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On the transfer map for the Hochschild cohomology of Frobenius algebras

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1 Introduction

We describe a transfer map between the complete Hochschild cohomologies of Frobenius algebras $\Lambda$ and $\Gamma$. For a Frobenius algebra $\Lambda$ over a commutative ring $R$ which is finitely generated projective $R$-module, we can define a complete Hochschild cohomology $H^r(\Lambda, M)$ with a coefficient $\Lambda$-bimodule $M$ (see [Na]). If, in addition, we assume that $\Gamma$ is a Frobenius extension of $\Lambda$, then $\Gamma$ is a Frobenius $R$-algebra. Under this assumption, we can define $\text{Res} : H^r(\Gamma, \Gamma M_{\Gamma}) \to H^r(\Lambda, M)$ and $\text{Cor} : H^r(\Lambda, \Gamma M_{\Gamma}) \to H^r(\Gamma, M)$. $\text{Res}$ for $r \geq 0$ and $\text{Cor}$ for $r \leq -1$ are defined naturally, and, particularly for Frobenius algebras, $\text{Res}$ and $\text{Cor}$ can also be defined for other integers $r$, which we may call them the transfer maps (see [S3], [S4], and also [No1], [No2]).

In this summary, we show the explicit description of $\text{Res}$ and $\text{Cor}$ by means of the standard resolutions of the Frobenius algebras above.

2 Complete Hochschild cohomology of Frobenius algebras

Let $R$ be a commutative ring with identity, $\Lambda$ an $R$-algebra which is finitely generated projective as $R$-module. $\Lambda^e = \Lambda \otimes_R \Lambda^{\text{opp}}$ denotes the enveloping algebra of $\Lambda$ and $Z\Lambda$ denotes the center of $\Lambda$.

If $M$ is a left $\Lambda^e$-module (i.e. $\Lambda$-bimodule), we define the Hochschild cohomology of $\Lambda$ with coefficient module $M$:

$$H^n(\Lambda, M) = \text{Ext}_{\Lambda^e}^n(\Lambda, M) \quad (n \geq 0).$$

It is easily verified that this is a $Z\Lambda$-module.

We denote $H^n(\Lambda, \Lambda)$ by $HH^n(\Lambda)$ in the following. By the definition, we see that

$$H^0(\Lambda, M) \cong M^\Lambda = \{x \in M \mid ax = xa \text{ for any } a \in \Lambda\}$$

and so we have $HH^0(\Lambda) = Z\Lambda$. 
2.1 Standard resolution, cup product

Let $n \geq 0$ be an integer, and we put $X_n = \Lambda \otimes \cdots \otimes \Lambda$ $(n + 2$-times tensor products over $R)$. Then we have the following $\Lambda^e$-projective resolution of $\Lambda$ which is called the standard resolution of $\Lambda$:

$$
\cdots \longrightarrow X_{n+1} \overset{d_{n+1}}{\longrightarrow} X_n \overset{d_n}{\longrightarrow} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \overset{d_1}{\longrightarrow} X_0 \overset{d_0}{\longrightarrow} \Lambda \longrightarrow 0,
$$

$$
d_n(x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1},
$$

$$
d_1(x_0 \otimes x_1 \otimes x_2) = x_0 x_1 \otimes x_2 - x_0 \otimes x_1 x_2,
$$

$$
d_0(x_0 \otimes x_1) = x_0 x_1
$$

We can define the cup product $H^i(\Lambda, M) \otimes H^j(\Lambda, N) \overset{\sim}{\longrightarrow} H^{i+j}(\Lambda, M \otimes_{\Lambda} N)$, which satisfies the anti-commutativity:

$$
\alpha \sim_{i,j} \beta = (-1)^{ij} \beta \sim_{j,i} \alpha \text{ for } \alpha \in HH^i(\Lambda), \beta \in HH^j(\Lambda, M).
$$

and the cup product $\mathcal{Z} \Lambda \otimes H^i(\Lambda, M) \overset{\sim}{\longrightarrow} H^i(\Lambda, M)$ gives the $\mathcal{Z}\Lambda$-module structure for $H^i(\Lambda, M)$.

Furthermore, we have $HH^i(\Lambda) \otimes HH^j(\Lambda) \overset{\vee}{\longrightarrow} HH^{i+j}(\Lambda)$, so this makes

$$
HH^*(\Lambda) := \bigoplus_{k \geq 0} HH^k(\Lambda)
$$

a ring containing $HH^0(\Lambda) = \mathcal{Z}\Lambda$ as a subring, which is called the Hochschild cohomology ring of $\Lambda$.

2.2 Frobenius extensions and Frobenius algebras

Let $\Gamma/\Lambda$ be a Frobenius extension. That is,

$$
\Gamma = a_1 \Lambda \oplus \cdots \oplus a_m \Lambda = \Lambda b_1 \oplus \cdots \oplus \Lambda b_m;
$$

$$
xa_i = \sum_{j=1}^{m} a_j \beta_{ji}(x), \quad b_j x = \sum_{i-1}^{m} \beta_{ji}(x)b_i \quad (x \in \Gamma, \beta_{ji}(x) \in \Lambda)
$$

and there exist the following isomorphisms:

$$
\phi_{\Gamma/\Lambda} : \Gamma \sim_{\Lambda} \text{Hom}_{\Lambda,-}(\Gamma, \Lambda), \quad \phi_{\Gamma/\Lambda}(a_i)(b_j) = \delta_{ij},
$$

$$
\phi'_{\Gamma/\Lambda} : \Lambda \Gamma \sim_{\Gamma} \text{Hom}_{\Lambda,-}(\Gamma, \Lambda), \quad \phi'_{\Gamma/\Lambda}(b_j)(a_i) = \delta_{ij}.
$$

We set

$$
\mu_{\Gamma/\Lambda} = \phi_{\Gamma/\Lambda}(1), \quad N_{\Gamma/\Lambda}(x) = \sum_{i=1}^{m} a_i x b_i \quad (x \in \Gamma).
$$
Then $\mu_{\Gamma/\Lambda} : \Gamma \rightarrow \Lambda$ is a two-sided $\Lambda$-module homomorphism and
\[
x = \sum_{i=1}^{m} \mu_{\Gamma/\Lambda}(xa_{i})b_{i} = \sum_{j=1}^{m} a_{j}\mu_{\Gamma/\Lambda}(b_{j}x) \quad (x \in \Gamma).
\]

Furthermore, let $R$ be a commutative ring and $\Lambda$ a Frobenius $R$-algebra which is finitely generated free $R$-module:
\[
\Lambda = u_{1}R \oplus \cdots \oplus u_{n}R = Rv_{1} \oplus \cdots \oplus Rv_{n};
\]
\[
yu_{i} = \sum_{j=1}^{n} u_{j}\alpha_{ji}(y), \quad v_{j}y = \sum_{i=1}^{n} \alpha_{ji}(y)v_{i} \quad (y \in \Lambda, \alpha_{ji}(y) \in R),
\]
\[
\phi_{\Lambda} : \Lambda \rightarrow \text{Hom}_{R}(\Lambda_{\Lambda}, R), \quad \phi_{\Lambda}(u_{i})(v_{j}) = \delta_{ij}.
\]

We set
\[
\mu_{\Lambda} = \phi_{\Lambda}(1), \quad N_{\Lambda}(y) = \sum_{i=1}^{n} u_{i}yv_{i},
\]
\[
y' = \sum_{i=1}^{n} \mu_{\Lambda}(u_{i})v_{i} \quad \text{(Nakayama automorphism of $\Lambda/R$)}.
\]

Then $\Gamma$ is a Frobenius $R$-algebra of rank $mn$ with $R$-bases $(a_{i}u_{j}), (v_{j}b_{i}) (1 \leq i \leq m, 1 \leq j \leq n)$:
\[
xa_{i}u_{j} = \sum_{k=1}^{m} \sum_{l=1}^{n} a_{k}u_{l}\beta_{ij}(\alpha_{ki}(x)),
\]
\[
v_{j}b_{k}x = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij}(\alpha_{ki}(x))v_{j}b_{i} \quad (x \in \Gamma),
\]
\[
\phi_{\Gamma} : \Gamma \rightarrow \text{Hom}_{R}(\Gamma_{\Gamma}, R), \quad \phi_{\Gamma}(a_{i}u_{j})(v_{j}b_{k}) = \delta_{(i,j),(k,l)}.
\]

If we set $\mu_{\Gamma} = \phi_{\Gamma}(1)$, then
\[
x = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_{\Gamma}(xa_{i}u_{j})v_{j}b_{i} = \sum_{k=1}^{m} \sum_{l=1}^{n} a_{k}u_{l}\mu_{\Gamma}(v_{l}b_{k}x) \quad (x \in \Gamma).
\]

We set
\[
N_{\Gamma}(x) := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i}u_{j}xv_{j}b_{i},
\]
\[
x' := \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_{\Gamma}(a_{i}u_{j}x)v_{j}b_{i} \quad (x \in \Gamma).
\]

Then we have
\[
N_{\Gamma/\Lambda} \circ N_{\Lambda} = N_{\Gamma}, \quad \mu_{\Lambda} \circ \mu_{\Gamma/\Lambda} = \mu_{\Gamma}, \quad \tau|_{\Lambda} = \tau'.
\]
3 Restriction and corestriction maps

We set

$$(X_{\Gamma})_{p} = \Gamma \otimes_{R} \cdots \otimes_{R} \Gamma \quad (p + 2 \text{ times tensor products of } \Gamma),$$

$$(X_{\Lambda})_{p} = \Lambda \otimes_{R} \cdots \otimes_{R} \Lambda \quad (p + 2 \text{ times tensor products of } \Lambda),$$

and we define

\[ d_{p} : (X_{\Gamma})_{p} \longrightarrow (X_{\Gamma})_{p-1}, \]

\[ d_{p}(x_{0} \otimes x_{1} \otimes \cdots \otimes x_{p} \otimes x_{p+1}) = x_{0}x_{1} \otimes \cdots \otimes x_{p} \otimes x_{p+1} \]

\[ + \sum_{i=1}^{p-1} (-1)^{i} x_{0} \otimes \cdots \otimes x_{i}x_{i+1} \otimes \cdots \otimes x_{p} \otimes x_{p+1} + (-1)^{p} x_{0} \otimes \cdots \otimes x_{p} \otimes 1 \otimes x_{p+1}. \]

Then we have the following commutative diagram:

\[ \cdots \longleftarrow \text{Hom}_{\Gamma}((X_{\Gamma})_{1}, M) \xrightarrow{d_{1}} M \xrightarrow{N_{\Gamma}} \quad \text{Hom}_{\Gamma}((X_{\Gamma})_{1}, \otimes_{\Gamma} M) \longleftarrow \cdots \]

\[ \longleftarrow \text{Hom}_{\Lambda}((X_{\Lambda})_{1}, M) \xrightarrow{d_{1}} M \xrightarrow{N_{\Lambda}} \quad \text{Hom}_{\Lambda}((X_{\Lambda})_{1}, \otimes_{\Lambda} M) \longleftarrow \cdots \]

Here, \((X_{\Gamma})_{\tau}^{r}\) is defined by \(w(x \otimes y^{\text{opp}}) = y^{r-1}wx\) for \(w \in (X_{\Gamma})_{p}, x \otimes y^{\text{opp}} \in \Gamma^{e}\), and

\[ \text{res}^{0} : M \rightarrow M, x \mapsto x, \quad \text{res} : M \rightarrow M, x \mapsto \sum_{i=1}^{m} b_{i}x_{i}^{r}, \]

\[ \text{res}_{q} : (X_{\Gamma})_{q} \otimes_{\Gamma^{e}} M \rightarrow (X_{\Lambda})_{q} \otimes_{\Lambda^{e}} M, 1 \otimes y_{1} \otimes \cdots \otimes y_{q} \otimes 1 \otimes_{\Gamma^{e}} x \mapsto \]

\[ \sum_{i_{1}, \ldots, i_{q+1}=1}^{m} 1 \otimes \mu_{\Gamma/\Lambda}(b_{i_{1}y_{1}a_{i_{2}}}) \otimes \cdots \otimes \mu_{\Gamma/\Lambda}(b_{i_{q}y_{q}a_{i_{q+1}}}) \otimes 1 \otimes_{\Lambda^{e}} b_{i_{q+1}}x_{i_{q+1}}^{r}, \]

\[ \text{res}^{p} \ (p \geq 1) \] is defined to be a natural homomorphism induced by \((X_{\Lambda})_{p} \rightarrow (X_{\Gamma})_{p}\). Then we have

\[ \text{Res}^{r} : H^{r}(\Gamma, \Gamma, M_{\Gamma}) \rightarrow H^{r}(\Lambda, M) \quad (r \in \mathbb{Z}). \]

Here, \(H^{r}(\Gamma, -)\) and \(H^{r}(\Lambda, -)\) denotes the complete Hochschild cohomology of \(\Gamma\) and \(\Lambda\), respectively, and these are obtained by the horizontal sequences.

On the other hand, we have the following commutative diagram:

\[ \cdots \longleftarrow \text{Hom}_{\Gamma}((X_{\Gamma})_{1}, M) \xrightarrow{d_{1}} M \xrightarrow{N_{\Gamma}} \quad \text{Hom}_{\Gamma}((X_{\Gamma})_{1}, \otimes_{\Gamma} M) \longleftarrow \cdots \]

\[ \longleftarrow \text{Hom}_{\Lambda}((X_{\Lambda})_{1}, M) \xrightarrow{d_{1}} M \xrightarrow{N_{\Lambda}} \quad \text{Hom}_{\Lambda}((X_{\Lambda})_{1}, \otimes_{\Lambda} M) \longleftarrow \cdots \]

Here,
\(\text{cor}_0 : M \to M, x \mapsto x, \quad \text{cor}^0 : M \to M, x \mapsto N_{\Gamma/\Lambda}(x),\)

\(\text{cor}^p : \text{Hom}_{\Lambda^e}((X_{\Lambda})_p, M) \to \text{Hom}_{\Gamma^e}((X_{\Gamma})_p, M),\)

\(\text{cor}^p(g)(y_0 \otimes y_1 \otimes \cdots \otimes y_p \otimes y_{p+1}) = \sum_{i_1, \ldots, i_{p+1}=1}^{m} y_0 a_{i_1} g(1 \otimes \mu_{\Gamma/\Lambda}(b_{i_1} y_1 a_{i_2}) \otimes \cdots \otimes \mu_{\Gamma/\Lambda}(b_{i_p} y_p a_{i_{p+1}}) \otimes 1) b_{i_{p+1}} y_{p+1},\)

and \(\text{cor}_q (q \geq 1)\) is defined to be a natural homomorphism induced by \((X_{\Lambda})_q \to (X_{\Gamma})_q\). Then we have

\(\text{Cor}^r : H^r(\Lambda, \Gamma M_{\Gamma}) \to H^r(\Gamma, M) \quad (r \in \mathbb{Z}).\)

**Proposition** We have following fundamental properties for \(\text{Res}\) and \(\text{Cor}\).

1. Given \(f : \Gamma_\ast M \to \Gamma_\ast N\), we have

\[
\begin{align*}
\text{f}^\ast \text{Res}^r &= \text{Res}^r \text{f}^\ast : H^r(\Gamma, M) \to H^r(\Lambda, N), \\
\text{f}^\ast \text{Cor}^r &= \text{Cor}^r \text{f}^\ast : H^r(\Lambda, M) \to H^r(\Gamma, N).
\end{align*}
\]

2. Given a short exact sequence \(0 \to L \to M \to N \to 0\) of \(\Gamma^e\)-modules, we have

\[
\begin{align*}
\partial \text{Res}^r &= \text{Res}^{r+1} \partial : H^r(\Gamma, N) \to H^{r+1}(\Lambda, L), \\
\partial \text{Cor}^r &= \text{Cor}^{r+1} \partial : H^r(\Lambda, N) \to H^{r+1}(\Gamma, L).
\end{align*}
\]

3. Given a \(\Gamma^e\)-module \(M\), we have

\[
\text{Cor}^r \text{Res}^r(w) = N_{\Gamma/\Lambda}(1)w \quad (w \in H^r(\Gamma, M)).
\]

Since \(\text{Res}\) preserves the cup product, it follows that we can define a ring homomorphism \(\text{HH}^*(\Gamma) \to \text{HH}^*(\Lambda, \Gamma)\). Moreover, using the embedding of \(\Lambda\)-bimodules \(\Lambda \to \Gamma\), we can define

\[
\text{HH}^*(\Lambda) \to \text{HH}^*(\Lambda, \Gamma) \xrightarrow{\text{Cor}} \text{HH}^*(\Gamma).
\]

In particular, we have \(\text{Z}/\text{N}_\Lambda(\Lambda) \to \text{Z}/\text{N}_\Gamma(\Gamma) : \overline{z} \mapsto \overline{N_{\Gamma/\Lambda}(z)}\) in the zero dimension. Note that the zero dimensional complete Hochschild cohomology is different from the ordinary one (cf. [Br]).

On the other hand, using the \(\Lambda\)-bimodule homomorphism \(\mu_{\Gamma/\Lambda} : \Gamma \to \Lambda\), we have

\[
\text{HH}^*(\Gamma) \xrightarrow{\text{Res}} \text{HH}^*(\Lambda, \Gamma) \xrightarrow{\mu_{\Gamma/\Lambda}} \text{HH}^*(\Lambda).
\]

In particular, we have \(\text{Z}/\text{N}_\Gamma(\Gamma) \to \text{Z}/\text{N}_\Lambda(\Lambda) : \overline{z} \mapsto \overline{\mu_{\Gamma/\Lambda}(z)}\) in the zero dimension.

We have some explicit calculations of \(\text{Res}\) and \(\text{Cor}\) for twisted group algebras and crossed products (see [S1] and [S2] for twisted group algebras, and [S4] for crossed products).
References


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