Hochschild cohomology ring of an order of a quaternion algebra

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Introduction

The cohomology theory of associative algebras was initiated by Hochschild [6], Cartan and Eilenberg [1] and MacLane [7]. Let $R$ be a commutative ring and $\Lambda$ an $R$-algebra which is a finitely generated projective $R$-module. If $M$ is a $\Lambda$-bimodule (i.e., a $\Lambda^e = \Lambda \otimes_R \Lambda^{op}$-module), then the $n$th Hochschild cohomology of $\Lambda$ with coefficients in $M$ is defined by $HH^n(\Lambda, M) := \text{Ext}^n_{\Lambda^e}(\Lambda, M)$. We set $HH^*(\Lambda) := \bigoplus_{n \geq 0} HH^n(\Lambda).$ The cup product gives $HH^*(\Lambda) := \bigoplus_{n \geq 0} HH^n(\Lambda)$ a graded ring structure with $1 \in Z\Lambda \simeq HH^0(\Lambda)$ where $Z\Lambda$ denotes the center of $\Lambda$. $HH^*(\Lambda)$ is called the Hochschild cohomology ring of $\Lambda$. It is known that the cup product coincides with the Yoneda product on the Ext-algebra. Note that the Hochschild cohomology ring $HH^*(\Lambda)$ is graded-commutative, that is, for $\alpha \in HH^n(\Lambda)$ and $\beta \in HH^q(\Lambda)$ we have $\alpha \beta = (-1)^{nq} \beta \alpha$. The Hochschild cohomology is an important invariant of algebras, however the Hochschild cohomology ring is difficult to compute in general.

Let $G$ denote the generalized quaternion 2-group of order $2^{r+2}$ for $r \geq 1$:

$$Q_{2^r} = \langle x, y \mid x^{2^{r+1}} = 1, x^{2^r} = y^2, yxy^{-1} = x^{-1} \rangle.$$

We set $e = (1 - x^{2^r})/2 \in \mathbb{Q}G$ and denote $xe$ by $\zeta$, a primitive $2^{r+1}$-th root of $e$. Then $e$ is a centrally primitive idempotent of $\mathbb{Q}G$. The simple component $\mathbb{Q}Ge$ is just the ordinary quaternion algebra over the field $K := \mathbb{Q}(\zeta + \zeta^{-1})$ with identity $e$, that is, $\mathbb{Q}Ge = K \oplus Ki \oplus Kj \oplus Kij$ where we set $i = x^{2^{r-1}}e$ and $j = ye$ (see [2, (7.40)]). Note that $\zeta^k j = j \zeta^{-k}$ and $\zeta^{2^r} = -e$ hold. In the following we set $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, the ring of integers of $K$, and we set $\Gamma = \mathbb{Z}Ge = R \oplus R\zeta \oplus Rj \oplus R\zeta j$. Note that $R$ is a commuting parameter ring, because $y$ commutes with $x + x^{-1}$. Then $\Gamma$ is an $R$-order of $\mathbb{Q}Ge$. In particular if $r = 1$, $\Gamma = \mathbb{Z}e \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij$ is just the ordinary quaternion algebra over $\mathbb{Z}$ with identity $e$.

We will give an efficient bimodule projective resolution of $\Gamma$, and we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ by calculating the Yoneda products using this bimodule projective resolution. This paper is a summary of [3].

1 A bimodule projective resolution of $\Gamma$

In this section, we state a $\Gamma^e$-projective resolution of $\Gamma$.

In general, $\Gamma \otimes \Gamma$ is a left $\Gamma^e$-module (i.e., a $\Gamma$-bimodule) by putting

$$(a \otimes b^o) \cdot (\gamma_1 \otimes \gamma_2) := a\gamma_1 \otimes \gamma_2 b.$$
for all \(a, b, \gamma_1, \gamma_2 \in \Gamma\). For each \(q \geq 0\), let \(Y_q\) be a direct sum of \(q + 1\) copies of \(\Gamma \otimes \Gamma\). As elements of \(Y_q\), we set

\[
c^q_s =\begin{cases} (0, \ldots, 0, e \otimes e, 0, \ldots, 0) & (if \ 1 \leq s \leq q + 1), \\ 0 & (otherwise). \end{cases}
\]

Then we have \(Y_q = \bigoplus_{k=1}^{q+1} \Gamma c^k_q \Gamma\). Let \(t = 2^r\). Define left \(\Gamma^e\)-homomorphisms \(\pi : Y_0 \to \Gamma; c^1_0 \mapsto e\) and \(\delta_q : Y_q \to Y_{q-1}\) \((q > 0)\) given by

\[
\delta_q(c^q_s) = \begin{cases} -\zeta c^q_{q-1} + c^q_{q-1} \zeta + (-1)^{(q-s)/2} \zeta j c^{q-1}_{q-1} j \zeta - c^{q-1}_{q-1} & (for \ q \ even, \ s \ even), \\
\sum_{l=0}^{t-1} \zeta^{t-1-l} c^q_{q-1-s} \zeta^l + (-1)^{(q-s-1)/2} j c^{q-1}_{q-1} j + c^{q-1}_{q-1} & (for \ q \ even, \ s \ odd), \\
-\sum_{l=0}^{t-1} \zeta^{t-1-l} c^q_{q-1-s} \zeta^l + (-1)^{(q-s-1)/2} j c^{q-1}_{q-1} j - c^{q-1}_{q-1} & (for \ q \ odd, \ s \ even), \\
(\zeta c^q_{q-1} - c^q_{q-1} \zeta + (-1)^{(q-s)/2} \zeta j c^{q-1}_{q-1} j + c^{q-1}_{q-1}) & (for \ q \ odd, \ s \ odd). \end{cases}
\]

**Theorem 1.** The above \((Y, \pi, \delta)\) is a \(\Gamma^e\)-projective resolution of \(\Gamma\).

**Proof.** By the direct calculations, we have \(\pi \cdot \delta_1 = 0\) and \(\delta_q \cdot \delta_{q+1} = 0\) \((q \geq 1)\).

To see that the complex \((Y, \pi, \delta)\) is acyclic, we state a contracting homotopy. In general, it suffices to define the homotopy as an abelian group homomorphism. However, we can see that there exists a homotopy as a right \(\Gamma\)-module, which permits us to cut down the number of cases. We define right \(\Gamma\)-homomorphisms \(T_{-1} : \Gamma \to Y_0\) and \(T_q : Y_q \to Y_{q+1}\) \((q \geq 0)\) as follows:

\[T_{-1}(\gamma) = c^1_0 \gamma \quad (for \ \gamma \in \Gamma).\]

If \(q(\geq 0)\) is even, then

\[
T_q(\zeta^k c^q_s) = \begin{cases} 0 & (k = 0, \ s = 1), \\
\sum_{l=0}^{k-1} \zeta^{k-1-l} c^1_{q+1} \zeta^l & (1 \leq k < t, \ s = 1), \\
0 & (s(\geq 2) \ even), \\
-\zeta^k c^s_{q+1} & (s(\geq 3) \ odd), \end{cases}
\]

\[
T_q(\zeta^k j c^q_s) = \begin{cases} (-1)^{q/2} c^2_{q+1} j & (k = 0, \ s = 1), \\
(-1)^{q/2} \left( \sum_{l=0}^{k-1} \zeta^{k-1-l} c^1_{q+1} \zeta^l j + \zeta^k c^2_{q+1} j \right) & (1 \leq k < t, \ s = 1), \\
\zeta^k j c^s_{q+1} & (s(\geq 2) \ even), \\
0 & (s(\geq 3) \ odd). \end{cases}
\]
If \( q \geq 1 \) is odd, then

\[
T_q(c^k c^s_q) = \begin{cases} 
0 & (0 \leq k \leq t-2, \ s = 1), \\
c^1_{q+1} & (k = t-1, \ s = 1), \\
0 & (s \geq 2 \text{ even}), \\
-\zeta^k c^s_{q+1} & (s \geq 3 \text{ odd}), 
\end{cases}
\]

\[
T_q(\zeta^k j c^s_q) = \begin{cases} 
(-1)^{(q-1)/2}(c^1_{q+1} j \zeta + \zeta^{t-1} c^2_{q+1} j \zeta) & (k = 0, \ldots \ (1 \leq k < t, \ s = 1), \\
\zeta^k j c^s_{q+1} & (s \geq 2 \text{ even}), \\
0 & (s \geq 3 \text{ odd}). 
\end{cases}
\]

Then by the direct calculations, we have

\[
\delta_{q+1} T_q + T_{q-1} \delta_q = \text{id}_{Y_q}
\]

for \( q \geq 0 \). Hence \((Y, \pi, \delta)\) is a \( \Gamma^e \)-projective resolution of \( \Gamma \).

2 Hochschild cohomology \( HH^*(\Gamma) \)

2.1 Module structure

In this section, we give the module structure of \( HH^*(\Gamma) \). This is obtained by using the \( \Gamma^e \)-projective resolution \((Y, \pi, \delta)\) of \( \Gamma \) stated in Theorem 1. In the following we denote a direct sum of \( q \) copies of a module \( M \) by \( M^q \).

First, we state the following lemma:

**Lemma 1.** Let \( \zeta \) be a primitive \( 2^{r+1} \)-th root of 1 for any positive integer \( r \geq 2 \) and \( K \) the maximal real subfield \( \mathbb{Q}(\zeta + \zeta^{-1}) \) of \( \mathbb{Q}(\zeta) \). Then \( (\zeta + \zeta^{-1})^2 \) divides 2 in \( R \), where \( R \) denotes \( \mathbb{Z}[\zeta + \zeta^{-1}] \), the ring of integers of \( K \).

**Proof.** See [4, Lemma 1]. Note that \( \zeta^{2^k} + \zeta^{-2^k} \) divides 2 in \( R \) for \( 0 \leq k \leq r - 2 \).

If \( r \geq 2 \), we set \( \eta_k = 2e/(\zeta^{2^k} + \zeta^{-2^k}) \) for \( 0 \leq k \leq r - 2 \) in the following. Let \( \eta = \eta_0 \).

In the following, we show that \( e - \eta^2 \) is an unit in \( R \). If \( r = 2 \), then we have \( e - \eta^2 = -e \).

If \( r \geq 3 \), then we have

\[
-(e - \eta^2) \prod_{k=1}^{r-2} (e + \eta_k)^2 = -(e - \eta_{r-2}^2) = e,
\]

because the equation \((e - \eta^2_{k-1})(e + \eta_k)^2 = e - \eta^2_k \) holds for \( 1 \leq k \leq r - 2 \). Therefore \( e - \eta^2 \) is an unit in \( R \).

As elements of \( \Gamma^{q+1} \), we set

\[
t^s_q = \begin{cases} 
(0, \ldots, 0, \delta, 0, \ldots, 0) & (\text{if } 1 \leq s \leq q + 1), \\
0 & (\text{otherwise}).
\end{cases}
\]
Then we have $\Gamma^{q+1} = \bigoplus_{k=1}^{q+1} \Gamma \iota_{q}^{k}$.

Applying the functor $\text{Hom}_{\Gamma^\epsilon}(-, \Gamma)$ to the resolution $(Y, \pi, \delta)$, we have the following complex, where we identify $\text{Hom}_{\Gamma^\epsilon}(Y_{q}, \Gamma)$ with $\Gamma^{q+1}$ using an isomorphism $\text{Hom}_{\Gamma^\epsilon}(Y_{q}, \Gamma) \rightarrow \Gamma^{q+1}; f \mapsto \sum_{k=1}^{q+1} f(\iota_{q}^{k}) \iota_{q}^{k}$:

$$(\text{Hom}_{\Gamma^\epsilon}(Y, \Gamma), \delta^\#) : 0 \rightarrow \Gamma \rightarrow \Gamma^2 \rightarrow \Gamma^3 \rightarrow \Gamma^4 \rightarrow \Gamma^5 \rightarrow \cdots,$$

where $\delta^\#_{q+1}(\gamma \iota_{q}^{s}) = \begin{cases} 
- \sum_{l=0}^{t-1} \zeta^{t-1-l} \gamma \iota_{q+1}^{s} + ((-1)^{(q-s)/2} \zeta j \gamma j \zeta + \gamma) \iota_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ even,} \\
- \sum_{l=0}^{t-1} \zeta^{t-1-l} \gamma \iota_{q+1}^{s} + ((-1)^{(q-s-1)/2} j \gamma j \zeta - \gamma) \iota_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ odd,} \\
- \zeta j \gamma j \zeta - \gamma \iota_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ even,} \\
- \zeta j \gamma j \zeta - \gamma \iota_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ odd.} 
\end{cases}$

In the above, note that

$$\gamma \iota_{q}^{s} = \begin{cases} 
(0, \ldots, 0, \check{\gamma}, 0, \ldots, 0) & (\text{if } 1 \leq s \leq q+1), \\
0 & (\text{otherwise}), 
\end{cases}$$

for $\gamma \in \Gamma$, and so on.

**Theorem 2.** (1) If $r = 1$, the $\mathbb{Z}$-module structure of $HH^{n}(\Gamma)$ is given as follows:

(i) If $n = 0$, then $HH^{0}(\Gamma) = \mathbb{Z}$.

(ii) If $n = 1$, then $HH^{1}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^3$ with generators $\zeta j \iota_{1}^{1}$, $j \iota_{1}^{1} + \zeta j \iota_{1}^{2}$, $\zeta \iota_{1}^{2}$.

(iii) If $n = 2$, then $HH^{2}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^5$ with generators $\zeta \iota_{2}^{1}$, $\iota_{2}^{1} + \zeta \iota_{2}^{2}$, $j \iota_{2}^{2} - j \iota_{2}^{3}$, $\iota_{2}^{3}$.

(iv) If $n = 3$, then $HH^{3}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^7$ with generators $j \iota_{3}^{1}$, $\zeta j \iota_{3}^{1} - j \iota_{3}^{2}$, $\iota_{3}^{2}$, $\zeta \iota_{3}^{2} - \iota_{3}^{3}$, $\zeta \iota_{3}^{3}$, $\iota_{3}^{3} + \zeta \iota_{3}^{4}$, $\zeta \iota_{3}^{2}$.

(v) If $n = 4k$ ($k \neq 0$), then $HH^{n}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ with generators

$$\iota_{n}^{4l+1}, \zeta \iota_{n}^{4l+1} - \iota_{n}^{4l+2}, \zeta j \iota_{n}^{4l+2}, j \iota_{n}^{4l+2} + \zeta j \iota_{n}^{4l+3}, \iota_{n}^{4l+3}, \zeta \iota_{n}^{4l+3} + \zeta j \iota_{n}^{4l+4},$$

where $l = 0, 1, 2, \ldots, k - 1$.

(vi) If $n = 4k + 1$ ($k \neq 0$), then $HH^{n}(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ with generators

$$\zeta j \iota_{n}^{4l+1}, \iota_{n}^{4l+1} + j \iota_{n}^{4l+2}, \iota_{n}^{4l+2} + \zeta \iota_{n}^{4l+3} + \zeta \iota_{n}^{4l+4} + j \iota_{n}^{4l+3} + \zeta \iota_{n}^{4l+4},$$

where $l = 0, 1, 2, \ldots, k$ and $m = 0, 1, 2, \ldots, k - 1$. 
(vii) If \( n = 4k + 2 \) \( (k \neq 0) \), then \( HH^n(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1} \) with generators
\[
\zeta_{l+1}^{4l+1}, \zeta_{l+1}^{4l+2}, j_{l+1}^{4l+2}, \zeta_{l+1}^{4l+3}, j_{l+1}^{4l+3} - j_{l+1}^{4l+3}, \zeta_{l+1}^{4m+3} - j_{l+1}^{4m+4}, \zeta_{l+1}^{4m+4}, j_{l+1}^{4m+4} + \zeta_{l+1}^{4m+5},
\]
where \( l = 0, 1, 2, \ldots, k \) and \( m = 0, 1, 2, \ldots, k - 1 \).

(viii) If \( n = 4k + 3 \) \( (k \neq 0) \), then \( HH^n(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1} \) with generators
\[
j_{l+1}^{4l+1}, \zeta_{l+1}^{4l+1} - j_{l+1}^{4l+2}, \zeta_{l+1}^{4l+2}, \zeta_{l+1}^{4l+3} - \zeta_{l+1}^{4l+3}, j_{l+1}^{4m+4} + \zeta_{l+1}^{4m+5},
\]
where \( l = 0, 1, 2, \ldots, k \) and \( m = 0, 1, 2, \ldots, k - 1 \).

(2) If \( r \geq 2 \), the \( R \)-module structure of \( HH^n(\Gamma) \) is as follows:

(i) If \( n = 0 \), then \( HH^0(\Gamma) = R \).

(ii) If \( n = 1 \), then \( HH^1(\Gamma) = (R/(\zeta + \zeta^{-1})R)^3 \) with generators \((j - \eta \zeta j)\zeta_1^1\), \((\zeta j - \eta j)\zeta_1^1 + (j - \eta \zeta j)\zeta_1^1\), \((e - \eta \zeta)\zeta_1^1\).

(iii) If \( n = 2 \), then \( HH^2(\Gamma) = R/2^R \oplus (R/(\zeta + \zeta^{-1})R)^4 \), where the \( R/2^R \) summand is generated by \((e - \eta \zeta)\zeta_1^3\) and the \((R/(\zeta + \zeta^{-1})R)^4 \) summands are generated by \(2^{-1} \eta \zeta_2^2 + \zeta_2^3, j_2^3, \zeta_2^3 - j_2^3, \zeta_2^3\).

(iv) If \( n = 3 \), then \( HH^3(\Gamma) = (R/(\zeta + \zeta^{-1})R)^7 \) with generators \(j_3^1, \zeta j_3^1 - j_3^2, \zeta_3^2, 2^{-1} \eta \zeta_3^2 + (\zeta - \eta)\zeta_3^2, (j - \eta \zeta j)\zeta_3^2, (\zeta j - \eta j)\zeta_3^2\).

(v) If \( n = 4k \) \( (k \neq 0) \), then \( HH^n(\Gamma) = R/2^R \oplus (R/(\zeta + \zeta^{-1})R)^{2n} \), where the \( R/2^R \) summand is generated by \(\eta \zeta_1^n\) and the \((R/(\zeta + \zeta^{-1})R)^{2n} \) summands are generated by
\[
2^{-1} \eta \zeta_1^n + (\zeta - \eta)\zeta_1^n, (j - \eta \zeta j)\zeta_1^n, \zeta_1^n - j_1^n, (e - \eta \zeta)\zeta_1^n.
\]

(vi) If \( n = 4k + 1 \) \( (k \neq 0) \), then \( HH^n(\Gamma) = (R/(\zeta + \zeta^{-1})R)^{2n+1} \) with generators
\[
(j - \eta \zeta j)\zeta_1^n + (\zeta j - \eta j)\zeta_1^n + (j - \eta \zeta j)\zeta_1^n, (e - \eta \zeta)\zeta_1^n + (j - \eta \zeta j)\zeta_1^n, 2^{-1} \eta \zeta_1^n + (\zeta - \eta)\zeta_1^n, j_1^n + (j - \eta \zeta j)\zeta_1^n,
\]
where \( l = 0, 1, 2, \ldots, k \) and \( m = 0, 1, 2, \ldots, k - 1 \).

(vii) If \( n = 4k + 2 \) \( (k \neq 0) \), then \( HH^n(\Gamma) = R/2^R \oplus (R/(\zeta + \zeta^{-1})R)^{2n} \), where the \( R/2^R \) summand is generated by \((e - \eta \zeta)\zeta_1^n\) and the \((R/(\zeta + \zeta^{-1})R)^{2n} \) summands are generated by
\[
2^{-1} \eta \zeta_1^n + (\zeta - \eta)\zeta_1^n, j_1^n + (j - \eta \zeta j)\zeta_1^n, \zeta_1^n + (\zeta - \eta)\zeta_1^n, 2^{-1} \eta \zeta_1^n + (\zeta - \eta)\zeta_1^n, (j - \eta \zeta j)\zeta_1^n + (j - \eta \zeta j)\zeta_1^n, (e - \eta \zeta)\zeta_1^n,
\]
where \( l = 0, 1, 2, \ldots, k \) and \( m = 0, 1, 2, \ldots, k - 1 \).
(viii) If \( n = 4k + 3 \) (\( k \neq 0 \)), then \( HH^n(\Gamma) = (R/(\zeta + \zeta^{-1})R)^{2n+1} \) with generators
\[
\delta_{n+4l+1} j_i^n, \quad \delta_{n+4l+2} j_i^n - \delta_{n+4l+2}, \quad \delta_{n+4l+3}, \quad 2^{r-1} \delta_{n+4l+2} + \delta_{n+4l+3}, \quad (j - \eta \zeta) j_i^n, \quad (j - \eta \zeta) j_i^n + (j - \eta \zeta) j_i^n,
\]
where \( l = 0, 1, 2, \ldots, k \) and \( m = 0, 1, 2, \ldots, k - 1 \).

**Proof.** The proof is straightforward. However it is complicated.

\( \square \)

## 2.2 Ring structure

In this subsection, we will determine the ring structure of the Hochschild cohomology ring \( HH^*(\Gamma) \).

Recall the Yoneda product in \( HH^*(\Gamma) \). Let \( \alpha \in HH^n(\Gamma) \) and \( \beta \in HH^m(\Gamma) \), where \( \alpha \) and \( \beta \) are represented by cocycles \( f_{\alpha} : Y_n \to \Gamma \) and \( f_{\beta} : Y_m \to \Gamma \), respectively. There exists the commutative diagram of \( \Gamma^e \)-modules:

\[
\cdots \delta_{n+4l+1} Y_{n+m} \xrightarrow{\delta_{n+4l+2}} \cdots \delta_{n+4l+2} Y_{n+m} \xrightarrow{\delta_{n+4l+3}} Y_{n+m} \xrightarrow{f_{\beta}} \Gamma \]

where \( \mu_l (0 \leq l \leq n) \) are liftings of \( f_{\beta} \). We define the product \( \alpha \cdot \beta \in HH^{n+m}(\Gamma) \) by the cohomology class of \( f_{\alpha} \mu_n \). This product is independent of the choice of representatives \( f_{\alpha} \) and \( f_{\beta} \), and liftings \( \mu_l (0 \leq l \leq n) \).

First, we consider the case \( r = 1 \). Note the Hochschild cohomology ring \( HH^*(\Gamma) \) is graded-commutative. From Theorem 2 (1), \( HH^*(\Gamma) \) is a commutative ring in this case.
We take generators of \( HH^1(\Gamma) \) as follows:

\[
A = \zeta j_1^2, \quad B = \zeta j_1^2, \quad C = j_1^2 + \zeta j_1^2.
\]

Then we have \( 2A = 2B = 2C = 0 \). We calculate the Yoneda products. Then \( HH^n(\Gamma) (n \geq 2) \) is multiplicatively generated by \( A, B \) and \( C \), and the equation \( A^2 + B^2 + C^2 = 0 \) holds. Moreover the relations are enough. Thus we can determine the ring structure of \( HH^*(\Gamma) \) in the case \( r = 1 \) (see [3, Section 3.1] for details).

Next, we consider the case \( r \geq 2 \). The computation is similar to the case where \( r = 1 \), however it is more complicated. By Theorem 2 (2), we take generators of \( HH^1(\Gamma) \) as follows:

\[
A = (e - \eta \zeta) j_1^2, \quad B = (j - \eta \zeta j) j_1^2, \quad C = (\zeta j - \eta j) j_1 + (j - \eta \zeta j) j_1^2.
\]

Then we have \( (\zeta + \zeta^{-1})A = (\zeta + \zeta^{-1})B = (\zeta + \zeta^{-1})C = 0 \). Note that products of \( A, B, C \) and \( X \in HH^n(\Gamma) (n \geq 0) \) are commutative, because \( HH^*(\Gamma) \) is graded-commutative and the equations \( 2A = 2B = 2C = 0 \) hold. By calculating the Yoneda products we have the following proposition.
Proposition 2. If \( r \geq 2 \), then the following equations hold in \( HH^2(\Gamma) \):

\[
\begin{align*}
A^2 &= j \iota_2^3, \\
AB &= j \iota_2^3, \\
AC &= \zeta j \iota_2^3 - j \iota_2^3, \\
B^2 &= 2^{r-1} \eta \zeta \iota_2^1 + \zeta \iota_2^2, \\
BC &= 2^{r-1} \eta (e - \eta \zeta) \iota_2^1, \\
C^2 &= 2^{r-1} \eta \zeta \iota_2^1 + \zeta \iota_2^2 + \iota_2^3.
\end{align*}
\]

In particular, generators of \( HH^2(\Gamma) \) except \( (e - \eta \zeta) \iota_2^1 \) are generated by the products of \( A, B \) and \( C \), and the equation \( A^2 + B^2 + C^2 = 0 \) holds.

In the following, we put \( D = (e - \eta \zeta) \iota_2^1 \) which is a generator of \( HH^2(\Gamma) \), and then we have \( 2^r D = 0 \) and \( BC = 2^{r-1} \eta D \). Similarly, we calculate the Yoneda products. Then \( HH^n(\Gamma) \ (n \geq 3) \) is multiplicatively generated by \( A, B, C \) and \( D \), and the relations are enough. Thus we can determine the ring structure of \( HH^*(\Gamma) \) in the case \( r \geq 2 \) (see [3, Section 3.2] for details).

We state the ring structure of the Hochschild cohomology ring \( HH^*(\Gamma) \) by summarizing these computations.

Theorem 3. (1) If \( r = 1 \), then the Hochschild cohomology ring \( HH^*(\Gamma) \) is isomorphic to

\[
\mathbb{Z}[A, B, C]/(2A, 2B, 2C, A^2 + B^2 + C^2),
\]

where \( \deg A = \deg B = \deg C = 1 \).

(2) If \( r \geq 2 \), then the Hochschild cohomology ring \( HH^*(\Gamma) \) is isomorphic to

\[
R[A, B, C, D]/((\zeta + \zeta^{-1})A, (\zeta + \zeta^{-1})B, (\zeta + \zeta^{-1})C, 2^r D, \\
A^2 + B^2 + C^2, BC - 2^{r-1} \eta D),
\]

where \( R = \mathbb{Z}[\zeta + \zeta^{-1}], \deg A = \deg B = \deg C = 1 \) and \( \deg D = 2 \).

Remark. In the case \( r = 1 \), this cohomology ring is already known by Sanada [8, Section 3.4]. In [8], he treats the Hochschild cohomology of crossed products over a commutative ring and its product structure using a spectral sequence of a double complex. As a special case, he determines the Hochschild cohomology ring of the quaternion algebra over \( \mathbb{Z} \).

References


