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Inverse eigenvalue problems for nonlinear ordinary differential equations

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1 Introduction

We consider the following problem

$$-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I, \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(0) = u(1) = 0, \tag{1.3}$$

where $I := (0, 1)$ and $\lambda > 0$ is a parameter. We assume the following conditions.

(**A.1**) $f(u)$ is a function of C^1 for $u \geq 0$ satisfying $f(0) = f'(0) = 0$.

(**A.2**) $g(u) := f(u)/u$ is strictly increasing for $u \geq 0$ ($g(0) := 0$).

(**A.3**) $g(u) \to \infty$ as $u \to \infty$.

The typical examples of $f(u)$ are as follows.

$$\begin{aligned} f(u) &= u^p \qquad (p > 1), \\ f(u) &= u^p \log(u+1) \qquad (p > 1), \\ f(u) &= u^p \left(1 - \frac{1}{1+u^q}\right) \qquad (p > 1, q > 1), \\ f(u) &= u^2 \left(1 - \frac{u-4}{2} e^{-u}\right), \\ f(u) &= u^p e^u \qquad (p > 1). \end{aligned}$$

The equation (1.1)–(1.3) has been studied by many authors. We refer to the papers in the references. We know that for any given $\alpha > 0$, there exists a unique solution pair of (1.1)–(1.3) $(\lambda, u) = (\lambda(\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{I})$ such that $\|u_\alpha\|_2 = \alpha$. Moreover, the set $\{(\lambda(\alpha), u_\alpha) : \alpha > 0\}$ gives all solutions of (1.1)–(1.3), which is an unbounded C^1-bifurcation curve emanating from $(\pi^2, 0)$ in $\mathbf{R}_+ \times L^2(I)$, and $\lambda(\alpha)$ is C^1 and strictly increasing for $\alpha > 0$. We know that for any given $\lambda > \pi^2$, there exists a unique solution $u_\lambda \in C^2(\bar{I})$. Further, for $\lambda \gg 1$,

$$\lambda = g(\|u_\lambda\|_\infty) + O(1). \tag{1.4}$$

For instance, let $f(u) = u^p$. Then since $g(u) = f(u)/u = u^{p-1}$, for $\lambda \gg 1$,

$$\lambda = \|u_\lambda\|_\infty^{p-1} + O(1). \tag{1.5}$$

More precisely, we know that as $\lambda \to \infty$

$$\lambda = \|u_\lambda\|_\infty^{p-1} + \lambda e^{-\sqrt{(p-1)\lambda}(1+o(1))/2}.$$

Further, we know that as $\lambda \to \infty$

$$\frac{u_\lambda(t)}{g^{-1}(\lambda)} \to 1 \tag{1.6}$$

uniformly on any compact set in I. Therefore,

$$\alpha = \|u_\alpha\|_2 = \left(\int_I g^{-1}(\lambda)^2 dt\right)^{1/2} (1 + o(1)) = g^{-1}(\lambda)(1 + o(1)).$$

Then in many cases, we have

$$\lambda(\alpha) = g(\alpha) + o(g(\alpha)). \tag{1.7}$$

For instance, if $f(u) = u^p$, then for $\alpha \gg 1$

$$\lambda(\alpha) = \alpha^{p-1} + o(\alpha^{p-1}). \tag{1.8}$$

We here consider L^2-inverse spectral problems. More precisely, it is valid that the L^2-bifurcation curve $\lambda(\alpha)$ is determined by the nonlinear term $f(u)$. Our problem here is, conversely, to investigate whether we determine $f(u)$ by the asymptotic formula for $\lambda(\alpha)$ as $\alpha \to \infty$ or not.

We know the following fact.

Theorem 1 [16]. *Let $f(u) = u^p$ $(p > 1)$. Then for any fixed $n \in \mathbf{N}_0$, as $\alpha \to \infty$:*

$$\lambda(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}} C_1^{k+2}\alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}),$$

where

$$C_1 = (p+3)\int_I \sqrt{\frac{p-1}{p+1} - s^2 + \frac{2}{p+1}s^{p+1}}ds$$

and $a_k(p)$ ($\deg a_k(p) \le k+1$) is a polynomial determined by $a_0 = 1, a_1, \cdots, a_{k-1}$.

Motivated by Theorem 1, we consider the following Problems.

Problem A. *Assume that the following asymptotic formula is valid as $\alpha \to \infty$.*

$$\lambda(\alpha) = \alpha^{p-1} + C_0\alpha^{(p-1)/2} + o(\alpha^{(p-1)/2}). \tag{1.9}$$

Then does $f(u) = u^p$ hold ?

Problem B. *Assume that the following asymptotic formula is valid as $\alpha \to \infty$.*

$$\lambda(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + \frac{1}{p-1}C_1^2 + o(1). \tag{1.10}$$

Then does $f(u) = u^p$ hold ?

Problem C. *Assume that the asymptotic formula in Theorem 1 with some $p > 1$ is valid for any $n \in N$ as $\alpha \to \infty$. Then can we conclude $f(u) = u^p$?*

Theorem 2. *For $p, q > 1$, let*

$$f(u) = u^p\left(1 - \frac{1}{1+u^q}\right).$$

(i) Assume that $(p-1)/2 < q < p+1$. Then (1.9) holds as $\alpha \to \infty$.
(ii) Assume that $p-1 < q < p+1$. Then (1.10) holds as $\alpha \to \infty$.

Theorem 3. *Assume that*

$$f(u) = u^2\left(1 - \frac{u-4}{2}e^{-u}\right).$$

Then the asymptotic formula for $\lambda(\alpha)$ in Theorem 1 with $p = 2$ holds for any $n \in \mathbf{N}$.

Therefore, unfortunately, we find that the assumptions in Problem A–C are not sufficient to obtain the desired results for L^2-inverse problems. The next problem we have to consider is to find the suitable setting for nonlinear inverse eigenvalue problems.

2 New and direct proof of Theorem 1

The proofs of Theorems 2 and 3 are variant of those used in [16]. We here introduce a new and direct proof of Theorem 1. We consider (λ, u_λ) for $\lambda \gg 1$. We put

$$\begin{aligned} R_\lambda(s) &:= 1 - s^2 - \frac{2}{p+1}\lambda^{-1}\|u_\lambda\|_\infty^{p-1}(1 - s^{p+1}), \\ S(s) &:= 1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}). \end{aligned}$$

Lemma 2.1. *For* $\lambda \gg 1$

$$\|u_\lambda\|_\infty^2 - \|u_\lambda\|_2^2 = \lambda^{-1/2}\|u_\lambda\|_\infty^2(C_2 + U_\lambda).$$

Here,

$$C_2 := 2\int_0^1 \frac{1 - s^2}{\sqrt{1 - s^2 - 2(1 - s^{p+1})/(p+1)}}ds,$$

$$U_\lambda := 2\int_0^1 \frac{(1 - s^2)(S(s) - R_\lambda(s))}{\sqrt{R_\lambda(s)}\sqrt{S(s)}(\sqrt{R_\lambda(s)} + \sqrt{S(s)})}ds.$$

Proof. For $0 \le t \le 1$,

$$\frac{d}{dt}\left[\frac{1}{2}u_\lambda'(t)^2 - \frac{1}{p+1}u_\lambda(t)^{p+1} + \frac{1}{2}\lambda u_\lambda(t)^2\right] = 0.$$

Then

$$\frac{1}{2}u_\lambda'(t)^2 - \frac{1}{p+1}u_\lambda(t)^{p+1} + \frac{1}{2}\lambda u_\lambda(t)^2 = \text{constant} = -\frac{1}{p+1}\|u_\lambda\|_\infty^{p+1} + \frac{1}{2}\lambda\|u_\lambda\|_\infty^2.$$

We put

$$M_\lambda(\theta) := \lambda(\|u_\lambda\|_\infty^2 - \theta^2) - \frac{2}{p+1}(\|u_\lambda\|_\infty^{p+1} - \theta^{p+1}).$$

Then for $0 \le t \le 1/2$,

(2.1) $$u_\lambda'(t) = \sqrt{M_\lambda(u_\lambda(t))}.$$

Then

$$\|u_\lambda\|_\infty^2 - \|u_\lambda\|_2^2 = 2\int_0^{1/2}(\|u_\lambda\|_\infty^2 - u_\lambda^2(t))\frac{u_\lambda'(t)}{\sqrt{M_\lambda(u_\lambda(t))}}dt$$

$$
\begin{aligned}
&= 2\int_0^{\|u_\lambda\|_\infty} (\|u_\lambda\|_\infty^2 - \theta^2)\frac{1}{\sqrt{M_\lambda(\theta)}}d\theta \\
&= 2\lambda^{-1/2}\|u_\lambda\|_\infty^2 \int_0^1 \frac{1-s^2}{\sqrt{R_\lambda(s)}}ds \\
&= \lambda^{-1/2}\|u_\lambda\|_\infty^2 \left(2\int_0^1 \frac{1-s^2}{\sqrt{S(s)}}ds + U_\lambda\right) \\
&= \lambda^{-1/2}\|u_\lambda\|_\infty^2 (C_2 + U_\lambda).
\end{aligned}
$$

Thus the proof is complete. ∎

Lemma 2.2. *For* $\lambda \gg 1$

$$|U_\lambda| \le C\lambda^{-1/2}e^{-\sqrt{(p-1)}(1+o(1))/(2\sqrt{\lambda})}.$$

The proof of Lemma 2.2 is long and tedious. So we omit the proof here. By using Lemmas 2.1 and 2.2, we easily obtain Theorem 1.

3 New example

In this section, we consider new example of $f(u)$. Let $f(u) = u^p e^u$ $(p > 1)$.

Theorem 4. *Assume that* $f(u) = u^p e^u$ $(p > 1)$ *in (1.1). Then as* $\alpha \to \infty$

$$\lambda(\alpha) = \alpha^{p-1}e^\alpha + \frac{\pi}{4}\alpha^{(p+1)/2}e^{\alpha/2}(1 + o(1)).$$

To prove Theorem 4, we begin with the fundamental properties of $\lambda(\alpha)$. We know

$$\frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} \le \lambda(\alpha) \le \frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} + \pi^2,$$

$$u_\alpha(t) = \|u_\alpha\|_\infty(1 + o(1)) = \alpha(1 + o(1)), \quad t \in I,$$

$$u_\alpha(t) = u_\alpha(1-t), \quad 0 \le t \le 1,$$

$$u_\alpha\left(\frac{1}{2}\right) = \max_{0\le t\le 1} u_\alpha(t) = \|u_\alpha\|_\infty,$$

$$u_\alpha'(t) > 0, \quad 0 \le t < \frac{1}{2}.$$

Lemma 3.1. *For* $\alpha \gg 1$

$$\|u_\alpha\|_\infty^2 - \alpha^2 = \frac{\pi}{2}(1+o(1))\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}}\|u_\alpha\|_\infty^2.$$

Proof. Put

$$F(u) := \int_0^u f(s)ds.$$

Then for $0 \le t \le 1$,

$$\frac{d}{dt}\left[\frac{1}{2}u_\alpha'(t)^2 - F(u_\alpha(t)) + \frac{1}{2}\lambda(\alpha)u_\alpha(t)^2\right] = 0.$$

Therefore, for $0 \le t \le 1$,

$$\frac{1}{2}u_\alpha'(t)^2 - F(u_\alpha(t)) + \frac{1}{2}\lambda(\alpha)u_\alpha(t)^2 = \text{constant} = -F(\|u_\alpha\|_\infty) + \frac{1}{2}\lambda(\alpha)\|u_\alpha\|_\infty^2.$$

We put

$$\begin{aligned} M_\alpha(\theta) &:= \lambda(\alpha)(\|u_\alpha\|_\infty - \theta^2) - 2(F(\|u_\alpha\|_\infty) - F(\theta)), \\ Q_\alpha(s) &:= \lambda(\alpha)\|u_\alpha\|_\infty^2(1-s^2) - 2(F(\|u_\alpha\|_\infty) - F(s\|u_\alpha\|_\infty))). \end{aligned}$$

Then for $0 \le t \le 1/2$

$$u_\alpha'(t) = \sqrt{M_\alpha(u_\alpha(t))}. \tag{3.1}$$

By putting $\theta := u_\alpha(t)$, $s = \theta/\|u_\alpha\|_\infty$

$$\begin{aligned} \|u_\alpha\|_\infty^2 - \alpha^2 &= 2\int_0^{1/2}(\|u_\alpha\|_\infty^2 - u_\alpha^2(t))\frac{u_\alpha'(t)}{\sqrt{M_\alpha(u_\alpha(t))}}dt \\ &= 2\int_0^{\|u_\alpha\|_\infty}(\|u_\alpha\|_\infty^2 - \theta^2)\frac{1}{\sqrt{M_\alpha(\theta)}}d\theta \\ &= 2\frac{\|u_\alpha\|_\infty^2}{\sqrt{\lambda(\alpha)}}\int_0^1 \frac{1-s^2}{\sqrt{Q_\alpha(s)/(\lambda(\alpha)\|u_\alpha\|_\infty^2)}}ds. \end{aligned}$$

Then we can show that as $\alpha \to \infty$

$$\int_0^1 \frac{1-s^2}{\sqrt{Q_\alpha(s)/(\lambda(\alpha)\|u_\alpha\|_\infty^2)}}ds \to \int_0^1 \sqrt{1-s^2}ds = \frac{\pi}{4}.$$

∎

Lemma 3.2. *For* $\alpha \gg 1$

$$\|u_\alpha\|_\infty - \alpha = \frac{\pi}{4}(1+o(1))\sqrt{\frac{\alpha}{f(\alpha)}}\alpha. \tag{3.2}$$

Proof. By Lemma 3.1, for $\alpha \gg 1$,

$$\|u_\alpha\|_\infty^2 \left(1 - \frac{\pi}{2}(1+o(1))\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}}\right) = \alpha^2.$$

By this and Taylor expansion, for $\alpha \gg 1$

$$\begin{aligned}
\|u_\alpha\|_\infty &= \alpha\left(1 - \frac{\pi}{2}(1+o(1))\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}}\right)^{-1/2} \\
&= \alpha\left(1 + \frac{\pi}{4}(1+o(1))\sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}}\right) \\
&= \alpha\left(1 + \frac{\pi}{4}(1+o(1))\sqrt{\frac{\alpha}{f(\|u_\alpha\|_\infty)}}\right).
\end{aligned}$$

For $\alpha \gg 1$, we can show

$$f(\|u_\alpha\|_\infty) = f(\alpha)(1+o(1)) \tag{3.3}$$

∎

Lemma 3.3. *For $\alpha \gg 1$*

$$f(\|u_\alpha\|_\infty) - f(\alpha) = \frac{\pi}{4}f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}}(1+o(1)).$$

Proof. For $\alpha \gg 1$, we have

$$f'(\|u_\alpha\|_\infty) = f'(\alpha)(1+o(1)) \tag{3.4}$$

Then

$$\begin{aligned}
f(\|u_\alpha\|_\infty) - f(\alpha) &= f'(\alpha_1)(\|u_\alpha\|_\infty - \alpha) \\
&\le f'(\|u_\alpha\|_\infty)(\|u_\alpha\|_\infty - \alpha) \\
&= \frac{\pi}{4}(1+o(1))f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}}, \\
f(\|u_\alpha\|_\infty) - f(\alpha) &= f'(\alpha_1)(\|u_\alpha\|_\infty - \alpha) \\
&\ge f'(\alpha)(\|u_\alpha\|_\infty - \alpha) \\
&= \frac{\pi}{4}(1+o(1))f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}}.
\end{aligned}$$

∎

Proof of Theorem 4. By Taylor expansion, for $\alpha \gg 1$

$$\begin{aligned}\lambda(\alpha) &= \frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} + O(1) \\ &= \frac{f(\alpha) + \frac{\pi}{4} f'(\alpha)\alpha\sqrt{\alpha/f(\alpha)}(1+o(1))}{\alpha(1+\frac{\pi}{4}\sqrt{\alpha/f(\alpha)}(1+o(1))} + O(1) \\ &= \frac{1}{\alpha}\left(f(\alpha) + \frac{\pi}{4} f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}}(1+o(1))\right)\left(1 - \frac{\pi}{4}\sqrt{\frac{\alpha}{f(\alpha)}}(1+o(1))\right) + O(1).\end{aligned}$$

Since for $\alpha \gg 1$,

$$f(\alpha)\sqrt{\frac{\alpha}{f(\alpha)}} \ll f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}},$$

by this, we obtain Theorem 4. ∎

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