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Asymptotic forms of slowly decaying positive solutions of second-order quasilinear ordinary differential equations

1 Introduction

Let us consider the quasilinear ODE
\[ (a(t)|u'|^{\alpha-1}u')' + b(t)|u|^\lambda u = 0, \quad \text{near } +\infty \] (A)
where we assume that \( \alpha > 0 \) and \( \lambda > 0 \) are constants, \( a(t) \) and \( b(t) \) are positive continuous functions satisfying \( \int^\infty a(t)^{-1/\alpha}dt < \infty \). Every positive solution \( u \) of (A) satisfies one of the following three asymptotic properties as \( t \to \infty \):
\[ u(t) \sim c_1 \quad \text{for some constant } c_1 > 0; \] (1.1)
\[ u(t) \sim c_2 \int_t^\infty a(s)^{-1/\alpha}ds \quad \text{for some constant } c_2 > 0; \] (1.2)
and
\[ u(t) \to 0 \quad \text{and} \quad \frac{u(t)}{\int_t^\infty a(s)^{-1/\alpha}ds} \to \infty. \] (1.3)

Asymptotic properties of solutions \( u \) satisfying either (1.1) or (1.2) were widely investigated. For example, necessary and sufficient conditions of existence of such solutions were established in [4, 7]. On the other hand there seems to be less information about qualitative properties of solutions \( u \) satisfying (1.3). Motivated by this fact, in the article we will discuss about asymptotic behavior of solutions \( u \) satisfying (1.3); in particular, we try to find exact asymptotic forms of such solutions near \( +\infty \). In what follows we refer solutions \( u \) satisfying (1.3) as slowly decaying solutions.

Remark 1. When \( \int^\infty a(t)^{-1/\alpha}dt = \infty \), Eq (A) reduces to the simpler one of the form
\[ (|u'|^{\alpha-1}u')' + \tilde{b}(t)|u|^\lambda u = 0 \quad \text{near } +\infty, \]
where \( \tilde{b}(t) \) is a positive continuous function. Studies of this equation were, for example, the main objective of [6]; and asymptotic properties of solutions have been fully established.
2 Preparatory observations and results

Asymptotic forms of slowly decaying solutions may be strongly affected by those of coefficient functions $a(t), b(t)$ and the exponents $\alpha$ and $\lambda$. Therefore let us consider the following ODE, which has more restrictive appearance than Eq (A):

$$(t^\beta|u'|^{\alpha-1}u')+t^\sigma(1+\epsilon(t))|u|^{\lambda-1}u=0 \text{ near } +\infty.$$  \hspace{1cm} (E)

In the sequel we assume the next conditions:

(A1) $\alpha, \beta, \lambda$ and $\sigma$ are constants satisfying $\lambda > \alpha > 0$ and $\beta > \alpha$;
(A2) $\epsilon(t)$ is a continuous (or $C^{1-}$) function defined near $+\infty$ satisfying $\lim_{t\to \infty}\epsilon(t) = 0$.

Additional conditions will be given later.

Since we can regard Eq (E) as a “perturbed equation” of the ODE

$$(t^\beta|u'|^{\alpha-1}u')+t^\sigma|u|^{\lambda-1}u=0 \text{ near } +\infty,$$  \hspace{1cm} (E_0)

we conjecture that slowly decaying solutions of Eq (E) and those of Eq (E_0) may have the same asymptotic behavior near $+\infty$ in some sense, if $\epsilon(t)$ is sufficiently small. It is easily seen that Eq (E_0) has an exact slowly decaying solution $u_0$ given by

$$u_0(t) = \hat{C}t^{-k},$$  \hspace{1cm} (2.1)

where

$$k = \frac{1+\sigma-(\beta-\alpha)}{\lambda-\alpha}, \text{ and } \hat{C}^{\lambda-\alpha} = k^\alpha\{\beta-\alpha(k+1)\}$$

if

$$(\beta-\alpha) - 1 < \sigma < \frac{\lambda}{\alpha}(\beta-\alpha) - 1.$$  \hspace{1cm} (2.2)

Below we always assume (2.2). We can show that our conjecture is true in various cases:

**Theorem 1.** Let $\alpha \leq 1$ and $\beta - \alpha(k+1) - k \neq 0$. If

$$\text{either } \int_0^\infty \frac{\epsilon(t)^2}{t}dt < \infty \text{ or } \int_0^\infty |\epsilon'(t)|dt < \infty,$$  \hspace{1cm} (2.3)

then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$, where $u_0(t)$ is given by (2.1).

**Theorem 2.** Let $\alpha \geq 1$ and $\beta - \alpha(k+1) - k \neq 0$. If

$$\lim_{t\to \infty} t\epsilon'(t) = 0 \text{ and } \int_0^\infty |\epsilon'(t)|dt < \infty,$$  \hspace{1cm} (2.4)

then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$.  \hspace{1cm}
Theorem 3. Let $\alpha \geq 1$ and $\alpha(2k+1) - \beta < 0$. If (2.3) holds, then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$.

Example 1. Let $N > m > 1$ and $N \geq 2$. Consider radial solutions $u = u(|x|)$ of the following quasilinear PDE in an exterior domain of $\mathbb{R}^N$:

$$\text{div}(|Du|^{m-2}Du) + |x|^\ell(1 + |x|^{-\theta})|u|^\lambda u = 0 \quad \text{near } \infty,$$

where $\lambda > m - 1, \ell \in \mathbb{R}, \theta > 0$, and $-m < \ell < \frac{\lambda}{m-1}(N - m) - N$. We know that $u$ solves the ODE

$$(r^{N-1}|u'|^{m-2}u')' + r^{N-1+\ell}(1 + r^{-\rho})|u|^\lambda u = 0 \quad \text{near } + \infty.$$

By Theorems 1 and 2, if $\lambda \neq (mN - N + m\ell)/(N - m)$, then every slowly decaying positive solution $u$ of this equation satisfies

$$u(r) \sim Ar^{-(\ell+m)/((\lambda-m+1)} \quad \text{as } r \to +\infty,$$

where $A$ is a positive constant given by

$$A^{\lambda-m+1} = \left( \frac{\ell + m}{\ell - m + 1} \right)^{m-1} \cdot \frac{N\lambda - Nm + N - m\ell - m\lambda + \ell}{\lambda - m + 1}.$$

Remark 1. For the autonomous equation $\text{div}(|Du|^{m-2}Du) + |u|^\lambda u = 0$, the assertion of Example 1 was established in [1] based on the theory of autonomous dynamical systems. Related results are found in [3, 5].

3 Sketches of the proof of the results

We give the outline of the proof of Theorems 1 and 2. We begin with several auxiliary results.

Lemma 1. Let $u(t)$ be a slowly decaying positive solution of (E). Then

$$u(t) = O(u_0(t)) \quad \text{and} \quad u'(t) = O(|u_0'(t)|) \quad \text{as } t \to \infty. \quad (3.1)$$

Proof. An integration of the both sides of Eq (E) on $[t_0, t]$ gives

$$t^\beta(-u'(t))^\alpha \geq \int_{t_0}^{t} r^\sigma(1 + \epsilon(r))u^\lambda dr,$$

where $t_0$ is a sufficiently large number. Since $u$ is a decreasing function, we have

$$t^\beta(-u'(t))^\alpha \geq u(t)^\lambda \int_{t_0}^{t} r^\sigma(1 + \epsilon(r))dr;$$
that is,

\[-u'(t)u(t)^{-\lambda/\alpha} \geq (t^{-\beta} \int_{t_0}^{t} r^\sigma (1 + \varepsilon(r)) dr)^{1/\alpha}.
\]

One more integration of the both sides gives the estimates for \(u\) in (3.1).

To get the estimates for \(u'\), it suffices to notice the inequality

\[t^\beta (-u'(t))^\alpha \leq C_1 \int_{t_0}^{t} r^\sigma u(r)^\lambda dr,\]

where \(C_1 > 0\) is a constant. Note that, to get this inequality, we must use the property \(\lim_{t \to \infty} t^\beta (-u'(t))^\alpha = \infty\).

**Lemma 2.** Let \(u(t)\) be a slowly decaying positive solution of (E). Put \(t = e^s\) and \(u/u_0 = v\). Then

(i) \(v\), and \(v\) are bounded, and \(v - kv < 0\) near \(+\infty\), where \(\cdot = d/ds;\)

(ii) \(v\) satisfies the ODE

\[(kv - \dot{v})^\alpha \cdot \{\beta - \alpha(k + 1)\} (kv - \dot{v})^\alpha - \hat{C}^{\lambda-\alpha} \{1 + \delta(s)\} v^\lambda = 0 \quad \text{near} \quad +\infty, \quad (3.2)\]

where \(\delta(s) = \varepsilon(e^s)\).

The proof of this lemma is based on direct computations; hence we omit it.

**Remark 2.** Equation (3.2) can be rewritten as

\[\dot{v} + \left(\frac{\beta}{\alpha} - 2k - 1\right) \dot{v} - k \left(\frac{\beta}{\alpha} - k - 1\right) v + \hat{C}^{\lambda-\alpha} \{1 + \delta(s)\} v^\lambda = 0. \quad (3.3)\]

**Lemma 3.** Let \(f(s)\) be a \(C^1\)-function near \(+\infty\) satisfying \(\dot{f}(s) = O(1)\) as \(s \to \infty\) and \(\int_{s_0}^{\infty} f(s)^2 ds < \infty\). Then \(\lim_{s \to \infty} f(s) = 0\).

The proof of this lemma will be found in [6].

**Proof of Theorem 1.** By the change of variables \((t, u) \mapsto (s, v)\) introduced in Lemma 2, we obtain Eq (3.2). We note that the integral conditions indicated in (2.3) are equivalent to

\[\int_{s_0}^{s} \delta(s)^2 ds < \infty \quad (3.4)\]

and

\[\int_{s_0}^{s} |\dot{\delta}(s)| ds < \infty, \quad (3.5)\]

respectively.

**Step 1.** We show that \(\int_{s_0}^{s} \dot{\delta}(s)^2 ds < \infty\). We multiply Eq (3.2) by \(\dot{v}\), and integrate the resulting equation on \([s_0, s]\) to obtain

\[\int_{s_0}^{s} \{(kv - \dot{v})^\alpha \cdot \dot{v} + \beta - \alpha(k + 1)\} \int_{s_0}^{s} (kv - \dot{v})^\alpha \dot{v} dr\]
\[ -\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1}v^{\lambda+1} - \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r)v^{\lambda} \dot{v}dr = \text{const.} \tag{3.6} \]

Since integral by parts implies that
\[ \int_{s_0}^{s} \{(kv - \dot{v})^\alpha\} \dot{v}dr = -\int_{s_0}^{s} \{(kv - \dot{v})^\alpha\} (kv - \dot{v})dr + k \int_{s_0}^{s} \{(kv - \dot{v})^\alpha\} \dot{v}dr \]
\[ = -\frac{\alpha}{\alpha+1}(kv - \dot{v})^{\alpha+1} + kv(kv - \dot{v})^\alpha - k \int_{s_0}^{s} (kv - \dot{v})^\alpha \dot{v}dr + \text{const}, \]
we obtain from (3.6)
\[ -\frac{\alpha}{\alpha+1}(kv - \dot{v})^{\alpha+1} + kv(kv - \dot{v})^\alpha + \{\beta - \alpha(k+1) - k\} \int_{s_0}^{s} (kv - \dot{v})^\alpha \dot{v}dr \]
\[ -\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1}v^{\lambda+1} - \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r)v^{\lambda} \dot{v}dr = \text{const.} \]

The boundedness of \( v \) and \( \dot{v} \) shown in Lemma 2 imply that
\[ \{\beta - \alpha(k+1) - k\} \int_{s_0}^{s} (kv - \dot{v})^\alpha \dot{v}dr - \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r)v^{\lambda} \dot{v}dr = O(1) \] as \( s \to \infty \). \tag{3.7}

Now, since \( 0 < \alpha \leq 1 \), the inequality
\[ (X^\alpha - Y^\alpha)(X - Y) \geq c_0(X - Y)^2(X + Y)^{\alpha-1} \]
for all \( X,Y \geq 0 \) with \( X + Y > 0 \) \tag{3.8}
holds for some constant \( c_0 > 0 \). Therefore we obtain
\[ \{(kv)^\alpha - (kv - \dot{v})^\alpha\} \dot{v} \geq c_0((kv) + (kv - \dot{v}))^{\alpha-1}\dot{v}^2; \]
that is,
\[ (kv - \dot{v})^\alpha \dot{v} \leq -c_1\dot{v}^2 + k^\alpha v^\alpha \dot{v}, \tag{3.9} \]
where \( c_1 > 0 \) is a constant. Let \( \beta - \alpha(k+1) - k > 0 \). From (3.7) and (3.9) we find that
\[ -c_1\{\beta - \alpha(k+1) - k\} \int_{s_0}^{s} \dot{v}^2dr + \{\beta - \alpha(k+1) - k\} \frac{k^\alpha}{\alpha+1}v^{\alpha+1} \]
\[ \geq \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r)v^{\lambda} \dot{v}dr + O(1) \] as \( s \to \infty \). \tag{3.10}

Suppose \( \int_{t}^{\infty} \epsilon(t)^2 dt/t < \infty \), that is, (3.4) holds. Schwarz's inequality and (3.10) imply that
\[ c_2 \int_{s_0}^{s} \dot{v}^2dr \leq c_3 - c_4 \int_{s_0}^{s} \delta(r)v^{\lambda} \dot{v}dr \]
\[ \leq c_3 + c_5 \left( \int_{s_0}^{s} \delta(r)^2 dr \right)^{1/2} \left( \int_{s_0}^{s} \dot{v}^2dr \right)^{1/2} \]
with some positive constants $c_2, c_3, c_4$ and $c_6$. We therefore obtain $\int^\infty \dot{v}^2 dr < \infty$. Suppose next $\int^\infty |\epsilon'(t)| dt < \infty$, that is, \((3.5)\) holds. We find from \((3.10)\) that

$$c_2 \int^s_0 \dot{v}^2 dr \leq c_3 - c_4 \int^s_0 \delta(r) \left( \frac{v^{\lambda+1}}{\lambda+1} \right) dr$$

$$\leq c_6 - \frac{c_4}{\lambda+1} \delta(s) v^{\lambda+1} - c_7 \int^s_0 \delta(r) dr,$$

where $c_6, c_7, c_8$ and $c_9$ are some positive constants. Hence we obtain $\int^\infty \dot{v}^2 dr < \infty$. The case where $\beta - \alpha(k+1) - k < 0$ can be treated similarly.

Since we have shown $\int^\infty \dot{v}^2 dr < \infty$, and $\alpha \leq 1$, Eq \((3.3)\) shows that $\dot{v} = O(1)$ as $s \to \infty$.

Hence we obtain $\int^\infty \dot{v}^2 dr < \infty$.

\textbf{Step 2. We show that} \(\liminf_{s\to\infty} v(s) > 0\). To see this by contradiction, we will derive a contradiction by assuming $\liminf_{s\to\infty} v(s) = 0$. The argument is divided into the two cases:

Case (a): $v(s)$ monotonically decreases to 0 (and so, $\dot{v}(s) \leq 0$);

Case (b): $\dot{v}(s)$ changes the sign in any neighborhood of $+\infty$.

Let case (a) occur. Put $v = x_1$ and $\dot{v} = x_2$, and $x = (x_1, x_2)$. Then, $x$ satisfies the binary system

$$\dot{x} = Ax + f(s, x),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ k \left( \frac{\beta}{\alpha} - k - 1 \right) & \left( \frac{\beta}{\alpha} - 2k - 1 \right) \end{pmatrix},$$

and

$$f(s, x) = \begin{pmatrix} 0 \\ -\beta \alpha^{-\alpha} \left( 1 + \delta(s) \right) \left( k|x_1| + |x_2| \right)^{1-\alpha} |x_1|^\lambda \end{pmatrix}.$$  

Here we have used the fact that $v(s) > 0$ and $\dot{v}(s) \leq 0$. Since

$$(k|x_1| + |x_2|)^{1-\alpha} |x_1|^\lambda \leq \left( \max \{ k \} \right)^{1-\alpha} \left( |x_1| + |x_2| \right)^{\lambda-\alpha+1},$$

and \((v(s), \dot{v}(s))\) corresponds to a solution $x(s)$ of system \((3.11)\) satisfying $\lim_{s\to\infty} x(s) = 0$, by \cite[Chapter 3, Theorem 5]{2} we have

$$\lim_{s\to\infty} \frac{\log \| x(s) \|}{s} = \Lambda,$$

where $\Lambda$ is the real part of an eigenvalue of $A$. All the eigenvalues of $A$ are $k$ and $-(\beta/\alpha - k - 1)$; the former is positive and the latter negative. Since $\|x(s)\| \to 0$, we have $\Lambda = -(\beta/\alpha - k - 1)$. By the assumption \((2.2)\) we find a small $\eta > 0$ satisfying $\sigma + \lambda(-\beta/\alpha + 1) + \lambda \eta < -1$. By \((3.12)\) we obtain

$$v(s) \leq e^{-(\beta/\alpha - k - 1) + \eta} s, \quad \text{near} + \infty.$$
This means that $u(t) \leq t^{-\beta/\alpha+1+\eta}$ near $+\infty$. Then

$$t^\beta(-u'(t))^{\alpha} \leq c_1 \int_0^t r^{\sigma+\lambda(-\beta/\alpha+1)+\lambda\eta} dr = O(1) \text{ as } t \to \infty.$$  

This contradicts the property of slowly decaying solution $\lim_{t \to \infty} t^\beta(-u'(t))^{\alpha} = \infty$. Hence Case (a) never occurs. As in the proof of [6, Theorem 1.3], we can show that Case (b) never occurs. Hence we have $\lim \inf_{s \to \infty} v(s) > 0$.

The remainder of the proof of the fact $\lim_{s \to \infty} v(s) = 1$ proceeds as in the proof of [6, Theorem 1.3]. We leave them to the reader.

**Proof of Theorem 2.** As in the proof of Theorem 1, we will show that $\lim_{s \to \infty} v(s) = 1$, where $v(s)$ is introduced in Lemma 2. Define

$$w = (kv - \dot{v})^{\alpha}. \quad (3.13)$$

By Eq (3.2) we know that

$$\dot{w} + \{\beta - \alpha(k+1)\}w - \hat{C}^{\lambda-\alpha}\{1+\delta(s)\}v^\lambda = 0.$$  

Let us rewrite this equation as

$$\dot{w} + aw - b\{1+\delta(s)\}v^\lambda = 0, \quad (3.14)$$

where we have put $\beta - \alpha(k+1) = a$ and $\hat{C}^{\lambda-\alpha} = b$. We therefore find that

$$v = b^{-1/\lambda}(1+\delta(s))^{-1/\lambda}(\dot{w}+aw)^{1/\lambda},$$

and $w$ satisfied the ODE

$$((1+\delta(s))^{-1/\lambda}(\dot{w}+aw)^{1/\lambda})' - k(1+\delta(s))^{-1/\lambda}(\dot{w}+aw)^{1/\lambda} + b^{1/\alpha}w^{1/\alpha} = 0. \quad (3.15)$$

We note, by the definition (3.13), (3.14), and Lemma 2, that $w, \dot{w} = O(1)$ as $s \to \infty$. By putting $(1+\delta(s))^{-1/\lambda} = h(s), 1/\lambda = \rho$, and $1/\alpha = \gamma$, we can rewrite (3.15) simply as

$$(h(s)(\dot{w} + aw)^{\rho})' - kh(s)(\dot{w} + aw)^{\rho} + b^\rho w^{1/\alpha} = 0. \quad (3.16)$$

We note that our assumptions (2.4) are equivalent to

$$\lim_{s \to \infty} \delta(s) = 0 \quad (3.17)$$

and

$$\int_0^{\infty} |\delta(s)| ds < \infty. \quad (3.18)$$

It should be emphasized that Eq (3.16) is equivalent to

$$\dot{w} + \left[a - \frac{k}{\rho} + \frac{h(s)}{\rho h(s)}\right] \dot{w} + \frac{a}{\rho} \left[\frac{h(s)}{h(s)} - k\right] w + \frac{b^\rho}{\rho h(s)}(\dot{w} + aw)^{1-\rho}w^{\gamma} = 0 \quad (3.19)$$

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By using (3.18) and computing as in the proof of Theorem 1, we find from Eq (3.16) that
\[ (a - k) \int_{s_{0}}^{s} h(r)(\dot{w} + aw)^{\rho}\dot{w} dr = O(1) \quad \text{as} \quad s \to \infty. \tag{3.20} \]

Notice that the assumption \( \beta - \alpha(k+1) - k \neq 0 \) means that \( a - k \neq 0 \). Since \( \alpha \geq 1 \) and \( \lambda > \alpha \), we have \( \rho < 1 \). So inequality (3.8) implies, as before, that
\[ \{(\dot{w} + aw)^{\rho} - (aw)^{\rho}\}\dot{w} \geq c_{0}\dot{w}^{2}\{(\dot{w} + aw) + |aw|\}^{\rho-1}; \]
that is,
\[ h(r)(\dot{w} + aw)^{\rho}\dot{w} \geq a^{\rho}h(r)w^{\rho}\dot{w} + c_{1}h(r)\dot{w}^{2} \]
for some constant \( c_{1} > 0 \). Hence by (3.20) and the fact that \( h(\infty) = 1 \), we find that
\[ c_{2} \int_{s_{0}}^{s} h(r)w^{\rho}\dot{w} dr + c_{3} \int_{s_{0}}^{s} \dot{w}^{2} dr = O(1) \quad \text{as} \quad s \to +\infty. \]

By integral by parts and by using this relation, we find that \( \int_{\infty}^{\infty} \dot{w}^{2} ds < \infty \). Moreover, since \( \rho < 1 \), we find that \( \lim_{s \to \infty} \dot{w}(s) = 0 \) as in the proof of Theorem 1.

We want to show that \( \liminf_{s \to \infty} w(s) > 0 \). The proof is done by a contradiction. Firstly suppose that \( w(s) \) decreases to \( 0 \) as \( s \to \infty \). Then, as in the proof of Theorem 1, we know by [2, Chapter 3, Theorem 5] that for every \( \eta > 0 \)
\[ w(s) \leq e^{(-\beta + \alpha(k+1)+\eta)s} \quad \text{as} \quad s \to \infty. \tag{3.21} \]
The definition (3.13) is equivalent to \( (e^{-ks}v) = -e^{-ks}w^{1/\alpha} \); and so
\[ v(s) = e^{ks} \int_{s}^{\infty} e^{-kr}w^{1/\alpha} dr. \tag{3.22} \]
Here we have employed the fact that \( \lim_{s \to \infty} v(s)/e^{ks} = 0 \). Combining (3.21) with (3.22), we get the estimate \( t^{\beta}|u'(t)| = O(1) \). Recall that this yields a contradiction.

Next, let \( \liminf_{s \to \infty} w(s) = 0 \) and \( \dot{w} \) change the sign in any neighborhood of \( +\infty \). Define the auxiliary function \( H(s) \) by
\[ H(s) = k^\alpha \left[1 - \frac{h(s)}{kh(s)}\right]^{\frac{\beta}{\alpha}}. \]
Then, in the region \( 0 < w < H(s) \), we have \( \dot{w} > 0 \). On the other hand in the region \( w > H(s) \), we have \( \dot{w} < 0 \). Hence, we can find out two sequences \( \{\xi_{n}\} \) and \( \{\eta_{n}\} \) satisfying
\[ \xi_{n} < \eta_{n} < \xi_{n+1} < \eta_{n+1} < \cdots; \lim_{n \to \infty} \xi_{n} = \lim_{n \to \infty} \eta_{n} = \infty; \]
and
\[ w(\eta_{n}) \to 0, \quad w(\xi_{n}) = H(\xi_{n}) \to k^\alpha \quad \text{as} \quad n \to \infty \quad \text{and} \quad \dot{w} \leq 0 \quad \text{on} \quad [\xi_{n}, \eta_{n}]. \tag{3.23} \]
Multiplying (3.19) by $\dot{w}$ and integrating the resulting equation on $[\xi_n, \eta_n]$, we have

$$\frac{1}{2}(\dot{w}(\eta_n)^2 - \dot{w}(\xi_n)^2) + \left(a - \frac{k}{\rho}\right)\int_{\xi_n}^{\eta_n} \dot{w}^2 dr + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr$$

$$- \frac{ak}{2\rho} (w(\eta_n)^2 - w(\xi_n)^2) + \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr + \frac{b^\rho}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} \dot{w} \dot{w}^\rho dr = 0.$$  

Noting the facts $\dot{w}(\infty) = 0$ and $\int_\infty \dot{w}^2 dr < \infty$, we have as $n \to \infty$

$$o(1) + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr - \frac{ak}{2\rho} (o(1) - k^{2\alpha})$$

$$+ \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr + \frac{a^{1-\rho}b}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} \dot{w}^{1+\gamma-\rho} \dot{w} dr \leq 0.$$  

(3.24)

Now, let us estimate each term of the above. We have firstly

$$\left| \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr \right| \leq C_0 \sup_{[\xi_n, \infty)} |\dot{h}| \int_\xi^{\infty} \dot{w}^2 dr = o(1) \quad \text{as} \quad n \to \infty;$$

and

$$\left| \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w} \dot{w} dr \right| = \left| \frac{\dot{h}(c_n)}{h(c_n)} \int_{\xi_n}^{\eta_n} \dot{w} \dot{w} dr \right| = \left| \frac{\dot{h}(c_n)}{2h(c_n)} (w(\xi_n)^2 - w(\eta_n)^2) \right| = o(1) \quad \text{as} \quad n \to \infty.$$  

Here $C_0 > 0$ is a constant, and we have used a variant of the mean value theorem for integrals; that is $c_n$ is a number satisfying $\xi_n < c_n < \eta_n$. Finally, we obtain

$$\int_{\xi_n}^{\eta_n} \frac{1}{h(r)} \dot{w}^{1+\gamma-\rho} \dot{w} dr = \int_{\xi_n}^{\eta_n} \frac{1}{h(r)^{-1} - 1} \dot{w}^{1+\gamma-\rho} \dot{w} dr + \frac{1}{2 + \gamma - \rho} (w(\eta_n)^{2+\gamma-\rho} - w(\xi_n)^{2+\gamma-\rho})$$

$$= (h(d_n)^{-1} - 1) \int_{\xi_n}^{\eta_n} \dot{w}^{1+\gamma-\rho} \dot{w} dr + \frac{1}{2 + \gamma - \rho} (o(1) - k^{a(2+\gamma-\rho)})$$

$$= o(1) - \frac{k^{a(2+\gamma-\rho)}}{2 + \gamma - \rho} \quad \text{as} \quad n \to \infty.$$  

Here $d_n$ is a number satisfying $\xi_n < d_n < \eta_n$. Therefore (3.24) can be simplified into

$$\frac{ak^{2\alpha+1}}{2\rho} + o(1) \leq \frac{a^{1-\rho}b}{\rho(2 + \gamma - \rho)} k^{a(2+\gamma-\rho)} \quad \text{as} \quad n \to \infty.$$  

This gives a contradiction. Hence we find that $\liminf_{s \to \infty} v(s) > 0$.

Arguing as in the proof of Theorem 1, we will show that $\lim_{s \to \infty} v(s) = 1$. The details are left to the reader.

To see Theorem 3, we will show that $\lim_{s \to \infty} v(s) = 1$, where $v(s)$ is introduced by Lemma 2, as before. However, we can not help omitting the proof for the lack of space.
References


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