Title
On Plane Curve Which Has Similar Caustic (Modeling and Complex analysis for functional equations)

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Citation
数理解析研究所講究録 (2008), 1582: 63-69

Issue Date
2008-02

URL
http://hdl.handle.net/2433/81454

Type
Departmental Bulletin Paper
On Plane Curve Which Has Similar Caustic

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1. What is a caustic?

A caustic is the envelope of rays reflected by a curve. For example, if we put a coffee cup on the table and we make parallel light rays on the coffee cup, then we will see a caustic on the surface of coffee. See Figure 1.

![Figure 1](image1.png)

Figure 1

The contents of this paper are as follows: In Section 2, we study how we calculate the caustic from a given curve. As examples, we show that the caustic of a half circle is an epicycloid and that the caustic of a cycloid is also a cycloid whose size is a half of the original cycloid. In Section 3, we study how we calculate the original curve from a given caustic. As an example, we show that, if the caustic is a cycloid, the original curve is also a cycloid. In Section 4, we prove that the cycloid is the unique curve whose caustic is similar to the original curve.

2. Parametrization by angle

Consider a smooth curve on $xy$-plane. Assume that light rays are parallel to the $y$-axis. Let $\theta$ be the angle between the $y$-axis and the tangent line of the curve at a point $P$. Assume that $\theta$ is increasing from 0 to $\pi$ as $P$ varies from end to end of the curve. So we can express the point $P$ by $\theta$. Let $\alpha(\theta) = (x(\theta), y(\theta))$ be a parametrization of a given curve.
How can we find the caustic from a given curve? By the definition of $\theta$, we have
\[
\frac{y'(\theta)}{x'(\theta)} = \cot \theta. \tag{1}
\]
Therefore, the equation of reflected ray from $P(x(\theta), y(\theta))$ is given by
\[
y = \cot 2\theta (x - x(\theta)) + y(\theta). \tag{2}
\]
By differentiating both sides with respect to $\theta$ and using (1), we have
\[
y'_{\theta} = -\frac{2}{\sin^2 2\theta} (x - x(\theta)) - \cot 2\theta x'(\theta) + y'(\theta)
\]
\[
= \frac{-2}{\sin^2 2\theta} (x - x(\theta)) - \frac{\cos 2\theta}{\sin 2\theta} x'(\theta) + \frac{\cos \theta}{\sin \theta} x'(\theta)
\]
\[
= \frac{-2}{\sin^2 2\theta} (x - x(\theta)) + \frac{1}{\sin 2\theta} x'(\theta).
\]
Setting $y'_{\theta} = 0$ gives the envelope. By setting $y'_{\theta} = 0$, we have
\[
x = x(\theta) + \frac{1}{2} \sin 2\theta x'(\theta) = x(\theta) + \sin \theta \cos \theta x'(\theta).
\]
By putting it to (2), we have
\[
y = y(\theta) + \frac{1}{2} \cos 2\theta x'(\theta) = y(\theta) + \frac{1}{2} \frac{\sin \theta (\cos^2 \theta - \sin^2 \theta)}{\cos \theta} y'(\theta).
\]
Therefore, if we put
Then \( \beta(\theta) = (u(\theta), v(\theta)) \) is the caustic of \( \alpha(\theta) \). By the definition of \( \theta \), we have
\[
\frac{v'(\theta)}{u'(\theta)} = \cot 2\theta.
\] (5)

**Example 1.** When \( \alpha(\theta) = (-\cos \theta, \sin \theta) \), find its caustic \( \beta(\theta) \).

**Solution.** Since \( \alpha(\theta) \) satisfies (1), we can apply our formulas to this example. By using (3) and (4), we have
\[
u(\theta) = \sin \theta + \frac{1}{2} \cos 2\theta \sin \theta = \frac{3}{4} \sin \theta + \frac{1}{4} \sin 3\theta.
\]
Thus we have \( \beta(\theta) = \left( -\frac{3}{4} \cos \theta - \frac{1}{4} \cos 3\theta, \frac{3}{4} \sin \theta + \frac{1}{4} \sin 3\theta \right) \). Therefore the caustic of a half circle is an epicycloid.

**Example 2.** When \( \alpha(\theta) = (2\theta - \sin 2\theta, 1 - \cos 2\theta) \), find its caustic \( \beta(\theta) \).

**Solution.** Since \( \alpha(\theta) \) satisfies (1), we can apply our formulas to this example. By using (3) and (4), we have
\[
u(\theta) = 1 - \cos 2\theta + \frac{1}{2} \cos 2\theta(2 - 2\cos 2\theta) = \left( 1 - \cos 2\theta \right) = \frac{1}{2}(1 - \cos 4\theta).
\]
Thus we have \( \beta(\theta) = \left( \frac{1}{2}(4\theta - \sin 4\theta), \frac{1}{2}(1 - \cos 4\theta) \right) \). Therefore the caustic of a cycloid is also a cycloid.

3. **Inverse problem**

From (3), we have
\[
x'(\theta) + \frac{1}{\sin \theta \cos \theta} x(\theta) = \frac{u(\theta)}{\sin \theta \cos \theta}
\]
The above equality is equivalent to
\[ \{x(\theta) \tan \theta\}' = \frac{u(\theta)}{\cos^2 \theta}. \] (6)

When \( 0 < \theta < \frac{\pi}{2} \), by integrating (6), we have

\[ x(\theta) \tan \theta = \int_0^\theta \frac{u(\phi)}{\cos^2 \phi} \, d\phi. \]

When \( \frac{\pi}{2} < \theta < \pi \), by integrating (6), we have

\[ -x(\theta) \tan \theta = \int_\theta^\pi \frac{u(\phi)}{\cos^2 \phi} \, d\phi. \]

Therefore we obtain

\[
\begin{cases}
  u(0) & (\theta = 0) \\
  \cot \theta \int_0^\theta \frac{u(\phi)}{\cos^2 \phi} \, d\phi & (0 < \theta < \frac{\pi}{2}) \\
  u \left( \frac{\pi}{2} \right) & (\theta = \frac{\pi}{2}) \\
  -\cot \theta \int_\theta^\pi \frac{u(\phi)}{\cos^2 \phi} \, d\phi & (\frac{\pi}{2} < \theta < \pi) \\
  u(\pi) & (\theta = \pi) 
\end{cases}
\]

(7)

**Example 3.** When \( \beta(\theta) = \left( \frac{1}{2}(4\theta - \sin 4\theta), \frac{1}{2}(1 - \cos 4\theta) \right) \), find the original curve \( \alpha(\theta) \).

**Solution.** Since \( \beta(\theta) \) satisfies (5), we can apply our formula to this example. When

0 < \( \theta < \frac{\pi}{2} \), by using (7), we have

\[
x(\theta) = \cot \theta \int_0^\theta \frac{(4\phi - \sin 4\phi)}{2\cos^2 \phi} \, d\phi
= \frac{1}{2} \cot \theta \left( \int_0^\theta \frac{4\phi}{\cos^2 \phi} \, d\phi - \int_0^\theta \frac{\sin 4\phi}{\cos^2 \phi} \, d\phi \right)
= \frac{1}{2} \cot \theta \left( \int_0^\theta \frac{4\phi}{\cos^2 \phi} \, d\phi - \int_0^\theta \frac{4\sin \phi \cos \phi (\cos^2 \phi - \sin^2 \phi)}{\cos^2 \phi} \, d\phi \right)
= \frac{1}{2} \cot \theta \left( 4\theta \tan \theta - 4 \int_0^\theta \tan \phi \, d\phi - 8 \int_0^\theta \sin \phi \cos \phi \, d\phi + 4 \int_0^\theta \tan \phi \, d\phi \right)
= \frac{1}{2} \cot \theta \left( 4\theta \tan \theta - 4\sin^2 \theta \right) = 2\theta - \sin 2\theta.
\]
When $\frac{\pi}{2} < \theta < \pi$, by using (7), we have

\[
x(\theta) = -\cot \theta \int_{\theta}^{\pi} \frac{4\phi - \sin 4\phi}{2\cos^2 \phi} d\phi
\]

\[
= -\frac{1}{2} \cot \theta \left( \int_{\theta}^{\pi} \frac{4\phi}{\cos^2 \phi} d\phi - \int_{\theta}^{\pi} \frac{4\sin \phi \cos \phi (\cos^2 \phi - \sin^2 \phi)}{\cos^2 \phi} d\phi \right)
\]

\[
= -\frac{1}{2} \cot \theta \left( -4\theta \tan \theta - 4\int_{\theta}^{\pi} \tan \phi d\phi - 8\int_{\theta}^{\pi} \sin \phi \cos \phi d\phi + 4\int_{\theta}^{\pi} \tan \phi d\phi \right)
\]

\[
= -\frac{1}{2} \cot \theta (-4\theta \tan \theta + 4\sin^2 \theta) = 2\theta - \sin 2\theta.
\]

Therefore we have $x(\theta) = 2\theta - \sin 2\theta$. By using (1), we have

\[
y'(\theta) = \cot \theta x'(\theta) = 2 \cot \theta \cdot (1 - \cos 2\theta) = 2 \sin 2\theta.
\]

Therefore we have

\[
y(\theta) = 2\int_{0}^{\theta} \sin 2\phi d\phi = 1 - \cos 2\theta.
\]

Thus we obtain $\alpha(\theta) = (2\theta - \sin 2\theta, 1 - \cos 2\theta)$.

4. **On plane curve which has similar caustic**

Example 2 says that the caustic of cycloid is also a cycloid. So a question arises: "Is there another curve which is similar to its caustic?" The following theorem is an answer of this problem.

**Theorem.** Suppose that a curve $\alpha(\theta)$ $(0 \leq \theta \leq \pi)$ with $\alpha(0) = (0, 0)$, $\alpha(\pi) = (2\pi, 0)$ has a caustic $\beta(\theta)$ which consists of two curves both similar to $\alpha(\theta)$ in ratio $\frac{1}{2}$, that is,

\[
\beta(\theta) = \begin{cases} 
\frac{1}{2} \alpha(2\theta) & (0 \leq \theta \leq \frac{\pi}{2}) \\
(\pi, 0) + \frac{1}{2} \alpha(2\theta - \pi) & (\frac{\pi}{2} \leq \theta \leq \pi), 
\end{cases}
\]

then $\alpha(\theta) = (2\theta - \sin 2\theta, 1 - \cos 2\theta)$.

**Proof.** Put $\alpha_0(\theta) = (x_0(\theta), y_0(\theta)) = (2\theta - \sin 2\theta, 1 - \cos 2\theta)$. In Example 2, we already proved that $\alpha_0(\theta)$ satisfies the assumption of the theorem. We assume that there is a curve
\( \alpha_i(\theta) = (x_i(\theta), y_i(\theta)) \) which also satisfies the assumption. Then by (7), both \( x_0(\theta) \) and \( x_1(\theta) \) satisfy

\[
x_i(\theta) = \begin{cases} 
0 & (\theta = 0) \\
\cot \theta \int_0^\theta \frac{x_i(2\phi)}{2\cos^2 \phi} \, d\phi & (0 < \theta < \frac{\pi}{2}) \\
\pi & (\theta = \frac{\pi}{2}) \\
\pi - \cot \theta \int_\theta^{\frac{\pi}{2}} \frac{x_i(2\phi - \pi)}{2\cos^2 \phi} \, d\phi & (\frac{\pi}{2} < \theta < \pi) \\
2\pi & (\theta = \pi).
\end{cases}
\]

Put \( M = \max_{0 \leq \theta \leq \pi} |x_1(\theta) - x_0(\theta)| \). Then we can calculate as follows:

\[
\sup_{0 \leq \theta \leq \frac{\pi}{2}} |x_1(\theta) - x_0(\theta)| = \sup_{0 \leq \theta \leq \frac{\pi}{2}} \left| \cot \theta \int_0^\theta \frac{x_i(2\phi)}{2\cos^2 \phi} \, d\phi - \cot \theta \int_0^\theta \frac{x_0(2\phi)}{2\cos^2 \phi} \, d\phi \right|
\leq \sup_{0 \leq \theta \leq \frac{\pi}{2}} \left\{ \cot \theta \int_0^\theta \frac{1}{2\cos^2 \phi} |x_i(2\phi) - x_0(2\phi)| \, d\phi \right\}
\leq \sup_{0 \leq \theta \leq \frac{\pi}{2}} \left\{ \cot \theta \int_0^\theta \frac{M}{2\cos^2 \phi} \, d\phi \right\} = \frac{M}{2},
\]

\[
\sup_{\frac{\pi}{2} \leq \theta \leq \pi} |x_1(\theta) - x_0(\theta)| = \sup_{\frac{\pi}{2} \leq \theta \leq \pi} \left| \pi - \cot \theta \int_\theta^{\frac{\pi}{2}} \frac{x_i(2\phi - \pi)}{2\cos^2 \phi} \, d\phi + \cot \theta \int_\theta^{\frac{\pi}{2}} \frac{x_0(2\phi - \pi)}{2\cos^2 \phi} \, d\phi \right|
\leq \sup_{\frac{\pi}{2} \leq \theta \leq \pi} \left\{ \cot \theta \int_\theta^{\frac{\pi}{2}} \frac{1}{2\cos^2 \phi} |x_i(2\phi - \pi) - x_0(2\phi - \pi)| \, d\phi \right\}
\leq \sup_{\frac{\pi}{2} \leq \theta \leq \pi} \left\{ \cot \theta \int_\theta^{\frac{\pi}{2}} \frac{M}{2\cos^2 \phi} \, d\phi \right\} = \frac{M}{2}.
\]

Therefore we have \( M \leq \max \left\{ 0, \frac{M}{2}, 0, \frac{M}{2} \right\} = \frac{M}{2} \). Thus we have \( M = 0 \), that is,

\( x_i(\theta) = x_0(\theta) \) for every \( \theta \). Since \( \frac{y_i'(\theta)}{x_i'(\theta)} = \frac{y_0'(\theta)}{x_0'(\theta)} = \cot \theta \), we have \( y_i'(\theta) = y_0'(\theta) \).

Since we have \( y_i(0) = y_0(0) \), we obtain \( y_i(\theta) = y_0(\theta) \) for every \( \theta \). Thus \( \alpha_0(\theta) \) is the only curve satisfying the assumption.

**Acknowledgement.** The author would like to thank Professor Kenzi Odani who helps him during this paper work.
References


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