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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2008), 1581: 28-33</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/81435">http://hdl.handle.net/2433/81435</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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On the transfer map for the Hochschild cohomology of Frobenius algebras

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1 Introduction

We describe a transfer map between the complete Hochschild cohomologies of Frobenius algebras $\Lambda$ and $\Gamma$. For a Frobenius algebra $\Lambda$ over a commutative ring $R$ which is finitely generated projective $R$-module, we can define a complete Hochschild cohomology $H^r(\Lambda, M)$ with a coefficient $\Lambda$-bimodule $M$ (see [Na]). If, in addition, we assume that $\Gamma$ is a Frobenius extension of $\Lambda$, then $\Gamma$ is a Frobenius $R$-algebra. Under this assumption, we can define $\text{Res} : H^r(\Gamma, \Gamma M_\Gamma) \to H^r(\Lambda, M)$ and $\text{Cor} : H^r(\Lambda, \Gamma M_\Gamma) \to H^r(\Gamma, M)$. $\text{Res}$ for $r \geq 0$ and $\text{Cor}$ for $r \leq -1$ are defined naturally, and, particularly for Frobenius algebras, $\text{Res}$ and $\text{Cor}$ can also be defined for other integers $r$, which we may call them the transfer maps (see [S3], [S4], and also [No1], [No2]).

In this summary, we show the explicit description of $\text{Res}$ and $\text{Cor}$ by means of the standard resolutions of the Frobenius algebras above.

2 Complete Hochschild cohomology of Frobenius algebras

Let $R$ be a commutative ring with identity, $\Lambda$ an $R$-algebra which is finitely generated projective as $R$-module. $\Lambda^e = \Lambda \otimes_R \Lambda^{opp}$ denotes the enveloping algebra of $\Lambda$ and $Z\Lambda$ denotes the center of $\Lambda$.

If $M$ is a left $\Lambda^e$-module (i.e. $\Lambda$-bimodule), we define the Hochschild cohomology of $\Lambda$ with coefficient module $M$:

$$H^n(\Lambda, M) = \text{Ext}^n_{\Lambda^e}(\Lambda, M) \quad (n \geq 0).$$

It is easily verified that this is a $Z\Lambda$-module.

We denote $H^n(\Lambda, \Lambda)$ by $HH^n(\Lambda)$ in the following. By the definition, we see that

$$H^0(\Lambda, M) \cong M^\Lambda = \{ x \in M \mid ax = xa \text{ for any } a \in \Lambda \}$$

and so we have $HH^0(\Lambda) = Z\Lambda$. 
2.1 Standard resolution, cup product

Let $n \geq 0$ be an integer, and we put $X_n = \Lambda \otimes \cdots \otimes \Lambda$ ($n + 2$-times tensor products over $R$). Then we have the following $\Lambda^e$-projective resolution of $\Lambda$ which is called the standard resolution of $\Lambda$:

\[
\cdots \xrightarrow{d_{n+1}} X_{n+1} \xrightarrow{d_n} X_n \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{d_0} \Lambda \xrightarrow{d_0} 0,
\]

where

\[
d_n(x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_{n+1},
\]

\[
d_1(x_0 \otimes x_1 \otimes x_2) = x_0 x_1 \otimes x_2 - x_0 \otimes x_1 x_2,
\]

\[
d_0(x_0 \otimes x_1) = x_0 x_1
\]

We can define the cup product $H^i(\Lambda, M) \otimes H^j(\Lambda, N) \xrightarrow{\cup} H^{i+j}(\Lambda, M \otimes \Lambda N)$, which satisfies the anti-commutativity:

\[\alpha \cup_{i,j} \beta = (-1)^{ij} \beta \cup_{j,i} \alpha \quad \text{for } \alpha \in HH^i(\Lambda), \beta \in HH^j(\Lambda, M).\]

and the cup product $Z \Lambda \otimes H^i(\Lambda, M) \xrightarrow{\cup} H^i(\Lambda, M)$ gives the $Z \Lambda$-module structure for $H^i(\Lambda, M)$.

Furthermore, we have $HH^i(\Lambda) \otimes HH^j(\Lambda) \xrightarrow{\cup} HH^{i+j}(\Lambda)$, so this makes

\[HH^*(\Lambda) := \bigoplus_{k \geq 0} HH^k(\Lambda)\]

a ring containing $HH^0(\Lambda) = Z \Lambda$ as a subring, which is called the Hochschild cohomology ring of $\Lambda$.

2.2 Frobenius extensions and Frobenius algebras

Let $\Gamma / \Lambda$ be a Frobenius extension. That is,

\[
\Gamma = a_1 \Lambda \oplus \cdots \oplus a_m \Lambda = \Lambda b_1 \oplus \cdots \oplus \Lambda b_m;
\]

\[
xa_i = \sum_{j=1}^{m} a_j \beta_{ji}(x), \quad b_j x = \sum_{i=1}^{m} \beta_{ji}(x) b_i \quad (x \in \Gamma, \beta_{ji}(x) \in \Lambda)
\]

and there exist the following isomorphisms:

\[
\phi_{\Gamma / \Lambda} : \Gamma \Lambda \xrightarrow{\sim} \text{Hom}_{\Lambda, -}(\Gamma, \Lambda \Lambda), \quad \phi_{\Gamma / \Lambda}(a_i) = \delta_{ij},
\]

\[
\phi'_{\Gamma / \Lambda} : \Lambda \Gamma \xrightarrow{\sim} \text{Hom}_{- \Lambda}(\Gamma, \Lambda \Lambda), \quad \phi'_{\Gamma / \Lambda}(b_j) = \delta_{ij}.
\]

We set

\[
\mu_{\Gamma / \Lambda} = \phi_{\Gamma / \Lambda}(1), \quad N_{\Gamma / \Lambda}(x) = \sum_{i=1}^{m} a_i x b_i \quad (x \in \Gamma).
\]
Then $\mu_{\Gamma/\Lambda} : \Gamma \rightarrow \Lambda$ is a two-sided $\Lambda$-module homomorphism and

$$x = \sum_{i=1}^{m} \mu_{\Gamma/\Lambda}(xa_{i})b_{i} = \sum_{j=1}^{m} a_{j}\mu_{\Gamma/\Lambda}(b_{j}x) \quad (x \in \Gamma).$$

Furthermore, let $R$ be a commutative ring and $\Lambda$ a Frobenius $R$-algebra which is finitely generated free $R$-module:

$$\Lambda = u_{1}R \oplus \cdots \oplus u_{n}R = Rv_{1} \oplus \cdots \oplus Rv_{n};$$

$$yu_{i} = \sum_{j=1}^{n} u_{j}\alpha_{ji}(y), \quad v_{j}y = \sum_{i=1}^{n} \alpha_{ji}(y)v_{i} \quad (y \in \Lambda, \alpha_{ji}(y) \in R),$$

$$\phi_{\Lambda} : \Lambda \xrightarrow{\sim} \text{Hom}_{R}(\Lambda_{\Lambda}, R), \quad \phi_{\Lambda}(u_{i})(v_{j}) = \delta_{ij}. $$

We set

$$\mu_{\Lambda} = \phi_{\Lambda}(1), \quad N_{\Lambda}(y) = \sum_{i=1}^{n} u_{i}yv_{i},$$

$$y^{\tau'} = \sum_{i=1}^{n} \mu_{\Lambda}(u_{i}y)v_{i} \text{ (Nakayama automorphism of } \Lambda/R).$$

Then $\Gamma$ is a Frobenius $R$-algebra of rank $mn$ with $R$-bases $(a_{i}u_{j}), (v_{j}b_{i})$ ($1 \leq i \leq m, 1 \leq j \leq n)$:

$$xa_{i}u_{j} = \sum_{k=1}^{m} \sum_{l=1}^{n} a_{k}u_{l}\beta_{ij}(\alpha_{ki}(x)),$$

$$v_{l}b_{k}x = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij}(\alpha_{ki}(x))v_{j}b_{i} \quad (x \in \Gamma),$$

$$\phi_{\Gamma} : \Gamma \xrightarrow{\sim} \text{Hom}_{R}(\Gamma_{\Gamma}, R), \quad \phi_{\Gamma}(a_{i}u_{j})(v_{l}b_{k}) = \delta_{(i,j),(k,l)}. $$

If we set $\mu_{\Gamma} = \phi_{\Gamma}(1)$, then

$$x = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_{\Gamma}(xa_{i}u_{j})v_{j}b_{i} = \sum_{k=1}^{m} \sum_{l=1}^{n} a_{k}u_{l}\mu_{\Gamma}(v_{l}b_{k}x) \quad (x \in \Gamma).$$

We set

$$N_{\Gamma}(x) := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i}u_{j}xv_{j}b_{i},$$

$$x^{\tau} := \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_{\Gamma}(a_{i}u_{j}x)v_{j}b_{i} \quad (x \in \Gamma).$$

Then we have

$$N_{\Gamma/\Lambda} \circ N_{\Lambda} = N_{\Gamma}, \quad \mu_{\Lambda} \circ \mu_{\Gamma/\Lambda} = \mu_{\Gamma}, \quad \tau|_{\Lambda} = \tau'. $$
3 Restriction and corestriction maps

We set

\[(X_{\Gamma})_{p} = \Gamma \otimes_{R} \cdots \otimes_{R} \Gamma \quad (p + 2 \text{ times tensor products of } \Gamma),\]
\[(X_{\Lambda})_{p} = \Lambda \otimes_{R} \cdots \otimes_{R} \Lambda \quad (p + 2 \text{ times tensor products of } \Lambda),\]

and we define

\[
d_{p} : (X_{\Gamma})_{p} \longrightarrow (X_{\Gamma})_{p-1},
\]
\[
d_{p}(x_{0} \otimes x_{1} \otimes \cdots \otimes x_{p} \otimes x_{p+1})
\]
\[
= x_{0}x_{1} \otimes \cdots \otimes x_{p} \otimes x_{p+1}
\]
\[
+ \sum_{i=1}^{p-1} (-1)^{i} x_{0} \otimes \cdots \otimes x_{i}x_{i+1} \otimes \cdots \otimes x_{p+1} + (-1)^{p} x_{0} \otimes \cdots \otimes x_{p-1} \otimes x_{p}x_{p+1}.
\]

Then we have the following commutative diagram:

\[
\cdots \longrightarrow \text{Hom}_{\Gamma*}(X_{\Gamma}), M \xrightarrow{d_{1}} M \xleftarrow{N_{\Gamma}} M \xrightarrow{d_{1} \otimes_{\Gamma} \cdot} (X_{\Gamma})^{1} \otimes_{\Gamma*} M \longrightarrow \cdots
\]
\[
\xrightarrow{\text{res}^{1}} \xleftarrow{\text{res}^{2}} \xrightarrow{\text{res}^{0}} \xrightarrow{\text{res}^{0}}
\]
\[
\cdots \longrightarrow \text{Hom}_{\Lambda*}(X_{\Lambda}), M \xrightarrow{d_{1}} M \xleftarrow{N_{\Lambda}} M \xrightarrow{d_{1} \otimes_{\Lambda} \cdot} (X_{\Lambda})^{1} \otimes_{\Lambda*} M \longrightarrow \cdots
\]

Here, \((X_{\Gamma})^{p}_{\tau}\) is defined by \(w(x \otimes y^{opp}) = y^{-1}wx\) for \(w \in (X_{\Gamma})_{p}, x \otimes y^{opp} \in \Gamma^{e}\), and

\[
\text{res}^{0} : M \rightarrow M, x \mapsto x, \quad \text{res}^{0} : M \rightarrow M, x \mapsto \sum_{i=1}^{m} b_{i}a_{i},
\]
\[
\text{res}_{q} : (X_{\Gamma})^{q}_{\tau} \otimes_{\Gamma*} M \rightarrow (X_{\Lambda})^{q'}_{\tau} \otimes_{\Lambda*} M, 1 \otimes y_{1} \otimes \cdots \otimes y_{q} \otimes 1 \otimes \gamma_{e} x \mapsto
\]
\[
\sum_{i_{1}, \ldots, i_{q+1}=1}^{m} 1 \otimes \mu_{\Gamma/\Lambda}(b_{i_{1}}y_{i_{1}}a_{i_{2}}) \otimes \cdots \otimes \mu_{\Gamma/\Lambda}(b_{i_{q}}y_{i_{q}}a_{i_{q+1}}) \otimes 1 \otimes b_{i_{q+1}}a_{i}^{r},
\]
\[
\text{res}^{p} (p \geq 1) \text{ is defined to be a natural homomorphism induced by } (X_{\Lambda})_{p} \rightarrow (X_{\Gamma})_{p}. \text{ Then we have}
\]
\[
\text{Res}^{r} : H^{r}(\Gamma, \Gamma, M_{\Gamma}) \rightarrow H^{r}(\Lambda, \Lambda, M) \quad (r \in \mathbb{Z}).
\]

Here, \(H^{r}(\Gamma, \cdot, -)\) and \(H^{r}(\Lambda, \cdot, -)\) denotes the complete Hochschild cohomology of \(\Gamma\) and \(\Lambda\), respectively, and these are obtained by the horizontal sequences.

On the other hand, we have the following commutative diagram:

\[
\cdots \longrightarrow \text{Hom}_{\Gamma*}(X_{\Gamma}), M \xrightarrow{d_{1}} M \xleftarrow{N_{\Gamma}} M \xrightarrow{d_{1} \otimes_{\Gamma} \cdot} (X_{\Gamma})^{1} \otimes_{\Gamma*} M \longrightarrow \cdots
\]
\[
\xrightarrow{\text{cor}^{1}} \xleftarrow{\text{cor}^{0}} \xrightarrow{\text{cor}_{0}} \xrightarrow{\text{cor}_{1}}
\]
\[
\cdots \longrightarrow \text{Hom}_{\Lambda*}(X_{\Lambda}), M \xrightarrow{d_{1}} M \xleftarrow{N_{\Lambda}} M \xrightarrow{d_{1} \otimes_{\Lambda} \cdot} (X_{\Lambda})^{1} \otimes_{\Lambda*} M \longrightarrow \cdots
\]

Here,
$\text{cor}_0 : M \rightarrow M, x \mapsto x, \quad \text{cor}^0 : M \rightarrow M, x \mapsto N_{\Gamma/\Lambda}(x),$

$\text{cor}^p : \text{Hom}_{\Lambda^e}((X_{\Lambda})_p, M) \rightarrow \text{Hom}_{\Gamma^e}((X_{\Gamma})_p, M),$

$\text{cor}^p(g)(y_0 \otimes y_1 \otimes \cdots \otimes y_p \otimes y_{p+1}) = \sum_{i_1, \ldots, i_p+1 = 1}^{m} y_0 a_{i_1} g(1 \otimes \mu_{\Gamma/\Lambda}(b_{i_1}y_1a_{\dot{j}2}) \otimes \cdots \otimes \mu_{\Gamma/\Lambda}(b_{i_p}y_pa_{i_{p+1}}) \otimes 1)b_{i_{p+1}}y_{p+1},$

and $\text{cor}_q (q \geq 1)$ is defined to be a natural homomorphism induced by $(X_{\Lambda})_q \rightarrow (X_{\Gamma})_q$. Then we have

$$\text{Cor}^r : H^r(\Lambda, \Gamma M_{\Gamma}) \rightarrow H^r(\Gamma, M) \quad (r \in \mathbb{Z}).$$

**Proposition** We have following fundamental properties for Res and Cor.

1. Given $f : \Gamma M \rightarrow \Gamma^e N$, we have

$$f^* \text{Res}^r = \text{Res}^r f^* : H^r(\Gamma, M) \rightarrow H^r(\Lambda, N),$$

$$f^* \text{Cor}^r = \text{Cor}^r f^* : H^r(\Lambda, M) \rightarrow H^r(\Gamma, N).$$

2. Given a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of $\Gamma^e$-modules, we have

$$\partial \text{Res}^r = \text{Res}^{r+1}\partial : H^r(\Gamma, N) \rightarrow H^{r+1}(\Lambda, L),$$

$$\partial \text{Cor}^r = \text{Cor}^{r+1}\partial : H^r(\Lambda, N) \rightarrow H^{r+1}(\Gamma, L).$$

3. Given a $\Gamma^e$-module $M$, we have

$$\text{Cor}^r \text{Res}^r(w) = N_{\Gamma/\Lambda}(1)w \quad (w \in H^r(\Gamma, M)).$$

Since Res preserves the cup product, it follows that we can define a ring homomorphism $HH^*(\Gamma) \rightarrow H^*(\Lambda, \Gamma)$. Moreover, using the embedding of $\Lambda$-bimodules $\Lambda \rightarrow \Gamma$, we can define

$$HH^*(\Lambda) \rightarrow H^*(\Lambda, \Gamma) \xrightarrow{\text{Cor}} HH^*(\Gamma).$$

In particular, we have $\mathbb{Z} \Lambda/N_{\Lambda}(\Lambda) \rightarrow \mathbb{Z} \Gamma/N_{\Gamma}(\Gamma) : \overline{z} \mapsto \overline{N_{\Gamma/\Lambda}(z)}$ in the zero dimension. Note that the zero dimensional complete Hochschild cohomology is different from the ordinary one (cf. [Br]).

On the other hand, using the $\Lambda$-bimodule homomorphism $\mu_{\Gamma/\Lambda} : \Gamma \rightarrow \Lambda$, we have

$$HH^*(\Gamma) \xrightarrow{\text{Res}} H^*(\Lambda, \Gamma) \xrightarrow{\mu_{\Gamma/\Lambda}} HH^*(\Lambda).$$

In particular, we have $\mathbb{Z} \Gamma/N_{\Gamma}(\Gamma) \rightarrow \mathbb{Z} \Lambda/N_{\Lambda}(\Lambda) : \overline{z} \mapsto \overline{\mu_{\Gamma/\Lambda}(z)}$ in the zero dimension.

We have some explicit calculations of Res and Cor for twisted group algebras and crossed products (see [S1] and [S2] for twisted group algebras, and [S4] for crossed products).
References


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