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Kyoto University
Asymptotic forms of slowly decaying positive solutions of second-order quasilinear ordinary differential equations

1 Introduction

Let us consider the quasilinear ODE

\[(a(t)|u'|^{\alpha-1}u')'+b(t)|u|^\lambda u=0, \quad \text{near } +\infty\]  \hspace{1cm} (A)

where we assume that $\alpha > 0$ and $\lambda > 0$ are constants, $a(t)$ and $b(t)$ are positive continuous functions satisfying $\int^{\infty}a(t)^{-1/\alpha}dt < \infty$. Every positive solution $u$ of (A) satisfies one of the following three asymptotic properties as $t \to \infty$:

1. $u(t) \sim c_1$ for some constant $c_1 > 0$; \hspace{1cm} (1.1)
2. $u(t) \sim c_2 \int_{t}^{\infty} a(s)^{-1/\alpha}ds$ for some constant $c_2 > 0$; \hspace{1cm} (1.2)

and

\[u(t) \to 0 \quad \text{and} \quad \frac{u(t)}{\int_{t}^{\infty} a(s)^{-1/\alpha}ds} \to \infty.\]  \hspace{1cm} (1.3)

Asymptotic properties of solutions $u$ satisfying either (1.1) or (1.2) were widely investigated. For example, necessary and sufficient conditions of existence of such solutions were established in [4, 7]. On the other hand there seems to be less information about qualitative properties of solutions $u$ satisfying (1.3). Motivated by this fact, in the article we will discuss about asymptotic behavior of solutions $u$ satisfying (1.3); in particular, we try to find exact asymptotic forms of such solutions near $+\infty$. In what follows we refer solutions $u$ satisfying (1.3) as slowly decaying solutions.

Remark 1. When $\int^{\infty} a(t)^{-1/\alpha}dt = \infty$, Eq (A) reduces to the simpler one of the form

\[(|u'|^{\alpha-1}u')'+\tilde{b}(t)|u|^\lambda u=0, \quad \text{near } +\infty,\]

where $\tilde{b}(t)$ is a positive continuous function. Studies of this equation were, for example, the main objective of [6]; and asymptotic properties of solutions have been fully established
2 Preparatory observations and results

Asymptotic forms of slowly decaying solutions may be strongly affected by those of coefficient functions $a(t), b(t)$ and the exponents $\alpha$ and $\lambda$. Therefore let us consider the following ODE, which has more restrictive appearance than Eq (A):

$$(t^\beta|u'|^{\alpha-1}u')' + t^\sigma(1 + \epsilon(t))|u|^\lambda u = 0 \quad \text{near } +\infty.$$  (E)

In the sequel we assume the next conditions:

(A$_1$) $\alpha, \beta, \lambda$ and $\sigma$ are constants satisfying $\lambda > \alpha > 0$ and $\beta > \alpha$;

(A$_2$) $\epsilon(t)$ is a continuous (or $C^1$-) function defined near $+\infty$ satisfying $\lim_{t \to \infty} \epsilon(t) = 0$.

Additional conditions will be given later.

Since we can regard Eq (E) as a "perturbed equation" of the ODE

$$(t^\beta|u'|^{\alpha-1}u')' + t^\sigma|u|^\lambda u = 0 \quad \text{near } +\infty,$$  (E$_0$)

we conjecture that slowly decaying solutions of Eq (E) and those of Eq (E$_0$) may have the same asymptotic behavior near $+\infty$ in some sense, if $\epsilon(t)$ is sufficiently small. It is easily seen that Eq (E$_0$) has an exact slowly decaying solution $u_0$ given by

$$u_0(t) = \hat{C} t^{-k},$$  (2.1)

where

$$k = \frac{1 + \sigma - (\beta - \alpha)}{\lambda - \alpha}, \quad \text{and} \quad \hat{C}^{\lambda - \alpha} = k^\alpha \{\beta - \alpha(k + 1)\}$$

if

$$(\beta - \alpha) - 1 < \sigma < \frac{\lambda}{\alpha}(\beta - \alpha) - 1.$$  (2.2)

Below we always assume (2.2). We can show that our conjecture is true in various cases:

**Theorem 1.** Let $\alpha \leq 1$ and $\beta - \alpha(k + 1) - k \neq 0$. If

- either $\int \frac{\epsilon(t)^2}{t} dt < \infty$ or $\int |\epsilon'(t)| dt < \infty,$

then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$, where $u_0(t)$ is given by (2.1).

**Theorem 2.** Let $\alpha \geq 1$ and $\beta - \alpha(k + 1) - k \neq 0$. If

$$\lim_{t \to \infty} t^\epsilon(t) = 0 \quad \text{and} \quad \int |\epsilon'(t)| dt < \infty,$$  (2.4)

then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$. 
Theorem 3. Let \( \alpha \geq 1 \) and \( \alpha(2k+1)-\beta < 0 \). If (2.3) holds, then every slowly decaying positive solution \( u \) of Eq (E) satisfies \( u(t) \sim u_0(t) \) as \( t \to \infty \).

Example 1. Let \( N > m > 1 \) and \( N \geq 2 \). Consider radial solutions \( u = u(|x|) \) of the following quasilinear PDE in an exterior domain of \( \mathbb{R}^N \):

\[
\text{div}(|Du|^{m-2}Du) + |x|^{\ell}(1 + |x|^{-\theta})|u|^{\lambda-1}u = 0 \quad \text{near} \quad \infty,
\]

where \( \lambda > m - 1, \ell \in \mathbb{R}, \theta > 0, \) and \( -m < \ell < \frac{\lambda}{m-1}(N - m) - N \). We know that \( u \) solves the ODE

\[
(r^{N-1}|u'|^{m-2}u')' + r^{N-1+\ell}(1 + r^{-\rho})|u|^{\lambda-1}u = 0 \quad \text{near} \quad + \infty.
\]

By Theorems 1 and 2, if \( \lambda \neq (mN - N + m\ell)/(N - m) \), then every slowly decaying positive solution \( u \) of this equation satisfies

\[
u(r) \sim Ar^{-(\ell+m)/(\lambda-m+1)} \quad \text{as} \quad r \to +\infty,
\]

where \( A \) is a positive constant given by

\[
A^{\lambda-m+1} = \left( \frac{\ell + m}{\ell - m + 1} \right)^{m-1} \cdot \frac{N\lambda -Nm + N - m\ell - m\lambda + \ell}{\lambda - m + 1}.
\]

Remark 1. For the autonomous equation \( \text{div}(|Du|^{m-2}Du) + |u|^{\lambda-1}u = 0 \), the assertion of Example 1 was established in [1] based on the theory of autonomous dynamical systems. Related results are found in [3, 5].

3 Sketches of the proof of the results

We give the outline of the proof of Theorems 1 and 2. We begin with several auxiliary results.

Lemma 1. Let \( u(t) \) be a slowly decaying positive solution of (E). Then

\[
u(t) = O(u_0(t)) \quad \text{and} \quad u'(t) = O(|u_0'(t)|) \quad \text{as} \quad t \to \infty.
\]

(3.1)

Proof. An integration of both sides of Eq (E) on \([t_0, t]\) gives

\[
t^\beta(-u'(t))^\alpha \geq \int_{t_0}^{t} r^\sigma(1 + \epsilon(r))u^\lambda dr,
\]

where \( t_0 \) is a sufficiently large number. Since \( u \) is a decreasing function, we have

\[
t^\beta(-u'(t))^\alpha \geq u(t)^\lambda \int_{t_0}^{t} r^\sigma(1 + \epsilon(r))dr;
\]
that is,

$$-u'(t)u(t)^{-\lambda/\alpha} \geq \left( t^{-\beta} \int_{t_0}^{t} r^\sigma (1 + \epsilon(r)) dr \right)^{1/\alpha}.$$ 

One more integration of the both sides gives the estimates for $u$ in (3.1).

To get the estimates for $u'$, it suffices to notice the inequality

$$t^\beta (-u'(t))^\alpha \leq C_1 \int_{t_0}^{t} r^\sigma u(r)^\lambda dr,$$

where $C_1 > 0$ is a constant. Note that, to get this inequality, we must use the property

$$\lim_{t \to \infty} t^\beta (-u'(t))^\alpha = \infty.$$

**Lemma 2.** Let $u(t)$ be a slowly decaying positive solution of (E). Put $t = e^s$ and $u/u_0 = v$. Then

(i) $v$, and $\dot{v}$ are bounded, and $\dot{v} - kv < 0$ near $+\infty$, where $\cdot = d/ds$;

(ii) $v$ satisfies the ODE

$$(kv - \dot{v})^\alpha + \{\beta - \alpha(k + 1)\}(kv - \dot{v})^\alpha - \hat{C}^{\lambda - \alpha}\{1 + \delta(s)\} v^\lambda = 0 \text{ near } + \infty,$$

where $\delta(s) = \epsilon(e^s)$.

The proof of this lemma is based on direct computations; hence we omit it.

**Remark 2.** Equation (3.2) can be rewritten as

$$\dot{v} + \left( \frac{\beta}{\alpha} - 2k - 1 \right) \dot{v} - k \left( \frac{\beta}{\alpha} - k - 1 \right) v + \hat{C}^{\lambda - \alpha}\{1 + \delta(s)\} v^\lambda = 0. \quad (3.3)$$

**Lemma 3.** Let $f(s)$ be a $C^1$-function near $+\infty$ satisfying $f'(s) = O(1)$ as $s \to \infty$ and $\int^\infty f(s)^2 ds < \infty$. Then $\lim_{s \to \infty} f(s) = 0$.

The proof of this lemma will be found in [6].

**Proof of Theorem 1.** By the change of variables $(t, u) \mapsto (s, v)$ introduced in Lemma 2, we obtain Eq (3.2). We note that the integral conditions indicated in (2.3) are equivalent to

$$\int^{\infty} \delta(s)^2 ds < \infty \quad (3.4)$$

and

$$\int^{\infty} |\delta(s)| ds < \infty, \quad (3.5)$$

respectively.

**Step 1.** We show that $\int^{\infty} \dot{v}(s)^2 ds < \infty$. We multiply Eq (3.2) by $\dot{v}$, and integrate the resulting equation on $[s_0, s]$ to obtain

$$\int_{s_0}^{s} \{(kv - \dot{v})^\alpha\} \dot{v} dr + \{\beta - \alpha(k + 1)\} \int_{s_0}^{s} (kv - \dot{v})^\alpha \dot{v} dr$$
\[
-\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1}v^{\lambda+1} - \hat{C}^{\lambda-\alpha}\int_{s_0}^{s}\delta(r)v^\lambda\dot{v}dr = \text{const.}
\] (3.6)

Since integral by parts implies that
\[
\int_{s_0}^{s}\{(kv - \dot{v})^\alpha\}\dot{v}dr = -\int_{s_0}^{s}\{(kv - \dot{v})^\alpha\}(kv - \dot{v})dr + k\int_{s_0}^{s}\{(kv - \dot{v})^\alpha\}\dot{v}dr
\]
we obtain from (3.6)
\[
-\frac{\alpha}{\alpha+1}(kv - \dot{v})^{\alpha+1} + kv(kv - \dot{v})^\alpha - k\int_{s_0}^{s}(kv - \dot{v})^\alpha\dot{v}dr + \text{const},
\]
The boundedness of \(v\) and \(\dot{v}\) shown in Lemma 2 imply that
\[
\{\beta - \alpha(k+1) - k\}\int_{s_0}^{s}(kv - \dot{v})^\alpha\dot{v}dr - \hat{C}^{\lambda-\alpha}\int_{s_0}^{s}\delta(r)v^\lambda\dot{v}dr = O(1)
\] as \(s \to \infty\). (3.7)

Now, since \(0 < \alpha \leq 1\), the inequality
\[
(X^\alpha - Y^\alpha)(X - Y) \geq c_0(X - Y)^2(X + Y)^{\alpha-1}
\]
for all \(X,Y \geq 0\) with \(X + Y > 0\) holds for some constant \(c_0 > 0\). Therefore we obtain
\[
\{(kv)^\alpha - (kv - \dot{v})^\alpha\}\dot{v} \geq c_0((kv) + (kv - \dot{v}))^{\alpha-1}\dot{v}^2;
\]
that is,
\[
(kv - \dot{v})^\alpha\dot{v} \leq -c_1\dot{v}^2 + k^\alpha v^\alpha\dot{v},
\] (3.9)

where \(c_1 > 0\) is a constant. Let \(\beta - \alpha(k+1) - k > 0\). From (3.7) and (3.9) we find that
\[
-c_1\{\beta - \alpha(k+1) - k\}\int_{s_0}^{s}\dot{v}^2dr + \{\beta - \alpha(k+1) - k\}k^\alpha \frac{v^{\alpha+1}}{\alpha + 1}
\]
\[
\geq \hat{C}^{\lambda-\alpha}\int_{s_0}^{s}\delta(r)v^\lambda\dot{v}dr + O(1) \quad \text{as} \quad s \to \infty.
\] (3.10)

Suppose \(\int_{0}^{\infty} e(t)^2dt/t < \infty\), that is, (3.4) holds. Schwarz's inequality and (3.10) imply that
\[
c_2\int_{s_0}^{s}\dot{v}^2dr \leq c_3 - c_4\int_{s_0}^{s}\delta(r)v^\lambda\dot{v}dr
\]
\[
\leq c_3 + c_5\left(\int_{s_0}^{s}\delta(r)^2dr\right)^{1/2}\left(\int_{s_0}^{s}\dot{v}^2dr\right)^{1/2}
\]
with some positive constants $c_2, c_3, c_4$ and $c_5$. We therefore obtain $\int^\infty \dot{v}^2 dr < \infty$. Suppose next $\int^\infty |\varepsilon(t)| dt < \infty$, that is, (3.5) holds. We find from (3.10) that

$$c_2 \int^s \dot{v}^2 dr \leq c_3 - c_4 \int^s \delta(r) \left( \frac{v^{\lambda+1}}{\lambda+1} \right) dr$$

$$\leq c_6 - \frac{c_4}{\lambda+1} \delta(s) v^{\lambda+1} - c_7 \int^s \delta(r) dr,$$

where $c_6, c_7, c_8$ and $c_9$ are some positive constants. Hence we obtain $\int^\infty \dot{v}^2 dr < \infty$. The case where $\beta - \alpha(k+1) - k < 0$ can be treated similarly.

Since we have shown $\int^\infty \dot{v}^2 dr < \infty$, and $\alpha \leq 1$, Eq (3.3) shows that $\ddot{v} = O(1)$ as $s \to \infty$. Therefore by Lemma 3 we find that $\lim_{s \to \infty} \dot{v}(s) = 0$.

**Step 2.** We show that $\liminf_{s \to \infty} v(s) > 0$. To see this by contradiction, we will derive a contradiction by assuming $\liminf_{s \to \infty} v(s) = 0$. The argument is divided into the two cases:

Case (a): $v(s)$ monotonically decreases to 0 (and so, $\dot{v}(s) \leq 0$);

Case (b): $\dot{v}(s)$ changes the sign in any neighborhood of $+\infty$.

Let case (a) occur. Put $v = x_1$ and $\dot{v} = x_2$, and $x = (x_1, x_2)$. Then, $x$ satisfies the binary system

$$\dot{x} = Ax + f(s, x),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ k \left( \frac{\beta}{\alpha} - k - 1 \right) - \left( \frac{\beta}{\alpha} - 2k - 1 \right) & 0 \end{pmatrix},$$

and

$$f(s, x) = \begin{pmatrix} 0 \\ -\frac{\alpha^{\lambda-\alpha}}{\alpha} \{1 + \delta(s)\} (k|x_1| + |x_2|)^{1-\alpha}|x_1|^\lambda \end{pmatrix}.$$ 

Here we have used the fact that $v(s) > 0$ and $\dot{v}(s) \leq 0$. Since

$$(k|x_1| + |x_2|)^{1-\alpha}|x_1|^\lambda \leq (\max\{1, k\})^{1-\alpha} (|x_1| + |x_2|)^{\lambda-\alpha+1},$$

and $(v(s), \dot{v}(s))$ corresponds to a solution $x(s)$ of system (3.11) satisfying $\lim_{s \to \infty} x(s) = 0$, by [2, Chapter 3, Theorem 5] we have

$$\lim_{s \to \infty} \frac{\log \|x(s)\|}{s} = \Lambda,$$

(3.12)

where $\Lambda$ is the real part of an eigenvalue of $A$. All the eigenvalues of $A$ are $k$ and $-(\beta/\alpha - k - 1)$; the former is positive and the latter negative. Since $\|x(s)\| \to 0$, we have $\Lambda = -(\beta/\alpha - k - 1)$. By the assumption (2.2) we find a small $\eta > 0$ satisfying $\sigma + \lambda(-\beta/\alpha + 1) + \lambda \eta < -1$. By (3.12) we obtain

$$v(s) \leq e^{-(\beta/\alpha - k - 1)\eta} s \text{ near } +\infty.$$
This means that $u(t) \leq t^{-\beta/\alpha+1+\eta}$ near $+\infty$. Then
\[ t^\beta u'(t) \leq c_1 \int_0^t r^{\sigma+\lambda(-\beta/\alpha+1)+\lambda\eta} dr = O(1) \quad \text{as} \quad t \to \infty. \]

This contradicts the property of slowly decaying solution $\lim_{t \to \infty} t^\beta u'(t) = \infty$. Hence Case (a) never occurs. As in the proof of [6, Theorem 1.3], we can show that Case (b) never occurs. Hence we have $\liminf_{s \to \infty} v(s) > 0$.

The remainder of the proof of the fact $\lim_{s \to \infty} v(s) = 1$ proceeds as in the proof of [6, Theorem 1.3]. We leave them to the reader.

**Proof of Theorem 2.** As in the proof of Theorem 1, we will show that $\lim_{s \to \infty} v(s) = 1$, where $v(s)$ is introduced in Lemma 2. Define
\[
 w = (kv - \dot{v})^\alpha. \tag{3.13}
\]
By Eq (3.2) we know that
\[
 \dot{w} + \{\beta - \alpha(k + 1)\}w - \hat{C}^{\lambda-\alpha}\{1+\delta(s)\}v^\lambda = 0.
\]
Let us rewrite this equation as
\[
 \dot{w} + aw - b\{1+\delta(s)\}v^\lambda = 0, \tag{3.14}
\]
where we have put $\beta - \alpha(k + 1) = a$ and $\hat{C}^{\lambda-\alpha} = b$. We therefore find that
\[
 v = b^{-1/\lambda}(1+\delta(s))^{-1/\lambda}(\dot{w}+aw)^{1/\lambda},
\]
and $w$ satisfied the ODE
\[
 (1+\delta(s))^{-1/\lambda}(\dot{w}+aw)^{1/\lambda} - k(1+\delta(s))^{-1/\lambda}(\dot{w}+aw)^{1/\lambda} + b^{1/\lambda}w^{1/\alpha} = 0. \tag{3.15}
\]
We note, by the definition (3.13), (3.14), and Lemma 2, that $w, \dot{w} = O(1)$ as $s \to \infty$. By putting $(1+\delta(s))^{-1/\lambda} = h(s), 1/\lambda = \rho$, and $1/\alpha = \gamma$, we can rewrite (3.15) simply as
\[
 (h(s)(\dot{w}+aw)^\rho) - kh(s)(\dot{w}+aw)^\rho + b^\rho w^{1/\alpha} = 0. \tag{3.16}
\]
We note that our assumptions (2.4) are equivalent to
\[
 \lim_{s \to \infty} \delta(s) = 0 \tag{3.17}
\]
and
\[
 \int_\infty^\infty |\delta(s)| ds < \infty. \tag{3.18}
\]
It should be emphasized that Eq (3.16) is equivalent to
\[
 \dot{w} + \left[ a - \frac{k}{\rho} + \frac{h(s)}{\rho h(s)} \right] \dot{w} + \frac{a}{\rho} \left[ \frac{h(s)}{h(s)} - k \right] w + \frac{b^\rho}{\rho h(s)}(\dot{w}+aw)^{1-\rho}w^\gamma = 0 \tag{3.19}
\]
By using (3.18) and computing as in the proof of Theorem 1, we find from Eq (3.16) that
\[(a - k) \int_{s_0}^{s} h(r)(\dot{w} + aw)^{\rho}\dot{w} dr = O(1) \quad \text{as} \quad s \to \infty. \tag{3.20}\]

Notice that the assumption $\beta - \alpha(k + 1) - k \neq 0$ means that $a - k \neq 0$. Since $\alpha \geq 1$ and $\lambda > \alpha$, we have $\rho < 1$. So inequality (3.8) implies, as before, that
\[
\{(\dot{w} + aw)^{\rho} - (aw)^{\rho}\}\dot{w} \geq c_0\dot{w}^2\{(\dot{w} + aw) + |aw|\}^{\rho-1};
\]
that is,
\[h(r)(\dot{w} + aw)^{\rho} \dot{w} \geq a^\rho h(r)w^{\rho}\dot{w} + c_1 h(r)\dot{w}^{2}\]
for some constant $c_1 > 0$. Hence by (3.20) and the fact that $h(\infty) = 1$, we find that
\[c_2 \int_{s_0}^{s} h(r)w^{\rho}\dot{w} dr + c_3 \int_{s_0}^{s} \dot{w}^2 dr = O(1) \quad \text{as} \quad s \to +\infty. \tag{3.21}\]

By integral by parts and by using this relation, we find that $\int_{s_0}^{\infty} \dot{w}^2 ds < \infty$. Moreover, since $\rho < 1$, we find that $\lim_{s \to \infty} \dot{w}(s) = 0$ as in the proof of Theorem 1.

We want to show that $\liminf_{s \to \infty} w(s) > 0$. The proof is done by a contradiction. Firstly suppose that $w(s)$ decreases to 0 as $s \to \infty$. Then, as in the proof of Theorem 1, we know by [2, Chapter 3, Theorem 5] that for every $\eta > 0$
\[w(s) \leq e^{-(\beta+\alpha(k+1)+\eta)s} \quad \text{as} \quad s \to \infty. \tag{3.22}\]

The definition (3.13) is equivalent to $(e^{-ks}v) = -e^{-\kappa s}w^{1/\alpha}$; and so
\[v(s) = e^{ks} \int_{s}^{\infty} e^{-kr}w^{1/\alpha} dr. \tag{3.23}\]

Here we have employed the fact that $\lim_{s \to \infty} v(s)/e^{ks} = 0$. Combining (3.21) with (3.22), we get the estimate $t^\beta |u'(t)| = O(1)$. Recall that this yields a contradiction.

Next, let $\lim_{s \to \infty} w(s) = 0$ and $\dot{w}$ change the sign in any neighborhood of $+\infty$. Define the auxiliary function $H(s)$ by
\[H(s) = k^\alpha \left[ 1 - \frac{\dot{h}(s)}{kh(s)} \right]^{\rho/\alpha}. \tag{3.24}\]

Then, in the region $0 < w < H(s)$, we have $\dot{w} > 0$. On the other hand in the region $w > H(s)$, we have $\dot{w} < 0$. Hence, we can find out two sequences $\{\xi_n\}$ and $\{\eta_n\}$ satisfying
\[\xi_n < \eta_n < \xi_{n+1} < \eta_{n+1} < \cdots; \lim_{n \to \infty} \xi_n = \lim_{n \to \infty} \eta_n = \infty;\]
and
\[w(\eta_n) \to 0, \quad w(\xi_n) = H(\xi_n) \to k^\alpha \quad \text{as} \quad n \to \infty \quad \text{and} \quad \dot{w} \leq 0 \quad \text{on} \quad [\xi_n, \eta_n]. \tag{3.25}\]
Multiplying (3.19) by $\dot{w}$ and integrating the resulting equation on $[\xi_n, \eta_n]$, we have

$$\frac{1}{2}(\dot{w}(\eta_n)^2 - \dot{w}(\xi_n)^2) + \left(a - \frac{k}{\rho}\right) \int_{\xi_n}^{\eta_n} \dot{w}^2 dr + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr$$

$$\frac{-a k}{2 \rho} (w(\eta_n)^2 - w(\xi_n)^2) + \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w} dr + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} (w + aw)^{2-\rho} w^\gamma \dot{w} dr = 0.$$  

Noting the facts $\dot{w}(\infty) = 0$ and $\int^\infty \dot{w}^2 dr < \infty$, we have as $n \to \infty$

$$o(1) + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr - \frac{a k}{2 \rho} (o(1) - k^2)$$

$$+ \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w \dot{w} dr + \frac{a^{2-\rho}}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} w^{2+\gamma-\rho} \dot{w} dr \leq 0.$$  

Now, let us estimate each term of the above. We have firstly

$$\left| \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr \right| \leq C_0 \sup_{[\xi_n, \infty)} |\dot{h}| \int_{\xi_n}^{\infty} \dot{w}^2 dr = o(1) \quad \text{as} \quad n \to \infty;$$

and

$$\left| \int_{\xi_n}^{\eta_n} \frac{h(r)}{h(c_n)} \dot{w}^2 dr \right| = \frac{h(c_n)}{h(h_c)} \int_{\xi_n}^{\eta_n} \dot{w}^2 dr = \frac{h(c_n)}{2h(c_n)} (w(\xi_n)^2 - w(\eta_n)^2) = o(1) \quad \text{as} \quad n \to \infty.$$  

Here $C_0 > 0$ is a constant, and we have used a variant of the mean value theorem for integrals; that is $c_n$ is a number satisfying $\xi_n < c_n < \eta_n$. Finally, we obtain

$$\int_{\xi_n}^{\eta_n} \frac{1}{h(r)} w^{2+\gamma-\rho} \dot{w} dr = \int_{\xi_n}^{\eta_n} [h(r)^{-1} - 1] w^{2+\gamma-\rho} \dot{w} dr$$

$$= (h(d_n)^{-1} - 1) \int_{\xi_n}^{\eta_n} w^{2+\gamma-\rho} \dot{w} dr + \frac{1}{2+\gamma-\rho} (w(\eta_n)^{2+\gamma-\rho} - w(\xi_n)^{2+\gamma-\rho})$$

$$= o(1) - \frac{k^{2+\gamma-\rho}}{2+\gamma-\rho} \quad \text{as} \quad n \to \infty.$$  

Here $d_n$ is a number satisfying $\xi_n < d_n < \eta_n$. Therefore (3.24) can be simplified into

$$\frac{ak^{2\alpha+1}}{2\rho} + o(1) \leq \frac{a^{1-\rho} k^{2+\gamma-\rho}}{\rho(2+\gamma-\rho)} \quad \text{as} \quad n \to \infty.$$  

This gives a contradiction. Hence we find that $\liminf_{s \to \infty} v(s) > 0$.

Arguing as in the proof of Theorem 1, we will show that $\lim_{s \to \infty} v(s) = 1$. The details are left to the reader.

To see Theorem 3, we will show that $\lim_{s \to \infty} v(s) = 1$, where $v(s)$ is introduced by Lemma 2, as before. However, we can not help omitting the proof for the lack of space.
References


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