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Mackey functor and cohomology of finite groups

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1 Composition factors of Mackey functors

Let G be a finite group and k a field of characteristic $p>0$. P. Symonds [4] determined the comoposition factors of the cohomology as a cohomological Mackey functor for G. We consider more deteiled structure of $H^{*}(-, k)$.

First, we recall the definition of the Mackey functor for G.

Definition A cohomological Mackey functor M for G is the following specification.
• k-vector space $M(H)$ for $H\leq G$.
• k-linear map

$$I_{K}^{H}: M(K)\longrightarrow M(H)$$
$$R_{K}^{H}: M(H)\longrightarrow M(K)$$
$$c_{g}: M(H)\longrightarrow M({}^{g}H)$$

for $K\leq H\leq G$, $g\in G$ such that

(i) I_{H}^{H}, R_{H}^{H}, c_{h} $(h\in H)$: identity maps on $M(H)$.
(ii)

$$I_{K}^{H}I_{J}^{K}=I_{J}^{H},\ R_{J}^{K}R_{K}^{H}=R_{J}^{H},\ c_{g}c_{h}=c_{gh}$$
$$I_{{}^{g}K}^{{}^{g}H}c_{g}=c_{g}I_{K}^{H},\ R_{{}^{g}K}^{{}^{g}H}c_{g}=c_{g}R_{K}^{H}$$

for $J\leq K\leq H\leq G$, $g, h\in G$.
(iii)

$$R_{J}^{H}I_{K}^{H}=\sum_{x\in J\backslash H/K}I_{J\cap{}^{x}K}^{J}R_{J\cap{}^{x}K}^{{}^{x}K}c_{x}$$

for $J, K\leq H\leq G$.
(iv)

$$I_{K}^{H}R_{K}^{H}=|H:K|$$

for $K\leq H$.

A global Mackey functor is a functor defined on all finite groups. Its restriction to a finite group G is a Mackey functor for G (see [7]).

Example (1) Let M be a kG-module. Then $H^n(-, M)$ is a cohomological Mackey functor for G.
(2) $H^n(-, k)$ is a global cohomological Mackey functor.

We can consider simple (global) Mackey functors and composition factors of Mackey functors. Simple cohomological Mackey functors for G are classified by Yoshida [8], Thévenaz-Webb [5]. They are parameterized by the pairs (P, V), where P is a p-subgroup of G and V is a simple $kN_G(P)/P$-module (up to iso. and conjugation). Let $S^G_{P,V}$ be the simple cohomological Mackey functor corresponding to the pair (P, V).

On the other hand, simple cohomological global Mackey functors are classified by Webb [7]. They are parameterized by the pairs (P, V), where P is a p-group and V is a simple $k(\mathrm{Out}(P))$-module (up to iso.).

Symonds [4] determines the composition factors of $H^*(-, k)$ as a (global) Mackey functor.

Theorem 1.1 ([4]) $H^*(-, k)$ contains every simple global cohomological Mackey functor as a composition factor.

Let P be a p-subgroup of G. If V be a simple $k(N_G(P)/PC_G(P))$- module, then $N_G(P)/PC_G(P)$ is a subgroup of $\mathrm{Out}(P)$ and there exists a simple $k(\mathrm{Out}(P))$-module W such that the restricton of W to $N_G(P)/PC_G(P)$ contains V as a composition factor. So we have the following corollary.

Corollary 1.2 Let P be a p-subgroup of G and V a simple $k(N_G(P)/P)$-module. Then $H^*(-, k)$ contains the simple cohomological Mackey functor $S^G_{P,V}$ (for G) as a composition factor if and only if $C_G(P)$ acts trivially on V.

Remark (1) For Theorem 1.1, we need a result from topology :

$$A(P, P) \longrightarrow \{(BP_+)^\wedge_p, (BP_+)^\wedge_p\} \otimes k \longrightarrow \mathrm{End}(H^*(P, k))$$

has nilpotent kernel, see [2].
(2) For Corollary 1.2, we have algebraic proofs (see [1], [3]).

2 Indecomposable direct summands of cohomology as a Mackey functor

Let G be a finite group. Cohomological Mackey functors for G are equivalent to the modules for a certain finite dimensional algebra, called cohomological Mackey algebra (see [6], [8]). We consider indecomposable direct summands of $H^n = H^n(-, k)$

as a cohomological Mackey functor for G.

Example 2.1 (1) Let G be a cyclic p-group. Then

$$H^n \simeq H^{n+2}$$

for $n > 0$.

(2) Assume that $p = 2$. Let G be a cyclic group of order 4. Then every conjugation is trivial and H^n is a module for the finite dimensional algebra Λ defined by the following quiver and relations :

$$0 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 1, \qquad \alpha\beta = \beta\alpha = 0.$$

Then,

$$H^{2n-1} \simeq \begin{pmatrix} S_1 \\ S_0 \end{pmatrix}, \qquad H^{2n} \simeq \begin{pmatrix} S_0 \\ S_1 \end{pmatrix}$$

for $n > 0$, where S_i is a simple Λ-module corresponding to the vertex i.

Example 2.2 Let $p = 2$ and $G = C_2 \times C_2$. Then every conjugation and every transfer are trivial, so H^n $(n > 0)$ is a module for the path algebra kQ,

$$Q = \begin{array}{ccccc} & & 0 & & \\ & \swarrow & \downarrow & \searrow & \\ 1 & & 2 & & 3 \end{array}.$$

The cohomology algebra $H^*(G, k)$ is a polynomial algebra

$$H^*(G, k) = k[x_1, x_2]$$

where $\deg x_i = 1$. Let H_j $(j = 1, 2, 3)$ be the subgroups of order 2. Then

$$H^*(H_j, k) = k[y_j]$$

where $\deg y_i = 1$. The restrictions from G to H_j are as follows :

$$\begin{array}{ccccccccc} & & & x_1 & & & & x_2 & \\ H^1 & & \swarrow & & \searrow & & \swarrow & & \searrow \\ & y_1 & & & & y_2 & & & y_3 \end{array}$$

$$\begin{array}{cccc} & x_1^2 + x_1x_2 & x_1x_2 & x_1x_2 + x_2^2 \\ H^2 & \downarrow & \downarrow & \downarrow \\ & y_1^2 & y_2^2 & y_3^2 \end{array}$$

$$\begin{array}{cccccc}
 & x_1^3 + x_1^2x_2 & x_1^2x_2 & x_1x_2^2 + x_2^3 & x_1^2x_2 + x_1x_2^2 & \\
H^3 & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & y_1^3 & y_2^3 & y_3^3 & 0 &
\end{array}$$

$$\begin{array}{cccccc}
 & x_1^4 + x_1^3x_2 & x_1^3x_2 & x_1x_2^3 + x_2^4 & x_1^3x_2 + x_1^2x_2^2 & x_1^2x_2^2 + x_1x_2^3 \\
H^4 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & y_1^4 & y_2^4 & y_3^4 & 0 & 0
\end{array}$$

$$\cdots\cdots$$

Let S_i $(0 \leq i \leq 3)$ be simple kQ-modules. We have the following kQ-module structure of H^n.

$$H^1 \simeq \begin{pmatrix} \;\; S_0 \;\; S_0 \;\; \\ S_1 \; S_2 \; S_3 \end{pmatrix}$$

$$H^2 \simeq \begin{pmatrix} S_0 \\ S_1 \end{pmatrix} \bigoplus \begin{pmatrix} S_0 \\ S_2 \end{pmatrix} \bigoplus \begin{pmatrix} S_0 \\ S_3 \end{pmatrix}$$

$$H^3 \simeq H^2 \bigoplus S_0$$

$$H^4 \simeq H^2 \bigoplus S_0 \bigoplus S_0$$

$$\cdots\cdots$$

Definition Let Λ be a finite dimensional k-algebra and M_n $(n \geq 0)$ finitely generated Λ-modules. Let $\mathrm{Ind}(\bigoplus M_n)$ be the set of isomorphism classes of the indecomposable direct summands of M_n $(n \geq 0)$. Namely

$$\mathrm{Ind}(\bigoplus M_n) = \{ \text{ indec. direct summands of } \bigoplus M_n \} / \simeq .$$

Remark To show that $\mathrm{Ind}(\bigoplus H^n)$ is a finite set for a finite group G, we may assume that k is a finite field.

The fact that $\mathrm{Ind}(\bigoplus H^n)$ is a finite set for the elementary abelian 2-group of order 4 (Example 2.2) is explained by the following Lemma.

Lemma 2.3 Let k be a finite field and Λ a finite dimensional k-algebra. Let $N_n \subseteq M_n$, $(n \geq 0)$ be finitely generated Λ-modules. Suppose that $\mathrm{Ind}(\bigoplus(M_n/N_n))$ is finite and there is $d > 0$ such that $\dim N_n \leq d$ for any n. Then $\mathrm{Ind}(\bigoplus M_n)$ is a finite set.

Using this Lemma and its dual, we have the following.

Example 2.4 Let $G = C_p \times C_p$ or $G = C_p \times C_p \times C_p$. Then $\mathrm{Ind}(\bigoplus H^n)$ is a finite set.

In this example, we do not know the explicit structure of H^n. On the other hand, for elementary abelian p-groups of arbitrary rank, we do not know even whether $\mathrm{Ind}(\bigoplus H^n)$ is a finite set or not.

Question If G is an elementary abelian p-group, then $\mathrm{Ind}(\bigoplus H^n)$ is a finite set?

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