数理解析研究所講究録
第 1582 巻 2008 年 18-22

An identity for a quasilinear ordinary differential equation and its applications

岡山理科大学・理学部　田中　敏 (Satoshi Tanaka)
Faculty of Science,
Okayama University of Science

We consider the two-point boundary value problem for the quasilinear ordinary differential equation

$$(\varphi_p(u'))' + a(x)f(u) = 0, \quad x_0 < x < x_1, \tag{1}$$

$$u(x_0) = u(x_1) = 0, \tag{2}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $a \in C^1[x_0, x_1]$, $a(x) > 0$ for $x \in [x_0, x_1]$, and $f \in C^1(\mathbf{R})$.

Recently there has been considerable investigation concerning two-point boundary value problems for quasilinear ordinary differential equations. For example, we refer the reader to [1], [4], [7], [8], [11], [15], [16], [18], [19], [20], [23], [24], [26], and [27]. In order to find the exact number of solutions of problem (1)–(2), the linearized equation

$$(\varphi_p'(u')w')' + a(x)f'(u)w = 0 \tag{3}$$

is studied frequently, where u is a solution of (1) and $\varphi_p'(s) = (p-1)|s|^{p-2}$ for $s \neq 0$. See, for example, [9], [10], [11], [12], [13], [14], [21], and [25].

The main result of this paper is the following identity. For the case $p = 2$, this identity has been obtained in [25], by using the idea due to Korman and Ouyang [12]. (See also [11, Lemma 4.1].)

Proposition 1. *Let u and w be solutions of* (1) *and* (3), *respectively. Suppose that $g \in C^2[x_0, x_1]$. Then*

$$\begin{aligned}\Big[g\varphi_p(u')w' + (p-1)^{-1}gaf(u)w - g'\varphi_p(u')w\Big]' \\ = -g''\varphi_p(u')w + (p-1)^{-1}(pg'a + ga')f(u)w\end{aligned} \tag{4}$$

for $u'(x) \neq 0$. In particular, if $a \in C^2[x_0, x_1]$ and $g(x) = [a(x)]^{-1/p}$, then

$$\Big[g\varphi_p(u')w' + (p-1)^{-1}gaf(u)w - g'\varphi_p(u')w\Big]' = -g''\varphi_p(u')w \tag{5}$$

for $u'(x) \neq 0$.

A straightforward calculation yields (4).

By using (4) we can obtain uniqueness results for problem (1)–(2). First we study the uniqueness of positive solutions of (1)–(2) for the case where

$$f \in C^1[0,\infty), \quad f(s) > 0 \quad \text{and} \quad f'(s) > (p-1)\frac{f(s)}{s} \quad \text{for } s > 0. \tag{6}$$

For example, the function $f(s) = s^{q-1}$, $q > \max\{p, 2\}$ satisfies (6).

For existence of solutions of (1)–(2) under condition (6), we refer to [1], [4], [7], [8], [15], [16], [23], and [26]. In particular, by results in [8], [23] or [26], we see that if

$$\lim_{s\to 0} \frac{f(s)}{s^{p-1}} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty,$$

then (1)–(2) has at least one positive solution. More precise conditions for the existence of a positive solution of (1)–(2) have been obtained in [7] and [20].

We note that if (6) holds, then $s^{1-p}f(s)$ is increasing on $(0,\infty)$. On the other hand, it is known by Naito [19] and Wong [27] that if $s^{1-p}f(s)$ is decreasing on $(0,\infty)$, then problem (1)–(2) has at most one positive solution for each $a \in C^1[x_0, x_1]$ satisfying $a(x) > 0$ on $[x_0, x_1]$.

We can obtain the following result.

Theorem 1. *Let* $1 < p \le 2$. *Suppose that* (6) *holds and*

$$pa(x) + (x - x_0)a'(x) \ge 0, \quad x_0 \le x \le x_1, \tag{7}$$

$$-pa(x) + (x_1 - x)a'(x) \le 0, \quad x_0 \le x \le x_1. \tag{8}$$

Then problem (1)–(2) *has at most one positive solution.*

Theorem 1 with $p = 2$ has been established by Kwong [13] and [14]. Uniqueness results of positive solutions for the case $p = 2$ also can be found in [2], [3], [10], and [21]. In the case where $p > 1$ and $a(x) \equiv 1$, Sánchez and Ubilla [24] showed that (1)–(2) has at most one positive solution. Nabana [18] gave the uniqueness result of positive solutions of (1)–(2) when $p \ge 2$ and $a(x) \not\equiv 1$. However it seems that very little is known about the uniqueness of positive solutions of (1)–(2) for the case where $1 < p < 2$ and $a(x) \not\equiv 1$.

Next we are concerned with sign-changing solutions of problem (1)–(2) in the case where

$$f \in C^1(\mathbf{R}), \quad sf(s) > 0 \quad \text{and} \quad f'(s) > (p-1)\frac{f(s)}{s} \quad \text{for } s \ne 0. \tag{9}$$

Since $f \in C^1(\mathbf{R})$ and $a \in C^1[x_0, x_1]$, we note that the solution of (1) with the initial condition

$$u(\xi) = \alpha, \quad u'(\xi) = \beta$$

exists on $[x_0, x_1]$ and it is unique for arbitrary $\xi \in [x_0, x_1]$ and α, $\beta \in \mathbf{R}$. (See Reichel and Walter [22].) Therefore we see that zeros of every nontrivial solution of (1)–(2) are simple, so that the problem (1)–(2) with $u'(x_0) = 0$ has only the trivial solution. For simplicity we assume $u'(x_0) > 0$. The case $u'(x_0) < 0$ is can be treated similarly.

We also note that the number of zeros of a nontrivial solution u of (1) in $[x_0, x_1]$ is finite. Indeed, if u has infinitely many zeros in $[x_0, x_1]$, then we can conclude that $u(x) \equiv 0$ on $[x_0, x_1]$, by the uniqueness of initial value problems.

Hence we consider the problem

$$(10) \qquad \begin{cases} (\varphi_p(u'))' + a(x)f(u) = 0, \quad x_0 < x < x_1, \\ u(x_0) = u(x_1) = 0, \quad u'(x_0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (x_0, x_1), \end{cases}$$

where k is a positive integer.

Recently Lee and Sim [16] proved that if

$$\lim_{s \to 0} \frac{f(s)}{\varphi_p(s)} = 0 \quad \text{and} \quad \lim_{|s| \to \infty} \frac{f(s)}{\varphi_p(s)} = \infty,$$

then problem (10) has at least one solution for each $k \in \mathbf{N}$. See also [20]. In the case where $a(x) \equiv 1$, by using the result of Sánchez and Ubilla [24], we can conclude that if (9) holds and $f(-s) = -f(s)$ for $s > 0$, then problem (10) has at most one solution. For uniqueness of solutions of problem (10) with $p = 2$, we refer the reader to [2], [21], [25] and [28]. However very little is known about the uniqueness of solutions of (10) when $a(x) \not\equiv 1$ and $p \neq 2$.

We can establish the next theorem.

Theorem 2. *Let* $1 < p \le 2$ *and* $k \in \mathbf{N}$. *Assume that* (9) *and the following* (11) *hold:*

$$(11) \qquad a \in C^2[x_0, x_1] \quad \text{and} \quad ([a(x)]^{-1/p})'' \le 0, \quad x_0 \le x \le x_1.$$

Then problem (10) *has at most one solution.*

In Theorems 1 and 2, we can not remove conditions (7), (8) and (11), respectively. Indeed we have the following Theorem 3. In particular it is emphasized that the uniqueness of solutions of (10) is not caused by the smoothness of the function $a(x)$.

Theorem 3. *Let* $p > 1$. *Assume that* $f(s) = |s|^{q-2}s$, $q > p$. *For each* $k \in \mathbf{N}$, *there exists* $a \in C^\infty[x_0, x_1]$ *such that* $a(x) > 0$ *for* $x_0 \le x \le x_1$ *and that* (10) *has at least three solutions.*

In the case $p = 2$, Moore and Nehari [17] proved that there exists a piecewise continuous function $a(x)$ such that (1)–(2) has three positive solutions. By using their idea and the shooting method, we can show Theorem 3. Gaudenzi, Habets and Zanolin [5] and [6] also proved the existence of at least three positive solutions of (1)–(2) with $p = 2$ for some $a(x)$ having both positive part and negative part.

REFERENCES

[1] R. P. Agarwal, H. Lü and D. O'Regan, Eigenvalues and the one-dimensional p-Laplacian, *J. Math. Anal. Appl.* **266** (2002), 383–400.

[2] C. V. Coffman, On the positive solutions of boundary-value problems for a class of nonlinear differential equations, *J. Differential Equations* **3** (1967), 92–111.

[3] R. Dalmasso, Uniqueness of positive solutions of nonlinear second-order equations, *Proc. Amer. Math. Soc.* **123** (1995), 3417–3424.

[4] M. del Pino, M. Elgueta, R. Manasevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$. *J. Differential Equations* **80** (1989), 1–13.

[5] M. Gaudenzi, P. Habets and F. Zanolin, An example of a superlinear problem with multiple positive solutions, *Atti Sem. Mat. Fis. Univ. Modena* **51** (2003), 259–272.

[6] M. Gaudenzi, P. Habets and F. Zanolin, A seven-positive-solutions theorem for a superlinear problem, *Adv. Nonlinear Stud.* **4** (2004), 149–164.

[7] N.B. Huy, T.D. Thanh, On an eigenvalue problem involving the one-dimensional p-Laplacian, *J. Math. Anal. Appl.* **290** (2004), 123–131.

[8] H. G. Kaper, M. Knaap and M. K. Kwong, Existence theorems for second order boundary value problems, *Differential Integral Equations* **4** (1991), 543–554.

[9] I. I. Kolodner, Heavy rotating string—a nonlinear eigenvalue problem, Comm. Pure Appl. Math. 8 (1955), 395–408.

[10] P. Korman, On the multiplicity of solutions of semilinear equations, *Math. Nachr.* **229** (2001), 119–127.

[11] P. Korman, *Global solution branches and exact multiplicity of solutions for two point boundary value problems*, Handbook of differential equations, Ordinary differential equations, vol. 3, 547–606, Elsevier/North-Holland, 2006.

[12] P. Korman and T. Ouyang, Solution curves for two classes of boundary-value problems, *Nonlinear Anal.* **27** (1996), 1031–1047.

[13] M. K. Kwong, On the Kolodner-Coffman method for the uniqueness problem of Emden-Fowler BVP, *Z. Angew. Math. Phys.* **41** (1990), 79–104.

[14] M. K. Kwong, Uniqueness results for Emden-Fowler boundary value problems, *Nonlinear Anal.* **16** (1991), 435–454.

[15] Y.-H. Lee and I. Sim, Global bifurcation phenomena for singular one-dimensional p-Laplacian, *J. Differential Equations* **229** (2006), 229–256.

[16] Y.-H. Lee and I. Sim, Existence results of sign-changing solutions for singular one-dimensional p-Laplacian problems, *Nonlinear Analysis.* (to appear)

[17] R. A. Moore and Z. Nehari, Nonoscillation theorems for a class of nonlinear differential equations, *Trans. Amer. Math. Soc.* **93** (1959), 30–52.

[18] E. Nabana, Uniqueness for positive solutions of p-Laplacian problem in an annulus, *Ann. Fac. Sci. Toulouse Math.* (6) **8** (1999), 143–154.

[19] Y. Naito, Uniqueness of positive solutions of quasilinear differential equations. *Differential Integral Equations* **8** (1995), 1813–1822.

[20] Y. Naito and and S. Tanaka, Sharp conditions for the existence of sign-changing solutions to equations involving the one-dimensional p-Laplacian, *Nonlinear Analysis.* (to appear)

[21] W.-M. Ni and R. D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u,r) = 0$, *Comm. Pure Appl. Math.* **38** (1985), 67–108.

[22] W. Reichel and W. Walter, Radial solutions of equations and inequalities involving the p-Laplacian. *J. Inequal. Appl.* **1** (1997), 47–71.

[23] J. Sánchez, Multiple positive solutions of singular eigenvalue type problems involving the one-dimensional p-Laplacian. *J. Math. Anal. Appl.* **292** (2004), 401–414.

[24] J. Sánchez and P. Ubilla, Uniqueness results for the one-dimensional m-Laplacian considering superlinear nonlinearities. *Nonlinear Anal.* **54** (2003), 927–938.

[25] S. Tanaka, On the uniqueness of solutions with prescribed numbers of zeros for a two-point boundary value problem, *Differential Integral Equations* **20** (2007), 93–104.

[26] J. Wang, The existence of positive solutions for the one-dimensional p-Laplacian, *Proc. Amer. Math. Soc.* **125** (1997), 2275–2283.

[27] F. H. Wong, Uniqueness of positive solutions for Sturm-Liouville boundary value problems, *Proc. Amer. Math. Soc.* **126** (1998), 365–374.

[28] E. Yanagida, Sturmian theory for a class of nonlinear second-order differential equations, *J. Math. Anal. Appl.* **187** (1994), 650–662.