\((*)\) There exists a 1-1 correspondence between the eigenvalues of $C$ and the elementary divisors of $C$ preserving the valuation $v$. i.e. the correspondants have the same valuations.

\((**)\) There exists $R$ in $\text{GL}_4(O)$ such that $R^{-1}CR$ is a diagonal matrix.

We remark that \((**)\) implies \((*)\) and that \((*)\) implies Conjecture 3 (except (3)). But \((*)\) does not hold in general, as the example $G = \text{SL}(2,5), \ p = 5$ shows. So we should study the following.

(Problem 1) What is the condition under which \((*)\) holds?

We can prove the following.

**Proposition 5** If $G$ is $p$-solvable and $l(B) = 2$, then \((**)\) holds.

So natural question arises.

(Problem 2) If $G$ is $p$-solvable, then is it true that \((**)\) holds?
References

