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Kyoto University
Asymptotic forms of slowly decaying positive solutions of second-order quasilinear ordinary differential equations

1 Introduction

Let us consider the quasilinear ODE

\[(a(t)|u'|^{\alpha-1}u')'+b(t)|u|^{\lambda-1}u=0, \quad \text{near } +\infty\]  

(A)

where we assume that \(\alpha > 0\) and \(\lambda > 0\) are constants, \(a(t)\) and \(b(t)\) are positive continuous functions satisfying \(\int^\infty a(t)^{-1/\alpha}dt < \infty\). Every positive solution \(u\) of (A) satisfies one of the following three asymptotic properties as \(t \to \infty\):

1. \(u(t) \sim c_1\) for some constant \(c_1 > 0\);
2. \(u(t) \sim c_2 \int_t^\infty a(s)^{-1/\alpha}ds\) for some constant \(c_2 > 0\);

and

\[u(t) \to 0 \quad \text{and} \quad \frac{u(t)}{\int_t^\infty a(s)^{-1/\alpha}ds} \to \infty.\]  

Asymptotic properties of solutions \(u\) satisfying either (1.1) or (1.2) were widely investigated. For example, necessary and sufficient conditions of existence of such solutions were established in [4, 7]. On the other hand there seems to be less information about qualitative properties of solutions \(u\) satisfying (1.3). Motivated by this fact, in the article we will discuss about asymptotic behavior of solutions \(u\) satisfying (1.3); in particular, we try to find exact asymptotic forms of such solutions near \(+\infty\). In what follows we refer solutions \(u\) satisfying (1.3) as slowly decaying solutions.

Remark 1. When \(\int^\infty a(t)^{-1/\alpha}dt = \infty\), Eq (A) reduces to the simpler one of the form

\[(|u'|^{\alpha-1}u')'+\tilde{b}(t)|u|^\lambda u=0, \quad \text{near } +\infty,\]

where \(\tilde{b}(t)\) is a positive continuous function. Studies of this equation were, for example, the main objective of [6]; and asymptotic properties of solutions have been fully established.
2 Preparatory observations and results

Asymptotic forms of slowly decaying solutions may be strongly affected by those of coefficient functions $a(t), b(t)$ and the exponents $\alpha$ and $\lambda$. Therefore let us consider the following ODE, which has more restrictive appearance than Eq (A):

$$
(t^\beta|u'|^{\alpha-1}u')' + t^\sigma (1 + \epsilon(t))|u|^\lambda u = 0 \quad \text{near } +\infty.
$$

(E)

In the sequel we assume the next conditions:

(A1) $\alpha, \beta, \lambda$ and $\sigma$ are constants satisfying $\lambda > \alpha > 0$ and $\beta > \alpha$;

(A2) $\epsilon(t)$ is a continuous (or $C^{1-}$) function defined near $+\infty$ satisfying $\lim_{t\to\infty}\epsilon(t) = 0$.

Additional conditions will be given later.

Since we can regard Eq (E) as a "perturbed equation" of the ODE

$$
(t^\beta|u'|^{\alpha-1}u')' + t^\sigma |u|^\lambda u = 0 \quad \text{near } +\infty,
$$

(E0)

we conjecture that slowly decaying solutions of Eq (E) and those of Eq (E0) may have the same asymptotic behavior near $+\infty$ in some sense, if $\epsilon(t)$ is sufficiently small. It is easily seen that Eq (E0) has an exact slowly decaying solution $u_0$ given by

$$
u_0(t) = \hat{C}t^{-k},$$

(2.1)

where

$$
k = \frac{1 + \sigma - (\beta - \alpha)}{\lambda - \alpha}, \quad \text{and} \quad \hat{C}^{\lambda-\alpha} = k^\alpha \{\beta - \alpha(k+1)\}
$$

if

$$(\beta - \alpha) - 1 < \sigma < \frac{\lambda}{\alpha}(\beta - \alpha) - 1. \quad (2.2)$$

Below we always assume (2.2). We can show that our conjecture is true in various cases:

**Theorem 1.** Let $\alpha \leq 1$ and $\beta - \alpha(k+1) - k \neq 0$. If

either $\int_{0}^{\infty} \frac{\epsilon(t)^2}{t} dt < \infty$ or $\int_{0}^{\infty} |\epsilon'(t)| dt < \infty,$

(2.3)

then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$, where $u_0(t)$ is given by (2.1).

**Theorem 2.** Let $\alpha \geq 1$ and $\beta - \alpha(k+1) - k \neq 0$. If

$$
\lim_{t \to \infty} t\epsilon'(t) = 0 \quad \text{and} \quad \int_{0}^{\infty} |\epsilon'(t)| dt < \infty,
$$

(2.4)

then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$. 
Theorem 3. Let $\alpha \geq 1$ and $\alpha(2k+1)-\beta < 0$. If (2.3) holds, then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$.

Example 1. Let $N > m > 1$ and $N \geq 2$. Consider radial solutions $u = u(|x|)$ of the following quasilinear PDE in an exterior domain of $\mathbb{R}^N$:
\[
div(|Du|^{m-2}Du) + |x|^{\ell}(1 + |x|^{-\theta})|u|^\lambda u = 0 \quad \text{near} \quad \infty,
\]
where $\lambda > m - 1, \ell \in \mathbb{R}, \theta > 0$, and $-m < \ell < \frac{\lambda}{m-1}(N - m) - N$. We know that $u$ solves the ODE
\[
(r^{N-1}|u'|^{m-2}u')' + r^{N-1+\ell}(1 + r^{-\rho})|u|^\lambda u = 0 \quad \text{near} \quad +\infty.
\]
By Theorems 1 and 2, if $\lambda \neq (mN - N + m\ell)/(N - m)$, then every slowly decaying positive solution $u$ of this equation satisfies
\[
u(r) \sim Ar^{-(\ell+m)/\lambda}\quad \text{as} \quad r \to +\infty,
\]
where $A$ is a positive constant given by
\[
A^{\lambda-m+1} = \left(\frac{\ell + m}{\ell - m + 1}\right)^{m-1} \cdot \frac{N\lambda -Nm+N-m\ell-m\lambda+\ell}{\lambda-m+1}.
\]

Remark 1. For the autonomous equation $\text{div}(|Du|^{m-2}Du) + |u|^\lambda u = 0$, the assertion of Example 1 was established in [1] based on the theory of autonomous dynamical systems. Related results are found in [3, 5].

3 Sketches of the proof of the results

We give the outline of the proof of Theorems 1 and 2. We begin with several auxiliary results.

Lemma 1. Let $u(t)$ be a slowly decaying positive solution of (E). Then
\[
u(t) = O(u_0(t)) \quad \text{and} \quad u'(t) = O(|u_0'(t)|) \quad \text{as} \quad t \to \infty. \quad (3.1)
\]

Proof. An integration of the both sides of Eq (E) on $[t_0, t]$ gives
\[
t^\beta(-u'(t))^\alpha \geq \int_{t_0}^{t} r^\sigma(1 + \epsilon(r))u^\lambda dr,
\]
where $t_0$ is a sufficiently large number. Since $u$ is a decreasing function, we have
\[
t^\beta(-u'(t))^\alpha \geq u(t)^\lambda \int_{t_0}^{t} r^\sigma(1 + \epsilon(r))dr;
\]
that is,

\[-u'(t)u(t)^{-\lambda/\alpha} \geq \left(t^{-\beta} \int_{t_{0}}^{t} r^{\sigma}(1 + \epsilon(r))dr\right)^{1/\alpha}.

One more integration of the both sides gives the estimates for \(u\) in (3.1).

To get the estimates for \(u'\), it suffices to notice the inequality

\[t^{\beta}(-u'(t))^{\alpha} \leq C_{1} \int_{t_{0}}^{t} r^{\sigma}u(r)^{\lambda}dr,
\]

where \(C_{1} > 0\) is a constant. Note that, to get this inequality, we must use the property \(\lim_{t\to\infty} t^{\beta}(-u'(t))^{\alpha} = \infty\).

**Lemma 2.** Let \(u(t)\) be a slowly decaying positive solution of (E). Put \(t = e^{s}\) and \(u/u_{0} = v\). Then

(i) \(v\), and \(\dot{v}\) are bounded, and \(\dot{v} - kv < 0\) near \(+\infty\), where \(\cdot = d/ds\);

(ii) \(v\) satisfies the ODE

\[(kv - \dot{v})^{\alpha} + (\beta - \alpha(k + 1))(kv - \dot{v})^{\alpha} - \dot{C}^{\lambda - \alpha}(1 + \delta(s))v^{\lambda} = 0\]

near \(+\infty\), (3.2)

where \(\delta(s) = \epsilon(e^{s})\).

The proof of this lemma is based on direct computations; hence we omit it.

**Remark 2.** Equation (3.2) can be rewritten as

\[\dot{v} + \left(\frac{\beta}{\alpha} - 2k - 1\right)\dot{v} - k\left(\frac{\beta}{\alpha} - k - 1\right)v + \dot{C}^{\lambda - \alpha}(1 + \delta(s))v^{\lambda} = 0.\]

**Lemma 3.** Let \(f(s)\) be a \(C^{1}\)-function near \(+\infty\) satisfying \(f'(s) = O(1)\) as \(s \to \infty\) and \(\int f(s)^2 ds < \infty\). Then \(\lim_{s \to \infty} f(s) = 0\).

The proof of this lemma will be found in [6].

**Proof of Theorem 1.** By the change of variables \((t, u) \to (s, v)\) introduced in Lemma 2, we obtain Eq (3.2). We note that the integral conditions indicated in (2.3) are equivalent to

\[\int_{e^{s_{0}}}^{\infty} \delta(s)^{2}ds < \infty\]

(3.4)

and

\[\int_{e^{s_{0}}}^{\infty} |\dot{\delta}(s)|ds < \infty,
\]

(3.5)

respectively.

**Step 1.** We show that \(\int \dot{v}(s)^2 ds < \infty\). We multiply Eq (3.2) by \(\dot{v}\), and integrate the resulting equation on \([s_{0}, s]\) to obtain

\[\int_{s_{0}}^{s} ((kv - \dot{v})^{\alpha}) \dot{v}dr + (\beta - \alpha(k + 1)) \int_{s_{0}}^{s} (kv - \dot{v})^{\alpha} \dot{v}dr\]
\[
-\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1}v^{\lambda+1} - \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r)v^\lambda \dot{v}dr = \text{const.}
\] (3.6)

Since integral by parts implies that
\[
\int_{s_0}^{s} \{(kv - \dot{v})^\alpha\}\cdot \dot{v}dr = -\int_{s_0}^{s} \{(kv - \dot{v})^\alpha\}(kv - \dot{v})dr + k \int_{s_0}^{s} (kv - \dot{v})^\alpha \dot{v}dr + \text{const},
\]
we obtain from (3.6)
\[
-\frac{\alpha}{\alpha + 1} (kv - \dot{v})^{\alpha+1} + kv(kv - \dot{v})^\alpha - k \int_{s_0}^{s} (kv - \dot{v})^\alpha \dot{v}dr + \text{const},
\]
\[
= -\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1}v^{\lambda+1} - \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r)v^\lambda \dot{v}dr = \text{const.}
\]

The boundedness of \(v\) and \(\dot{v}\) shown in Lemma 2 imply that
\[
\{\beta - \alpha(k + 1) - k\} \int_{s_0}^{s} (kv - \dot{v})^\alpha \dot{v}dr - \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r)v^\lambda \dot{v}dr = O(1)\] as \(s \to \infty\). (3.7)

Now, since \(0 < \alpha \leq 1\), the inequality
\[
(X^\alpha - Y^\alpha)(X - Y) \geq c_0(X - Y)^2(X + Y)^{\alpha - 1}
\]
for all \(X,Y \geq 0\) with \(X + Y > 0\) holds for some constant \(c_0 > 0\). Therefore we obtain
\[
\{(kv)^\alpha - (kv - \dot{v})^\alpha\}\dot{v} \geq c_0((kv) + (kv - \dot{v}))^{\alpha - 1}\dot{v}\]
that is,
\[
(kv - \dot{v})^\alpha \dot{v} \leq -c_1 \dot{v}^2 + k^\alpha v^\alpha \dot{v},
\] (3.9)

where \(c_1 > 0\) is a constant. Let \(\beta - \alpha(k + 1) - k > 0\). From (3.7) and (3.9) we find that
\[
-c_1 \{\beta - \alpha(k + 1) - k\} \int_{s_0}^{s} \dot{v}^2dr + \{\beta - \alpha(k + 1) - k\} \frac{k^\alpha}{\alpha + 1} v^{\alpha+1}
\]
\[
\geq \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r)v^\lambda \dot{v}dr + O(1)\] as \(s \to \infty\). (3.10)

Suppose \(\int_{t_0}^{\infty} \epsilon(t)^2 dt/t < \infty\), that is, (3.4) holds. Schwarz's inequality and (3.10) imply that
\[
c_2 \int_{s_0}^{s} \dot{v}^2dr \leq c_3 - c_4 \int_{s_0}^{s} \delta(r)v^\lambda \dot{v}dr
\]
\[
\leq c_3 + c_6 \left( \int_{s_0}^{s} \delta(r)^2dr \right)^{1/2} \left( \int_{s_0}^{s} \dot{v}^2dr \right)^{1/2}
\]
with some positive constants $c_2, c_3, c_4$ and $c_5$. We therefore obtain $\int^\infty \dot{v}^2 dr < \infty$. Suppose next $\int^\infty |\dot{\varepsilon}(t)| dt < \infty$, that is, (3.5) holds. We find from (3.10) that

$$c_2 \int^s \dot{v}^2 dr \leq c_3 - c_4 \int^s \delta(r) \left( \frac{v^{\lambda+1}}{\lambda+1} \right) dr$$

$$\leq c_5 - \frac{c_4}{\lambda+1} \delta(s)v^{\lambda+1} + c_7 \int^s \delta(r) v^{\lambda+1} dr,$$

where $c_5, c_7, c_8$ and $c_9$ are some positive constants. Hence we obtain $\int^\infty \dot{v}^2 dr < \infty$. The case where $\beta - \alpha(k+1) - k < 0$ can be treated similarly.

Since we have shown $\int^\infty \dot{\varepsilon}^2 dr < \infty$, and $\alpha \leq 1$, Eq (3.3) shows that $\dot{\varepsilon} = O(1)$ as $s \to \infty$. Therefore by Lemma 3 we find that $\lim_{s \to \infty} \dot{v}(s) = 0$.

**Step 2.** We show that $\lim_{s \to \infty} v(s) > 0$. To see this by contradiction, we will derive a contradiction by assuming $\lim_{s \to \infty} v(s) = 0$. The argument is divided into the two cases:

Case (a): $v(s)$ monotonically decreases to 0 (and so, $\dot{v}(s) \leq 0$);
Case (b): $\dot{v}(s)$ changes the sign in any neighborhood of $+\infty$.

Let case (a) occur. Put $v = x_1$ and $\dot{v} = x_2$, and $x = (x_1, x_2)$. Then, $x$ satisfies the binary system

$$\dot{x} = Ax + f(s, x), \quad (3.11)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ k \left( \frac{\beta}{\alpha} - k - 1 \right) & \left( \frac{\beta}{\alpha} - 2k - 1 \right) \end{pmatrix},$$

and

$$f(s, x) = \begin{pmatrix} 0 \\ -C^{\lambda-a}_{\alpha} \{1 + \delta(s)\} (k|x_1| + |x_2|)^{1-a}|x_1|^\lambda \end{pmatrix}.$$

Here we have used the fact that $v(s) > 0$ and $\dot{v}(s) \leq 0$. Since

$$(k|x_1| + |x_2|)^{1-a}|x_1|^\lambda \leq (\max\{1, k\})^{1-a}(|x_1| + |x_2|)^{\lambda-a+1},$$

and $(v(s), \dot{v}(s))$ corresponds to a solution $x(s)$ of system (3.11) satisfying $\lim_{s \to \infty} x(s) = 0$, by [2, Chapter 3, Theorem 5] we have

$$\lim_{s \to \infty} \frac{\log \|x(s)\|}{s} = \Lambda, \quad (3.12)$$

where $\Lambda$ is the real part of an eigenvalue of $A$. All the eigenvalues of $A$ are $k$ and $-(\beta/\alpha - k - 1)$; the former is positive and the latter negative. Since $\|x(s)\| \to 0$, we have $\Lambda = -((\beta/\alpha - k - 1)$. By the assumption (2.2) we find a small $\eta > 0$ satisfying $\sigma + \lambda(-\beta/\alpha + 1) + \lambda\eta < -1$. By (3.12) we obtain

$$v(s) \leq e^{-(\beta/\alpha-k-1)+\eta} \quad \text{near } +\infty.$$
This means that \(u(t) \leq t^{-\beta/\alpha+1+\eta}\) near \(+\infty\). Then
\[
t^\beta(-u'(t))^\alpha \leq c_1 \int_{t_0}^{t} r^{\sigma+\lambda(-\beta/\alpha+1)+\lambda\eta} dr = O(1) \quad \text{as } t \to \infty.
\]
This contradicts the property of slowly decaying solution \(\lim_{t \to \infty} t^\beta(-u'(t))^\alpha = \infty\). Hence Case (a) never occurs. As in the proof of [6, Theorem 1.3], we can show that Case (b) never occurs. Hence we have \(\lim \inf_{s \to \infty} v(s) > 0\).

The remainder of the proof of the fact \(\lim_{s \to \infty} v(s) = 1\) proceeds as in the proof of [6, Theorem 1.3]. We leave them to the reader.

**Proof of Theorem 2.** As in the proof of Theorem 1, we will show that \(\lim_{s \to \infty} v(s) = 1\), where \(v(s)\) is introduced in Lemma 2. Define
\[
w = (kv - \dot{v})^\alpha.
\]
By Eq (3.2) we know that
\[
\dot{w} + \{\beta - \alpha(k + 1)\} w - \hat{C}^{\lambda-\alpha}\{1 + \delta(s)\} v^\lambda = 0.
\]
Let us rewrite this equation as
\[
\dot{w} + aw - b\{1 + \delta(s)\} v^\lambda = 0,
\]
where we have put \(\beta - \alpha(k + 1) = a\) and \(\hat{C}^{\lambda-\alpha} = b\). We therefore find that
\[
v = b^{-1/\lambda}(1 + \delta(s))^{-1/\lambda}(\dot{w} + aw)^{1/\lambda},
\]
and \(w\) satisfied the ODE
\[
((1 + \delta(s))^{-1/\lambda}(\dot{w} + aw)^{1/\lambda}) - k(1 + \delta(s))^{-1/\lambda}(\dot{w} + aw)^{1/\lambda} + b^{1/\lambda}w^{1/\alpha} = 0.
\]
We note, by the definition (3.13), (3.14), and Lemma 2, that \(w, \dot{w} = O(1)\) as \(s \to \infty\). By putting \((1 + \delta(s))^{-1/\lambda} = h(s), 1/\lambda = \rho, \text{ and } 1/\alpha = \gamma\), we can rewrite (3.15) simply as
\[
(h(s)(\dot{w} + aw)^\rho) - kh(s)(\dot{w} + aw)^\rho + b^\rho w^{1/\alpha} = 0.
\]
We notc that our assumptions (2.4) are equivalent to
\[
\lim_{s \to \infty} \delta(s) = 0
\]
and
\[
\int_{t_0}^{\infty} |\delta(s)| ds < \infty.
\]
It should be emphasized that Eq (3.16) is equivalent to
\[
\dot{w} + \left[ a - \frac{k}{\rho} + \frac{\dot{h}(s)}{\rho h(s)} \right] \dot{w} + \frac{a}{\rho} \left[ \frac{\dot{h}(s)}{h(s)} - k \right] w + \frac{b^\rho}{\rho h(s)} (\dot{w} + aw)^{1-\rho} w^\gamma = 0
\]
By using (3.18) and computing as in the proof of Theorem 1, we find from Eq (3.16) that
\[(a - k) \int_{s_0}^{s} h(r)(\dot{w} + aw)^{\rho} \dot{w}dr = O(1) \quad as \quad s \to \infty. \]  
(3.20)

Notice that the assumption \(\beta - \alpha(k + 1) - k \neq 0\) means that \(a - k \neq 0\). Since \(\alpha \geq 1\) and \(\lambda > \alpha\), we have \(\rho < 1\). So inequality (3.8) implies, as before, that
\[
\{(\dot{w} + aw)^{\rho} - (aw)^{\rho}\}\dot{w} \geq c_0 \dot{w}^2\{|\dot{w} + aw| + |aw|\}^{\rho - 1};
\]
that is,
\[h(r)(\dot{w} + aw)^{\rho} \dot{w} \geq a^{\rho} h(r) w^{\rho} \dot{w} + c_1 h(r) \dot{w}^2\]
for some constant \(c_1 > 0\). Hence by (3.20) and the fact that \(h(\infty) = 1\), we find that
\[c_2 \int_{s_0}^{s} h(r) w^{\rho} \dot{w}dr + c_3 \int_{s_0}^{s} \dot{w}^2dr = O(1) \quad as \quad s \to +\infty.
\]
By integral by parts and by using this relation, we find that \(\int_{0}^{\infty} \dot{w}^2ds < \infty\). Moreover, since \(\rho < 1\), we find that \(w(s)\) changes the sign in any neighborhood of \(+\infty\).

Define the auxiliary function \(H(s)\) by
\[H(s) = k^\alpha \left[1 - \frac{\dot{h}(s)}{kh(s)}\right]^{\frac{\rho}{\alpha}}.
\]
(3.22)

Then, in the region \(0 < w < H(s)\), we have \(\dot{w} > 0\). On the other hand in the region \(w > H(s)\), we have \(\dot{w} < 0\). Hence, we can find out two sequences \(\{\xi_n\}\) and \(\{\eta_n\}\) satisfying
\[
\xi_n < \eta_n < \xi_{n+1} < \eta_{n+1} < \cdots; \lim_{n \to \infty} \xi_n = \lim_{n \to \infty} \eta_n = \infty;
\]
and
\[w(\eta_n) \to 0, \quad w(\xi_n) = H(\xi_n) \to k^\alpha \quad as \quad n \to \infty \quad and \quad \dot{w} \leq 0 \quad on [\xi_n, \eta_n]. \]  
(3.23)

The definition (3.13) is equivalent to \(e^{-ks}v) = -e^{-k}w^{1/a}\); and so
\[v(s) = e^{k^\epsilon} \int_{0}^{\infty} e^{-kr}w^{1/\alpha}dr.
\]
(3.22)
Multiplying (3.19) by $\dot{w}$ and integrating the resulting equation on $[\xi_n, \eta_n]$, we have

\[
\frac{1}{2}(\dot{w}(\eta_n)^2 - \dot{w}(\xi_n)^2) + \left(a - \frac{k}{\rho}\right) \int_{\xi_n}^{\eta_n} \dot{w}^2 dr + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{h(r)}{h(r)} \dot{w}^2 dr
\]

\[-\frac{ak}{2\rho} (w(\eta_n)^2 - w(\xi_n)^2) + \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} (w + aw)^{1-\rho} w^\gamma \dot{w} dr = 0.
\]

Noting the facts $\dot{w}(\infty) = 0$ and $\int_{\xi_n}^{\eta_n} \dot{w}^2 dr < \infty$, we have as $n \to \infty$

\[
o(1) + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr - \frac{ak}{2\rho} (o(1) - k^{2\alpha})
\]

\[+ \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w \dot{w} dr + \frac{a^{1-\rho} \nu}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} (\dot{w} + aw)^{1-\rho} w^\gamma \dot{w} dr \leq 0.
\]

(3.24)

Now, let us estimate each term of the above. We have firstly

\[
\left| \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr \right| \leq C_0 \sup_{[\xi_n, \infty)} |\dot{h}| \int_{\xi_n}^{\infty} \dot{w}^2 dr = o(1) \quad \text{as} \quad n \to \infty;
\]

and

\[
\left| \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w \dot{w} dr \right| = \left| \frac{\dot{h}(c_n)}{h(c_n)} \int_{\xi_n}^{\eta_n} w \dot{w} dr \right| = \left| \frac{\dot{h}(c_n)}{2h(c_n)} (w(\xi_n)^2 - w(\eta_n)^2) \right| = o(1) \quad \text{as} \quad n \to \infty.
\]

Here $C_0 > 0$ is a constant, and we have used a variant of the mean value theorem for integrals; that is $c_n$ is a number satisfying $\xi_n < c_n < \eta_n$. Finally, we obtain

\[
\int_{\xi_n}^{\eta_n} \frac{1}{h(r)} w^{1+\gamma-\rho} \dot{w} dr = \int_{\xi_n}^{\eta_n} \frac{h(r)}{h(c_n)} \int_{\xi_n}^{\eta_n} w \dot{w} dr = \frac{\dot{h}(c_n)}{2h(c_n)} (w(\xi_n)^2 - w(\eta_n)^2) = o(1) \quad \text{as} \quad n \to \infty.
\]

Here $d_n$ is a number satisfying $\xi_n < d_n < \eta_n$. Therefore (3.24) can be simplified into

\[
\frac{ak^{2\alpha + 1}}{2\rho} + o(1) \leq \frac{a^{1-\rho} \nu k^{(2+\gamma-\rho)}}{\rho(2 + \gamma - \rho)} \quad \text{as} \quad n \to \infty.
\]

This gives a contradiction. Hence we find that $\lim\inf_{s \to \infty} v(s) > 0$.

Arguing as in the proof of Theorem 1, we will show that $\lim_{s \to \infty} v(s) = 1$. The details are left to the reader.

To see Theorem 3, we will show that $\lim_{s \to \infty} v(s) = 1$, where $v(s)$ is introduced by Lemma 2, as before. However, we can not help omitting the proof for the lack of space.
References


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