Additive indecomposability of submodular set functions and its generalization (Information and mathematics of non-additivity and non-extensivity; contacts with nonlinearity and non-commutativity)

Author(s): Murofushi, Toshiaki; Fujimoto, Katsushige; Sawata, Yoshinari

Citation: 数理解析研究所講究録 (2008), 1585: 115-119

Issue Date: 2008-02

URL: http://hdl.handle.net/2433/81513

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
Additive indecomposability of submodular set functions and its generalization*

Toshiaki Murofushi (Tokyo Institute of Technology)
Katsushige Fujimoto (Fukushima University)
Yoshinari Sawata (Tokyo Institute of Technology)

1 Introduction

This paper deals with a decomposition of a submodular set function into a sum of submodular set functions on subdomains and its generalization. Submodular set functions have an important role in mathematical programming [3], and a supermodular set function, which is the conjugate of a submodular set function (see Section 2), also is an important concept called a convex game in cooperative game theory [6]. Therefore, the additive decomposition of submodular set functions has broad application possibilities.

This paper is organized as follows. Section 2 explains basic concepts such as inclusion-exclusion family, submodularity, and (weak) k-monotonicity. Section 3 shows the results on additive decompositions of set functions we have obtained so far. Section 4 gives the main results, that is, conditions for additive indecomposability, which provide a foothold for further investigation of additive decompositions.

2 Preliminaries

For a finite set $X$, the number of elements of $X$ is denoted by $|X|$, the power set of $X$ by $2^X$, and, for an integer $k$ such that $0 \leq k \leq |X|$, the family of $k$-element subsets of $X$ is denoted by $\binom{X}{k}$, i.e,

$$\binom{X}{k} \overset{\text{def}}{=} \{Y \in 2^X \mid |Y| = k\}.$$

Throughout this paper, $E$ is assumed to be a finite set.

A family $\mathcal{A}$ of subsets of $E$ is called an antichain if $A, A' \in \mathcal{A}$ and $A \subseteq A'$ together imply $A = A'$. For antichains $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \subseteq \mathcal{B}$ if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$; then $\subseteq$ is a partial ordering on the class of all antichains over $E$, and the class forms a lattice with the following meet:

$$\mathcal{A} \cap \mathcal{B} = \text{Max}\{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$

*This work is partially supported by grants from the Ministry of Education, Culture, Sports, Science and Technology, the 21st Century COE Program "Creation of Agent-Based Social Systems Sciences" and Grant-in-Aid for Scientific Research (C), 19510136, 2007.
where, for a family $\mathcal{F}$ of sets, $\text{Max}\mathcal{F}$ is defined by

$$\text{Max}\mathcal{F} \triangleq \{M \in \mathcal{F} \mid M \text{ is maximal in } \mathcal{F} \text{ with respect to set inclusion } \subseteq\}.$$ 

A function $f : 2^E \to \mathbb{R}$ satisfying $f(\emptyset) = 0$ is called a set function on $E$. An antichain $\mathcal{A}$ of subsets of $E$ is called an inclusion-exclusion family, or an inclusion-exclusion antichain, with respect to a set function $f$ on $E$ if $(\mathcal{A})$ below holds:

$$(\mathcal{A}) : \quad f(X) = \sum_{B \subseteq A, B \neq \emptyset} (-1)^{|B|+1} f\left(X \cap \bigcap B\right) \quad \text{for all } X \subseteq E.$$ 

If an antichain $\mathcal{A}$ contains a subset $A$ such that $f(X) = f(X \cap A)$ for every $X \subseteq E$, then $\mathcal{A}$ is an inclusion-exclusion family with respect to $f$, and $\mathcal{A}$ is called a trivial inclusion-exclusion family; for example, $\{E\}$ is a trivial inclusion-exclusion family with respect to any set function on $E$. For antichains $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A} \subseteq \mathcal{B}$, and if $\mathcal{A}$ is an inclusion-exclusion family with respect to a set function $f$, then so is $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are inclusion-exclusion antichains with respect to a set function $f$, then so is $\mathcal{A} \cap \mathcal{B}$. Therefore, every set function has its least (with respect to $\subseteq$) inclusion-exclusion antichain.

For a set function $f$ on $E$, the sign inversion $-f$ of $f$ and the conjugate, or dual, $f^\#$ of $f$ are defined as follows [3]:

$$(-f)(X) \triangleq -f(X), \quad f^\#(X) \triangleq f(E) - f(E \setminus X)$$

for every $X \subseteq E$. For any set function $f_A$ on $A \subseteq E$, we regard $f_A$ as a set function on $E$ by defining $f_A(X) = f_A(X \cap A)$ for every $X \in 2^E \setminus 2^A$. Let $\mathcal{A} \subseteq 2^E$ and $\{f_A\}_{A \in \mathcal{A}}$ be a collection of set functions $f_A$ on $A \in \mathcal{A}$. Then the following holds:

$$f = \sum_{A \in \mathcal{A}} f_A \iff -f = \sum_{A \in \mathcal{A}} (-f_A) \iff f^\# = \sum_{A \in \mathcal{A}} f^\#_A;$$

note that, for every set function $f_A$ on $A \subseteq E$, the conjugate $f_A(A) - f_A(A \setminus (\cdot))$ over $A$ coincides, as a set function on $E$, with the conjugate $f_A(E) - f_A(E \setminus (\cdot))$ over $E$.

A set function $f$ is said to be submodular if the following inequalities hold [3]:

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y) \quad \text{for all } X, Y \subseteq E.$$ 

A set function $f$ is said to be supermodular if $f^\#$ is submodular.

The difference function $\bigwedge f : 2^E \times \mathbb{N}^{(2^E)} \to \mathbb{R}$ of a set function $f$ on $E$ is defined recursively as follows [2]:

$$\bigwedge f(X, \emptyset) \triangleq f(X),$$

$$\bigwedge f(X, \mathcal{Y} \uplus \{Y\}) \triangleq \bigwedge f(X, \mathcal{Y}) - \bigwedge f(X \cap Y, \mathcal{Y}),$$

where $\mathbb{N}$ is the set of nonnegative integers, $\mathcal{Y}$ is a multiset over $2^E$—$\mathcal{Y} : 2^E \to \mathbb{N}$ and $\mathcal{Y}(Z) \in \mathbb{N}$ is the multiplicity of $Z \in 2^E$ in $\mathcal{Y}$—, $\uplus$ is the sum of multisets, and it holds that

$$(\mathcal{Y} \uplus \{Y\})(Z) = \begin{cases} \mathcal{Y}(Z) + 1 & \text{if } Z = Y, \\ \mathcal{Y}(Z) & \text{if } Z \neq Y. \end{cases}$$

When $|\mathcal{Y}| (\triangleq \sum_{Z \in 2^E} \mathcal{Y}(Z)) = k$, we write $\bigwedge f(X, \mathcal{Y})$ as $\bigwedge_k f(X, \mathcal{Y})$ also.
(i) [2] For a positive integer \(k\), a set function \(f\) is said to be \(k\)-\emph{monotone} if \(\bigwedge_k f \geq 0\), i.e., \(\bigwedge X f(X, Y) \geq 0\) whenever \(X \subseteq 2^E\) and \(Y \in \binom{2^E}{k}\) \(\triangleq \{X \in N^{(2^E)} \mid |X| = k\}\).

(ii) [1] For an integer \(k\) greater than 1, a set function \(f\) is said to be \emph{weakly \(k\)-monotone} if for every \(X \in \binom{2^E}{k}\)

\[
 f\left(\bigcup X\right) \geq \sum_{Y \subseteq X, Y \neq \emptyset} (-1)^{|Y|+1} f\left(\bigcap Y\right),
\]

where, for \(Z \in N^{(2^E)}, \bigcup Z \triangleq \bigcup_{Z \in Z} Z = \cup(suppZ), \bigcap Z \triangleq \bigcap_{Z \in Z} Z = \cap(suppZ)\), \(Z \in Z\) means \(Z(Z) > 0\), and \(suppZ\) is the ordinary set \(\{Z \mid Z(Z) > 0\} \subseteq 2^E\) called the support of \(Z\).

The 1-monotonicity is equivalent to the ordinary monotonicity, i.e., \(X \subseteq Y \implies f(X) \leq f(Y)\). The concept of weak 1-additivity is not defined. There are the following relations between submodularity and weak 2-monotonicity:

\[
 f\text{ is submodular} \iff -f\text{ is weakly 2-monotone} \iff f^\#\text{ is weakly 2-monotone}.
\]

For every integer \(k\) greater than 1, a set function \(f\) is \(k\)-monotone iff \(f\) is monotone and weakly \(k\)-monotone. If \(k\) and \(k'\) are integers such that \(1 \leq k \leq k'\), and if a set function \(f\) is \(k'\)-monotone, then \(f\) is \(k\)-monotone. If \(k\) and \(k'\) are integers such that \(2 \leq k \leq k'\), and if a set function \(f\) is weakly \(k'\)-monotone, then \(f\) is weakly \(k\)-monotone.

# 3 Additive decomposition

This paper deals with the following additive decomposition of a set function \(f\) on \(E\) with respect to an antichain \(A\) of subsets of \(E\).

\(\langle AD\rangle\): A set function \(f\) on \(E\) is decomposable into a sum of set functions \(f_A\) over all \(A \in A\), that is, there exists a collection \(\{f_A\}_{A \in A}\) such that each \(f_A\) is a set function on \(A\) and

\[
f = \sum_{A \in A} f_A. \tag{1}\]

A necessary and sufficient condition for the additive decomposition \(\langle AD\rangle\) is \(\langle IE\rangle\), that is, \(A\) is an inclusion-exclusion family with respect to \(f\) [5].

If \(f\) is a submodular set function, and if an antichain \(A\) is an inclusion-exclusion family with respect to \(f\), there does not always exist a collection \(\{f_A\}_{A \in A}\) of \emph{submodular} set functions satisfying Eq. (1), while there always exists a collection \(\{f_A\}_{A \in A}\) of set functions satisfying Eq. (1). That is to say, the antichain \(A\) being an inclusion-exclusion family is only a necessary condition and not a sufficient condition for a submodular set function \(f\) to be decomposable into a sum of submodular set functions \(f_A\) over all \(A \in A\).

So far, the authors have obtained two theorems showing sufficient conditions for the decomposition of submodular set functions into a sum of submodular set functions and their generalizations [4][7]. We show below the two generalized additive decomposition
theorems. For an antichain $\mathcal{A}$ of subsets of $E$, a set function $f$ on $E$ is said to have a $k$-monotone [resp. weakly $k$-monotone] $\mathcal{A}$-decomposition if there exists a collection $\{f_A\}_{A \in \mathcal{A}}$ such that each $f_A$ is a $k$-monotone [resp. weakly $k$-monotone] set function on $A$ and Eq. (1) holds. The two theorems deal with the following three types of conditions $\cap(k,l,\mathcal{A}), M(k',k,\mathcal{A})$, and $wM(k',k,\mathcal{A})$ on positive integers $k$, $k'$, and $l$ such that $k \leq k'$ and an antichain $\mathcal{A}$:

$$\cap(k,l,\mathcal{A}): |\cap\mathcal{B}| \leq k$$

for any $\mathcal{B} \in \binom{\mathcal{A}}{l}$.

$M(k',k,\mathcal{A})$: Every $k'$-monotone set function $f$ with $\mathcal{A}$ as an inclusion-exclusion family has a $k$-monotone $\mathcal{A}$-decomposition.

$wM(k',k,\mathcal{A})$: Every weakly $k'$-monotone set function $f$ with $\mathcal{A}$ as an inclusion-exclusion family has a weakly $k$-monotone $\mathcal{A}$-decomposition.

Condition $wM(k',1,\mathcal{A})$ is not considered.

**Theorem 1** (Generalized Additive Decomposition Theorem A). For a positive integer $k$ and an antichain $\mathcal{A}$, the three conditions $\cap(k,2,\mathcal{A})$, $M(k,k,\mathcal{A})$, and $wM(k,k,\mathcal{A})$ are equivalent to each other.

**Theorem 2** (Generalized Additive Decomposition Theorem B). Let $k$ and $k'$ be positive integers, $k \leq k'$, and $\mathcal{A}$ be an antichain. Then $\cap(k-1,k'-k+2,\mathcal{A})$ is a sufficient condition for each of $M(k',k,\mathcal{A})$ and $wM(k',k,\mathcal{A})$.

### 4 Indecomposability

Our present subject is the unification of Theorems 1 and 2, that is, necessary and sufficient conditions for $M(k',k,\mathcal{A})$ and $wM(k',k,\mathcal{A})$. We have found a cue to this subject, and we give it below. Note that, for every integer $k$ greater than 1, a monotone set function $f$ has a $k$-monotone $\mathcal{A}$-decomposition iff it has a weakly $k$-monotone $\mathcal{A}$-decomposition.

**Proposition 1.** Let $k$, $k'$, $l$, $n$ be positive integers such that $k \leq k' \leq l \leq n - 3$, and $E$ be an $n$-element set. If

$$(n - l)(l - k' + 1) - 2(l - k + 1) > 0,$$

then there exists a $k'$-monotone set function $f$ on $E$ with $E \choose l+2$ as the least inclusion-exclusion family such that $f$ does not have a $k$-monotone $E \choose l+2$-decomposition.

If $\mathcal{A}$ is a non-trivial inclusion-exclusion family with respect to a set function $f$, a $k$-monotone $\mathcal{A}$-decomposition of $f$ is said to be non-trivial.
Corollary 1. Let $k$ and $k'$ be positive integers such that $k \leq k'$, and $E$ be an $n$-element set. If

$$n > 3k' - 2k + 2,$$

then there exists a $k'$-monotone set function on $E$ with a non-trivial inclusion-exclusion family such that $f$ has no non-trivial $k$-monotone decomposition.

Corollary 2. Let $k$ and $k'$ be positive integers such that $k \leq k'$, and $E$ be an $n$-element set. If

$$n > k' + 1 + \sqrt{8(k' - k) + 1},$$

then there exist a $k'$-monotone set function $f$ and an inclusion-exclusion antichain $A$ with respect to $f$ such that $f$ does not have a $k$-monotone $A$-decomposition.

References


