### Title
Asymptotic forms of slowly decaying positive solutions of second-order quasilinear ordinary differential equations

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1 Introduction

Let us consider the quasilinear ODE

\[(a(t)|u'|^{\alpha-1}u')' + b(t)|u|^{\lambda-1}u = 0, \quad \text{near } + \infty\]  

(A)

where we assume that $\alpha > 0$ and $\lambda > 0$ are constants, $a(t)$ and $b(t)$ are positive continuous functions satisfying $\int^\infty a(t)^{-1/\alpha}dt < \infty$. Every positive solution $u$ of (A) satisfies one of the following three asymptotic properties as $t \to \infty$:

\[u(t) \sim c_1\]  

for some constant $c_1 > 0$; \hspace{1cm} (1.1)

\[u(t) \sim c_2 \int_t^\infty a(s)^{-1/\alpha}ds\]  

for some constant $c_2 > 0$; \hspace{1cm} (1.2)

and

\[u(t) \to 0 \quad \text{and} \quad \frac{u(t)}{\int_t^\infty a(s)^{-1/\alpha}ds} \to \infty.\] \hspace{1cm} (1.3)

Asymptotic properties of solutions $u$ satisfying either (1.1) or (1.2) were widely investigated. For example, necessary and sufficient conditions of existence of such solutions were established in [4, 7]. On the other hand there seems to be less information about qualitative properties of solutions $u$ satisfying (1.3). Motivated by this fact, in the article we will discuss about asymptotic behavior of solutions $u$ satisfying (1.3); in particular, we try to find exact asymptotic forms of such solutions near $+\infty$. In what follows we refer solutions $u$ satisfying (1.3) as slowly decaying solutions.

Remark 1. When $\int^\infty a(t)^{-1/\alpha}dt = \infty$, Eq (A) reduces to the simpler one of the form

\[(|u'|^{\alpha-1}u')' + \tilde{b}(t)|u|^{\lambda-1}u = 0, \quad \text{near } + \infty,\]

where $\tilde{b}(t)$ is a positive continuous function. Studies of this equation were, for example, the main objective of [6]; and asymptotic properties of solutions have been fully established.
2 Preparatory observations and results

Asymptotic forms of slowly decaying solutions may be strongly affected by those of coefficient functions $a(t), b(t)$ and the exponents $\alpha$ and $\lambda$. Therefore let us consider the following ODE, which has more restrictive appearance than Eq (A):

$$(t^{|u'|^{|\alpha-1}u'})' + t^{|u|^{|\lambda-1}u} = 0 \quad \text{near} \ + \infty. \quad (E)$$

In the sequel we assume the next conditions:

(A) $\alpha, \beta, \lambda$ and $\sigma$ are constants satisfying $\lambda > \alpha > 0$ and $\beta > \alpha$;
(B) $\epsilon(t)$ is a continuous (or $C^1$-)function defined near $+\infty$ satisfying $\lim_{t \to +\infty} \epsilon(t) = 0$.

Additional conditions will be given later.

Since we can regard Eq (E) as a "perturbed equation" of the ODE

$$(t^{|u'|^{|\alpha-1}u'})' + t^{|u|^{|\lambda-1}u} = 0 \quad \text{near} \ + \infty, \quad (E_0)$$

we conjecture that slowly decaying solutions of Eq (E) and those of Eq (E)$\_0$ may have the same asymptotic behavior near $+\infty$ in some sense, if $\epsilon(t)$ is sufficiently small. It is easily seen that Eq (E)$\_0$ has an exact slowly decaying solution $u_0$ given by

$$u_0(t) = \hat{C}t^{-k}, \quad (2.1)$$

where

$$k = \frac{1 + \sigma - (\beta - \alpha)}{\lambda - \alpha}, \quad \text{and} \quad \hat{C}^{\lambda - \alpha} = k^\alpha \{\beta - \alpha(k + 1)\}$$

if

$$(\beta - \alpha) - 1 < \sigma < \frac{\lambda}{\alpha}(\beta - \alpha) - 1. \quad (2.2)$$

Below we always assume (2.2). We can show that our conjecture is true in various cases:

**Theorem 1.** Let $\alpha \leq 1$ and $\beta - \alpha(k + 1) - k \neq 0$. If

$$\int_\infty^t \frac{1}{t} dt < \infty \quad \text{or} \quad \int_\infty^t |\epsilon'(t)| dt < \infty, \quad (2.3)$$

then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to +\infty$, where $u_0(t)$ is given by (2.1).

**Theorem 2.** Let $\alpha \geq 1$ and $\beta - \alpha(k + 1) - k \neq 0$. If

$$\lim_{t \to +\infty} t\epsilon'(t) = 0 \quad \text{and} \quad \int_\infty^t |\epsilon'(t)| dt < \infty, \quad (2.4)$$

then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to +\infty$. 


Theorem 3. Let $\alpha \geq 1$ and $\alpha(2k+1) - \beta < 0$. If (2.3) holds, then every slowly decaying positive solution $u$ of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \to \infty$.

Example 1. Let $N > m > 1$ and $N \geq 2$. Consider radial solutions $u = u(|x|)$ of the following quasilinear PDE in an exterior domain of $\mathbb{R}^N$:
\[ \text{div}(|Du|^{m-2}Du) + |x|^{\ell}(1 + |x|^{-\theta})|u|^\lambda u = 0 \quad \text{near} \quad \infty, \]
where $\lambda > m - 1, \ell \in \mathbb{R}, \theta > 0$, and $-m < \ell < \frac{\lambda}{m-1}(N - m) - N$. We know that $u$ solves the ODE
\[ (r^{N-1}|u'|^{m-2}u')' + r^{N-1+\ell}(1 + r^{-\rho})|u|^\lambda u = 0 \quad \text{near} \quad + \infty. \]
By Theorems 1 and 2, if $\lambda \neq (mN - N + m\ell)/(N - m)$, then every slowly decaying positive solution $u$ of this equation satisfies
\[ u(r) \sim Ar^{-(\ell+m)/(\lambda-m+1)} \quad \text{as} \quad r \to +\infty, \]
where $A$ is a positive constant given by
\[ A^{\lambda-m+1} = \left( \frac{\ell + m}{\ell - m + 1} \right)^{m-1} \cdot \frac{N\lambda - Nm + N - m\ell - m\lambda + \ell}{\lambda - m + 1}. \]

Remark 1. For the autonomous equation $\text{div}(|Du|^{m-2}Du) + |u|^\lambda u = 0$, the assertion of Example 1 was established in [1] based on the theory of autonomous dynamical systems. Related results are found in [3, 5].

3 Sketches of the proof of the results

We give the outline of the proof of Theorems 1 and 2. We begin with several auxiliary results.

Lemma 1. Let $u(t)$ be a slowly decaying positive solution of (E). Then
\[ u(t) = O(u_0(t)) \quad \text{and} \quad u'(t) = O(|u_0'(t)|) \quad \text{as} \quad t \to \infty. \hspace{1cm} (3.1) \]

Proof. An integration of the both sides of Eq (E) on $[t_0, t]$ gives
\[ t^\beta (-u'(t))^\alpha \geq \int_{t_0}^{t} r^\sigma (1 + \epsilon(r))u^\lambda dr, \]
where $t_0$ is a sufficiently large number. Since $u$ is a decreasing function, we have
\[ t^\beta (-u'(t))^\alpha \geq u(t)^\lambda \int_{t_0}^{t} r^\sigma (1 + \epsilon(r))dr; \]
that is,

\[-u'(t)u(t)^{-\lambda/\alpha} \geq (t^{-\beta} \int_{t_{0}}^{t} r^\sigma (1 + \epsilon(r)) dr)^{1/\alpha} \cdot\]

One more integration of the both sides gives the estimates for \( u \) in (3.1).

To get the estimates for \( u' \), it suffices to notice the inequality

\[ t^\beta (-u'(t))^\alpha \leq C_1 \int_{t_{0}}^{t} r^\sigma u(r)^\lambda dr, \]

where \( C_1 > 0 \) is a constant. Note that, to get this inequality, we must use the property \( \lim_{t \to \infty} t^\beta (-u'(t))^\alpha = \infty \).

**Lemma 2.** Let \( u(t) \) be a slowly decaying positive solution of (E). Put \( t = e^s \) and \( u/u_{0} = v \). Then

(i) \( v \), and \( \dot{v} \) are bounded, and \( \dot{v} - kv < 0 \) near \( +\infty \), where \( \cdot = d/ds \);

(ii) \( v \) satisfies the ODE

\[(kv - \dot{v})^\alpha + (\beta - \alpha (k + 1))(kv - \dot{v})^\alpha - \hat{C}^{\lambda - \alpha} (1 + \delta(s)) v^\lambda = 0 \quad \text{near} \quad +\infty, \quad (3.2)\]

where \( \delta(s) = \epsilon(e^s) \).

The proof of this lemma is based on direct computations; hence we omit it.

**Remark 2.** Equation (3.2) can be rewritten as

\[ \dot{v} + \left( \frac{\beta}{\alpha} - 2k - 1 \right) \dot{v} - k \left( \frac{\beta}{\alpha} - k - 1 \right) v + \hat{C}^{\lambda - \alpha} (1 + \delta(s)) v^\lambda = 0. \quad (3.3)\]

**Lemma 3.** Let \( f(s) \) be a \( C^1 \)-function near \( +\infty \) satisfying \( \dot{f}(s) = O(1) \) as \( s \to \infty \) and \( \int_{s_{0}}^{\infty} f(s)^{2} ds < \infty \). Then \( \lim_{s \to \infty} f(s) = 0 \).

The proof of this lemma will be found in [6].

**Proof of Theorem 1.** By the change of variables \((t, u) \mapsto (s, v)\) introduced in Lemma 2, we obtain Eq (3.2). We note that the integral conditions indicated in (2.3) are equivalent to

\[ \int_{s_{0}}^{\infty} \delta(s)^{2} ds < \infty \quad (3.4) \]

and

\[ \int_{s_{0}}^{\infty} |\delta(s)| ds < \infty, \quad (3.5) \]

respectively.

**Step 1.** We show that \( \int_{s_{0}}^{\infty} \dot{v}(s)^{2} ds < \infty \). We multiply Eq (3.2) by \( \dot{v} \), and integrate the resulting equation on \([s_{0}, s]\) to obtain

\[ \int_{s_{0}}^{s} ((kv - \dot{v})^\alpha \dot{v} dr + \{\beta - \alpha (k + 1)\} \int_{s_{0}}^{s} (kv - \dot{v})^\alpha \dot{v} dr \]
Since integral by parts implies that
\[
\int_{s_0}^{s} \left\{(kv - \dot{v})^{\alpha}\right\} \dot{v} \, dr = -\int_{s_0}^{s} \left\{(kv - \dot{v})^{\alpha}\right\} (kv - \dot{v}) \, dr + k \int_{s_0}^{s} \left\{(kv - \dot{v})^{\alpha}\right\} \dot{v} \, dr
\]
we obtain from (3.6)
\[
-\frac{\alpha}{\alpha+1} (kv - \dot{v})^{\alpha+1} + kv(kv - \dot{v})^{\alpha} - k \int_{s_0}^{s} \left\{(kv - \dot{v})^{\alpha}\right\} \dot{v} \, dr + \text{const}
\]
we obtain from (3.6)
\[
\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1} v^{\lambda+1} - \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r) v^{\lambda} \dot{v} \, dr = \text{const.}
\]

The boundedness of \(v\) and \(\dot{v}\) shown in Lemma 2 imply that
\[
\{\beta - \alpha(k+1) - k\} \int_{s_0}^{s} (kv - \dot{v})^{\alpha} \dot{v} \, dr - \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r) v^{\lambda} \dot{v} \, dr = O(1)
\] as \(s \to \infty\). (3.7)

Now, since \(0 < \alpha \leq 1\), the inequality
\[
(X^\alpha - Y^\alpha)(X - Y) \geq c_0 (X - Y)^2 (X + Y)^{\alpha-1}
\]
holds for some constant \(c_0 > 0\). Therefore we obtain
\[
\{(kv)^{\alpha} - (kv - \dot{v})^{\alpha}\} \dot{v} \geq c_0 ((kv) + (kv - \dot{v}))^{\alpha-1} \dot{v}^2
\]
that is,
\[
(kv - \dot{v})^{\alpha} \dot{v} \leq -c_1 \dot{v}^2 + k^{\alpha} v^{\alpha} \dot{v},
\]
where \(c_1 > 0\) is a constant. Let \(\beta - \alpha(k+1) - k > 0\). From (3.7) and (3.9) we find that
\[
-\frac{\beta - \alpha(k+1) - k}{\alpha+1} \int_{s_0}^{s} \dot{v}^2 \, dr + \{\beta - \alpha(k+1) - k\} \frac{k^{\alpha}}{\alpha+1} v^{\alpha+1} \geq \hat{C}^{\lambda-\alpha} \int_{s_0}^{s} \delta(r) v^{\lambda} \dot{v} \, dr + O(1) \text{ as } s \to \infty.
\] (3.10)

Suppose \(\int_{t_0}^{\infty} e(t)^2 \, dt / t < \infty\), that is, (3.4) holds. Schwarz's inequality and (3.10) imply that
\[
c_2 \int_{s_0}^{s} \dot{v}^2 \, dr \leq c_3 - c_4 \int_{s_0}^{s} \delta(r) v^{\lambda} \dot{v} \, dr
\]
\[
\leq c_3 + c_6 \left( \int_{s_0}^{s} \delta(r)^2 \, dr \right)^{1/2} \left( \int_{s_0}^{s} \dot{v}^2 \, dr \right)^{1/2}
\]
with some positive constants $c_2, c_3, c_4$ and $c_6$. We therefore obtain $\int_{0}^{\infty} \dot{v}^2 dr < \infty$. Suppose next $\int_{0}^{\infty} |\epsilon'(t)| dt < \infty$, that is, (3.5) holds. We find from (3.10) that

$$c_2 \int_{0}^{s} \dot{v}^2 dr \leq c_3 - c_4 \int_{0}^{s} \delta(r) \left( \frac{v^{\lambda+1}}{\lambda+1} \right) dr$$

$$\leq c_6 - \frac{c_4}{\lambda+1} \delta(s) v^{\lambda+1} - c_7 \int_{0}^{s} \delta(r) v^{\lambda+1} dr \leq c_8 + c_9 \int_{s_0}^{s} \dot{\delta}(r) v^{\lambda+1} dr$$

where $c_6, c_7, c_8$ and $c_9$ are some positive constants. Hence we obtain $\int_{0}^{\infty} \dot{v}^2 dr < \infty$. The case where $\beta - \alpha(k+1) - k < 0$ can be treated similarly.

Since we have shown $\int_{0}^{\infty} \dot{v}^2 dr < \infty$, and $\alpha \leq 1$, Eq (3.3) shows that $\dot{v} = O(1)$ as $s \to \infty$. Therefore by Lemma 3 we find that $\lim_{s \to \infty} \dot{v}(s) = 0$.

**Step 2. We show that** $\lim_{s \to \infty} v(s) > 0$. **To see this by contradiction, we will derive a contradiction by assuming** $\lim_{s \to \infty} v(s) = 0$. The argument is divided into the two cases:

Case (a): $v(s)$ monotonically decreases to 0 (and so, $\dot{v}(s) \leq 0$);

Case (b): $\dot{v}(s)$ changes the sign in any neighborhood of $+\infty$.

Let case (a) occur. Put $v = x_1$ and $\dot{v} = x_2$, and $x = \{x_1, x_2\}$. Then, $x$ satisfies the binary system

$$\dot{x} = Ax + f(s, x), \quad (3.11)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ k \left( \frac{\beta}{\alpha} - k - 1 \right) & -\left( \frac{\beta}{\alpha} - 2k - 1 \right) \end{pmatrix},$$

and

$$f(s, x) = \begin{pmatrix} 0 \\ -\frac{\alpha^{\lambda-\alpha}}{\alpha} \{1 + \delta(s)\} (k|x_1| + |x_2|)^{1-\alpha} |x_1|^\lambda \end{pmatrix}.$$
This means that $u(t) \leq t^{-\beta/\alpha+1+\eta}$ near $+\infty$. Then
\[
t^\beta(-u'(t))^\alpha \leq c_1 \int_0^t r^{\sigma+\lambda(-\beta/\alpha+1)+\lambda\eta} dr = O(1) \quad \text{as } t \to \infty.
\]
This contradicts the property of slowly decaying solution $\lim_{t \to \infty} t^\beta(-u'(t))^\alpha = \infty$. Hence Case (a) never occurs. As in the proof of [6, Theorem 1.3], we can show that Case (b) never occurs. Hence we have $\liminf_{s \to \infty} v(s) > 0$.

The remainder of the proof of the fact $\lim_{s \to \infty} v(s) = 1$ proceeds as in the proof of [6, Theorem 1.3]. We leave them to the reader.

\textbf{Proof of Theorem 2.} As in the proof of Theorem 1, we will show that $\lim_{s \to \infty} v(s) = 1$, where $v(s)$ is introduced in Lemma 2. Define
\[
w = (kv - \dot{v})^\alpha.
\]
By Eq (3.2) we know that
\[
\dot{w} + \{\beta - \alpha(k + 1)\} w - \dot{C}^{\lambda-\alpha}\{1 + \delta(s)\} v^\lambda = 0.
\]
Let us rewrite this equation as
\[
\dot{w} + \omega w - b\{1 + \delta(s)\} v^\lambda = 0,
\]
where we have put $\beta - \alpha(k + 1) = \omega$ and $\dot{C}^{\lambda-\alpha} = b$. We therefore find that
\[
v = b^{-1/\lambda}(1 + \delta(s))^{-1/\lambda}(\dot{w} + \omega w)^{1/\lambda},
\]
and $w$ satisfied the ODE
\[
((1 + \delta(s))^{-1/\lambda}(\dot{w} + \omega w)^{1/\lambda}) - k(1 + \delta(s))^{-1/\lambda}(\dot{w} + \omega w)^{1/\lambda} + b^{1/\lambda} w^{1/\alpha} = 0.
\]
We note, by the definition (3.13), (3.14), and Lemma 2, that $w, \dot{w} = O(1)$ as $s \to \infty$. By putting $(1 + \delta(s))^{-1/\lambda} = h(s), 1/\lambda = \rho$, and $1/\alpha = \gamma$, we can rewrite (3.15) simply as
\[
(h(s)(\dot{w} + \omega w)^\rho) - kh(s)(\dot{w} + \omega w)^\rho + b^\rho w^{1/\alpha} = 0.
\]
We notc that our assumptions (2.4) are equivalent to
\[
\lim_{s \to \infty} \delta(s) = 0
\]
and
\[
\int_0^\infty |\delta(s)| ds < \infty.
\]
It should be emphasized that Eq (3.16) is equivalent to
\[
\dot{w} + \left[a - \frac{k}{\rho} + \frac{\dot{h}(s)}{\rho h(s)}\right] \dot{w} + \frac{a}{\rho} \left[\frac{\dot{h}(s)}{h(s)} - k\right] w + \frac{b^\rho}{\rho h(s)}(\dot{w} + \omega w)^{1-\rho} w^\gamma = 0
\]
By using (3.18) and computing as in the proof of Theorem 1, we find from Eq (3.16) that

\[(a - k) \int_{s_0}^{s} h(r)(\dot{w} + aw)^{\rho} \dot{w} dr = O(1) \quad \text{as} \quad s \to \infty. \tag{3.20}\]

Notice that the assumption \(\beta - \alpha(k + 1) - k \neq 0\) means that \(a - k \neq 0\). Since \(\alpha \geq 1\) and \(\lambda > \alpha\), we have \(\rho < 1\). So inequality (3.8) implies, as before, that

\[\{(\dot{w} + aw)^{\rho} - (aw)^{\rho}\} \dot{w} \geq c_0 \dot{w}^2\{(\dot{w} + aw) + |aw|\}^{\rho - 1};\]

that is,

\[h(r)(\dot{w} + aw)^{\rho} \dot{w} \geq a^{\rho} h(r)w^{\rho} \dot{w} + c_1 h(r) \dot{w}^2\]

for some constant \(c_1 > 0\). Hence by (3.20) and the fact that \(h(\infty) = 1\), we find that

\[c_2 \int_{s_0}^{s} h(r)w^{\rho} \dot{w} dr + c_3 \int_{s_0}^{s} \dot{w}^2 dr = O(1) \quad \text{as} \quad s \to +\infty.\]

By integral by parts and by using this relation, we find that \(\int_{s_0}^{s} \dot{w}^2 ds < \infty\). Moreover, since \(\rho < 1\), we find that \(\lim_{s \to \infty} \dot{w}(s) = 0\) as in the proof of Theorem 1.

We want to show that \(\lim \inf_{s \to \infty} w(s) > 0\). The proof is done by a contradiction. Firstly suppose that \(w(s)\) decreases to 0 as \(s \to \infty\). Then, as in the proof of Theorem 1, we know by [2, Chapter 3, Theorem 5] that for every \(\eta > 0\)

\[w(s) \leq e^{-\beta + \alpha(k + 1) + \eta} s \quad \text{as} \quad s \to \infty. \tag{3.21}\]

The definition (3.13) is equivalent to \((e^{-ks}v) = -e^{-ks}w^{1/\alpha}\); and so

\[v(s) = e^{ks} \int_{s}^{\infty} e^{-kr}w^{1/\alpha} dr. \tag{3.22}\]

Here we have employed the fact that \(\lim_{s \to \infty} v(s)/e^{ks} = 0\). Combining (3.21) with (3.22), we get the estimate \(t^2|u'(t)| = O(1)\). Recall that this yields a contradiction.

Next, let \(\lim \inf_{s \to \infty} w(s) = 0\) and \(\dot{w}\) change the sign in any neighborhood of \(+\infty\). Define the auxiliary function \(H(s)\) by

\[H(s) = k^\alpha \left[1 - \frac{\dot{h}(s)}{kh(s)}\right]^{\frac{1}{\alpha}}.\]

Then, in the region \(0 < w < H(s)\), we have \(\dot{w} > 0\). On the other hand in the region \(w > H(s)\), we have \(\dot{w} < 0\). Hence, we can find out two sequences \(\{\xi_n\}\) and \(\{\eta_n\}\) satisfying

\[\xi_n < \eta_n < \xi_{n+1} < \eta_{n+1} < \cdots; \lim_{n \to \infty} \xi_n = \lim_{n \to \infty} \eta_n = \infty;\]

and

\[w(\eta_n) \to 0, \quad w(\xi_n) = H(\xi_n) \to k^\alpha \quad \text{as} \quad n \to \infty \quad \text{and} \quad \dot{w} \leq 0 \quad \text{on} \quad [\xi_n, \eta_n]. \tag{3.23}\]
Multiplying (3.19) by $\dot{w}$ and integrating the resulting equation on $[\xi_n, \eta_n]$, we have

$$\frac{1}{2}(\dot{w}(\eta_n)^2 - \dot{w}(\xi_n)^2) + \left( a - \frac{k}{\rho} \right) \int_{\xi_n}^{\eta_n} w^2 dr + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{h(r)}{h(r)} w^2 dr$$

$$- \frac{ak}{2\rho} (w(\eta_n)^2 - w(\xi_n)^2) + \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w^2 dr + \frac{b^\rho}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} (w + aw)^{1-\rho} w^\gamma dr = 0.$$  

Noting the facts $\dot{w}(\infty) = 0$ and $\int_\infty^\infty \dot{w}^2 dr < \infty$, we have as $n \to \infty$

$$o(1) + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w^2 dr - \frac{ak}{2\rho} (o(1) - k^{2a})$$

$$+ \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w dr + \frac{a^{1-\rho}\Psi}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} w^{1+\gamma-\rho} \dot{w} dr \leq 0.$$  

(3.24)

Now, let us estimate each term of the above. We have firstly

$$\left| \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w^2 dr \right| \leq C_0 \sup_{[\xi_n, \infty)} |\dot{h}| \int_{\xi_n}^{\infty} \dot{w}^2 dr = o(1) \quad \text{as} \quad n \to \infty;$$

and

$$\left| \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w^2 dr \right| \leq C_0 \sup_{[\xi_n, \infty)} |\dot{h}| \int_{\xi_n}^{\eta_n} \dot{w}^2 dr = o(1) \quad \text{as} \quad n \to \infty.$$  

Here $C_0 > 0$ is a constant, and we have used a variant of the mean value theorem for integrals; that is $c_n$ is a number satisfying $\xi_n < c_n < \eta_n$. Finally, we obtain

$$\int_{\xi_n}^{\eta_n} \frac{1}{h(r)} w^{1+\gamma-\rho} \dot{w} dr = \int_{\xi_n}^{\eta_n} [h(r)^{-1} - 1] w^{1+\gamma-\rho} \dot{w} dr + \frac{1}{2 + \gamma - \rho} (w(\eta_n)^{2+\gamma-\rho} - w(\xi_n)^{2+\gamma-\rho})$$

$$= (h(d_n)^{-1} - 1) \int_{\xi_n}^{\eta_n} w^{1+\gamma-\rho} \dot{w} dr + \frac{1}{2 + \gamma - \rho} (o(1) - k^{a(2+\gamma-\rho)})$$

$$= o(1) - \frac{k^{a(2+\gamma-\rho)}}{2 + \gamma - \rho} \quad \text{as} \quad n \to \infty.$$  

Here $d_n$ is a number satisfying $\xi_n < d_n < \eta_n$. Therefore (3.24) can be simplified into

$$\frac{ak^{2a+1}}{2\rho} + o(1) \leq \frac{a^{1-\rho}\Psi k^{a(2+\gamma-\rho)}}{\rho(2 + \gamma - \rho)} \quad \text{as} \quad n \to \infty.$$  

This gives a contradiction. Hence we find that $\liminf_{s \to \infty} v(s) > 0$.

Arguing as in the proof of Theorem 1, we will show that $\lim_{s \to \infty} v(s) = 1$. The details are left to the reader.

To see Theorem 3, we will show that $\lim_{s \to \infty} v(s) = 1$, where $v(s)$ is introduced by Lemma 2, as before. However, we can not help omitting the proof for the lack of space.
References


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