Additive indecomposability of submodular set functions and its generalization*

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1 Introduction

This paper deals with a decomposition of a submodular set function into a sum of submodular set functions on subdomains and its generalization. Submodular set functions have an important role in mathematical programming [3], and a supermodular set function, which is the conjugate of a submodular set function (see Section 2), also is an important concept called a convex game in cooperative game theory [6]. Therefore, the additive decomposition of submodular set functions has broad application possibilities.

This paper is organized as follows. Section 2 explains basic concepts such as inclusion-exclusion family, submodularity, and (weak) k-monotonicity. Section 3 shows the results on additive decompositions of set functions we have obtained so far. Section 4 gives the main results, that is, conditions for additive indecomposability, which provide a foothold for further investigation of additive decompositions.

2 Preliminaries

For a finite set $X$, the number of elements of $X$ is denoted by $|X|$, the power set of $X$ by $2^X$, and, for an integer $k$ such that $0 \leq k \leq |X|$, the family of $k$-element subsets of $X$ is denoted by $\binom{X}{k}$, i.e,

$$\binom{X}{k} \coloneqq \{Y \in 2^X \mid |Y| = k\}.$$

Throughout this paper, $E$ is assumed to be a finite set.

A family $\mathcal{A}$ of subsets of $E$ is called an antichain if $A, A' \in \mathcal{A}$ and $A \subsetneq A'$ together imply $A = A'$. For antichains $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \subseteq \mathcal{B}$ if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$; then $\subseteq$ is a partial ordering on the class of all antichains over $E$, and the class forms a lattice with the following meet:

$$\mathcal{A} \cap \mathcal{B} = \text{Max}\{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$

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where, for a family \( \mathcal{F} \) of sets, \( \text{Max}\mathcal{F} \) is defined by

\[
\text{Max\mathcal{F}} \eqdef \{ M \in \mathcal{F} \mid M \text{ is maximal in } \mathcal{F} \text{ with respect to set inclusion } \subseteq \}.
\]

A function \( f : 2^E \to \mathbb{R} \) satisfying \( f(\emptyset) = 0 \) is called a set function on \( E \). An antichain \( \mathcal{A} \) of subsets of \( E \) is called an inclusion-exclusion family, or an inclusion-exclusion antichain, with respect to a set function \( f \) on \( E \) if \( \langle \mathcal{A} \rangle \) below holds:

\[
\langle \mathcal{A} \rangle : \quad f(X) = \sum_{B \subseteq A, B \neq \emptyset} (-1)^{|B|+1} f \left( X \cap \bigcap B \right) \quad \text{for all } X \subseteq E.
\]

If an antichain \( \mathcal{A} \) contains a subset \( A \) such that \( f(X) = f(X \cap A) \) for every \( X \subseteq E \), then \( \mathcal{A} \) is an inclusion-exclusion family with respect to \( f \), and \( \mathcal{A} \) is called a trivial inclusion-exclusion family; for example, \( \{ E \} \) is a trivial inclusion-exclusion family with respect to any set function on \( E \). For antichains \( \mathcal{A} \) and \( \mathcal{B} \), if \( \mathcal{A} \subseteq \mathcal{B} \), and if \( \mathcal{A} \) is an inclusion-exclusion family with respect to a set function \( f \), then so is \( \mathcal{B} \). If \( \mathcal{A} \) and \( \mathcal{B} \) are inclusion-exclusion antichains with respect to a set function \( f \), then so is \( \mathcal{A} \cap \mathcal{B} \). Therefore, every set function has its least (with respect to \( \subseteq \) inclusion-exclusion antichain.

For a set function \( f \) on \( E \), the sign inversion \(-f\) of \( f \) and the conjugate, or dual, \( f^\# \) of \( f \) are defined as follows [3]:

\[
(-f)(X) \eqdef -f(X), \quad f^\#(X) \eqdef f(E) - f(E \setminus X)
\]

for every \( X \subseteq E \). For any set function \( f\alpha \) on \( A \subseteq E \), we regard \( f\alpha \) as a set function on \( E \) by defining \( f\alpha(X) = f\alpha(X \cap A) \) for every \( X \in 2^E \setminus 2^A \). Let \( \mathcal{A} \subseteq 2^E \) and \( \{ f\alpha \}_{A \in \mathcal{A}} \) be a collection of set functions \( f\alpha \) on \( A \in \mathcal{A} \). Then the following holds:

\[
f = \sum_{A \in \mathcal{A}} f\alpha \iff -f = \sum_{A \in \mathcal{A}} (-f\alpha) \iff f^\# = \sum_{A \in \mathcal{A}} f\alpha^\#;
\]

note that, for every set function \( f\alpha \) on \( A \subseteq E \), the conjugate \( f\alpha(A) - f\alpha(A \setminus \cdot) \) over \( A \) coincides, as a set function on \( E \), with the conjugate \( f\alpha(E) - f\alpha(E \setminus \cdot) \) over \( E \).

A set function \( f \) is said to be submodular if the following inequalities hold [3]:

\[
f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y) \quad \text{for all } X, Y \subseteq E.
\]

A set function \( f \) is said to be supermodular if \( f^\# \) is submodular.

The difference function \( \wedge f : 2^E \times \mathbb{N}^{2^E} \to \mathbb{R} \) of a set function \( f \) on \( E \) is defined recursively as follows [2]:

\[
\wedge f(X, \emptyset) \eqdef f(X), \quad \wedge f(X, \mathcal{Y} \cup \{ Y \}) \eqdef \wedge f(X, \mathcal{Y}) - \wedge f(X \cap Y, \mathcal{Y}),
\]

where \( \mathbb{N} \) is the set of nonnegative integers, \( \mathcal{Y} \) is a multiset over \( 2^E \)—— \( \mathcal{Y} : 2^E \to \mathbb{N} \) and \( \mathcal{Y}(Z) \in \mathbb{N} \) is the multiplicity of \( Z \in 2^E \) in \( \mathcal{Y} \)——, \( \cup \) is the sum of multisets, and it holds that

\[
(\mathcal{Y} \cup \{ Y \})|Z| = \begin{cases} \mathcal{Y}(Z) + 1 & \text{if } Z = Y, \\ \mathcal{Y}(Z) & \text{if } Z \neq Y. \end{cases}
\]

When \( |\mathcal{Y}| \eqdef \sum_{Z \in 2^E} \mathcal{Y}(Z) = k \), we write \( \wedge f(X, \mathcal{Y}) \) as \( \wedge_k f(X, \mathcal{Y}) \) also.
(i) [2] For a positive integer \( k \), a set function \( f \) is said to be \( k \)-monotone if \( \bigwedge_k f \geq 0 \), i.e., \( \bigwedge f(X, \mathcal{Y}) \geq 0 \) whenever \( X \subseteq 2^E \) and \( \mathcal{Y} \in (2^E)^k \) are \( |X| = k \).

(ii) [1] For an integer \( k \) greater than 1, a set function \( f \) is said to be weakly \( k \)-monotone if for every \( X \subseteq 2^E \) and \( \mathcal{Y} \subseteq 2^E \) the following condition is satisfied:

\[
f\left( \bigcup X \right) \geq \sum_{\mathcal{Y} \subseteq X, \mathcal{Y} \neq \emptyset} (-1)^{|\mathcal{Y}|+1} f\left( \bigcap \mathcal{Y} \right),
\]

where, for \( Z \subseteq 2^E \), \( \bigcup Z \) is the union of \( Z \), \( \bigcap Z \) is the intersection of \( Z \), \( \mathcal{Z} \in N^{(2^E)} \), \( \mathcal{Z}(Z) > 0 \), and \( \text{supp} Z \) is the ordinary set \( \{ Z \mid Z(Z) > 0 \} \subseteq 2^E \).

The 1-monotonicity is equivalent to the ordinary monotonicity, i.e., \( X \subseteq Y \Rightarrow f(X) \leq f(Y) \). The concept of weak 1-additivity is not defined. There are the following relations between submodularity and weak 2-monotonicity:

\[ f \text{ is submodular } \iff \neg f \text{ is weakly } 2\text{-monotone } \iff f^\# \text{ is weakly } 2\text{-monotone}. \]

For every integer \( k \) greater than 1, a set function \( f \) is \( k \)-monotone iff \( f \) is monotone and weakly \( k \)-monotone. If \( k \) and \( k' \) are integers such that \( 1 \leq k \leq k' \), and if a set function \( f \) is \( k' \)-monotone, then \( f \) is \( k \)-monotone. If \( k \) and \( k' \) are integers such that \( 2 \leq k \leq k' \), and if a set function \( f \) is weakly \( k' \)-monotone, then \( f \) is weakly \( k \)-monotone.

3 Additive decomposition

This paper deals with the following additive decomposition of a set function \( f \) on \( E \) with respect to an antichain \( \mathcal{A} \) of subsets of \( E \).

\( \langle AD \rangle \): A set function \( f \) on \( E \) is decomposable into a sum of set functions \( f_A \) over all \( A \in \mathcal{A} \), that is, there exists a collection \( \{ f_A \}_{A \in \mathcal{A}} \) such that each \( f_A \) is a set function on \( A \) and

\[
f = \sum_{A \in \mathcal{A}} f_A. \tag{1}\]

A necessary and sufficient condition for the additive decomposition \( \langle AD \rangle \) is \( \langle IE \rangle \), that is, \( \mathcal{A} \) is an inclusion-exclusion family with respect to \( f \) [5].

If \( f \) is a submodular set function, and if an antichain \( \mathcal{A} \) is an inclusion-exclusion family with respect to \( f \), there does not always exist a collection \( \{ f_A \}_{A \in \mathcal{A}} \) of submodular set functions satisfying Eq. (1), while there always exists a collection \( \{ f_A \}_{A \in \mathcal{A}} \) of set functions satisfying Eq. (1). That is to say, the antichain \( \mathcal{A} \) being an inclusion-exclusion family is only a necessary condition and not a sufficient condition for a submodular set function \( f \) to be decomposable into a sum of submodular set functions \( f_A \) over all \( A \in \mathcal{A} \).

So far, the authors have obtained two theorems showing sufficient conditions for the decomposition of submodular set functions into a sum of submodular set functions and their generalizations [4][7]. We show below the two generalized additive decomposition
theorems. For an antichain $A$ of subsets of $E$, a set function $f$ on $E$ is said to have a k-monotone [resp. weakly k-monotone] $A$-decomposition if there exists a collection $\{f_A\}_{A \in A}$ such that each $f_A$ is a k-monotone [resp. weakly k-monotone] set function on $A$ and Eq. (1) holds. The two theorems deal with the following three types of conditions $\cap(k, l, A), M(k', k, A)$, and wM($k', k, A$) on positive integers $k, k'$, and $l$ such that $k \leq k'$ and an antichain $A$:

$\cap(k, l, A): |\cap B| \leq k$ for any $B \in \binom{A}{l}$.

M($k', k, A$): Every $k'$-monotone set function $f$ with $A$ as an inclusion-exclusion family has a k-monotone $A$-decomposition.

wM($k', k, A$): Every weakly $k'$-monotone set function $f$ with $A$ as an inclusion-exclusion family has a weakly k-monotone $A$-decomposition.

Condition wM($k', 1, A$) is not considered.

**Theorem 1** (Generalized Additive Decomposition Theorem A). For a positive integer $k$ and an antichain $A$, the three conditions $\cap(k, 2, A)$, M($k', k, A$), and wM($k', k, A$) are equivalent to each other.

**Theorem 2** (Generalized Additive Decomposition Theorem B). Let $k$ and $k'$ be positive integers, $k \leq k'$, and $A$ be an antichain. Then $\cap(k - 1, k' - k + 2, A)$ is a sufficient condition for each of M($k', k, A$) and wM($k', k, A$).

4 Indecomposability

Our present subject is the unification of Theorems 1 and 2, that is, necessary and sufficient conditions for M($k', k, A$) and wM($k', k, A$). We have found a cue to this subject, and we give it below. Note that, for every integer $k$ greater than 1, a monotone set function $f$ has a k-monotone $A$-decomposition iff it has a weakly k-monotone $A$-decomposition.

**Proposition 1.** Let $k, k', l, n$ be positive integers such that $k \leq k' \leq l \leq n - 3$, and $E$ be an $n$-element set. If

$$(n - l)(l - k' + 1) - 2(l - k + 1) > 0,$$  (2)

then there exists a $k'$-monotone set function $f$ on $E$ with $\binom{E}{l + 2}$ as the least inclusion-exclusion family such that $f$ does not have a k-monotone $\binom{E}{l + 2}$-decomposition.

If $A$ is a non-trivial inclusion-exclusion family with respect to a set function $f$, a k-monotone $A$-decomposition of $f$ is said to be non-trivial.
Corollary 1. Let $k$ and $k'$ be positive integers such that $k \leq k'$, and $E$ be an $n$-element set. If

$$n > 3k' - 2k + 2,$$

then there exists a $k'$-monotone set function on $E$ with a non-trivial inclusion-exclusion family such that $f$ has no non-trivial $k$-monotone decomposition.

Corollary 2. Let $k$ and $k'$ be positive integers such that $k \leq k'$, and $E$ be an $n$-element set. If

$$n > k' + 1 + \sqrt{8(k'-k)+1},$$

then there exist a $k'$-monotone set function $f$ and an inclusion-exclusion antichain $A$ with respect to $f$ such that $f$ does not have a $k$-monotone $A$-decomposition.

References


