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数理解析研究所講究録 2008, 1585: 91-105

http://hdl.handle.net/2433/81515
Strong and Weak Convergence Theorems for Equilibrium Problems and Nonlinear Mappings in Banach Spaces

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Abstract. In this article, we prove strong and weak convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Next, we prove two strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the normal hybrid method and a new hybrid method called the shrinking projection method. Further, we obtain a necessary and sufficient condition for the existence of solutions of the equilibrium problem by using the metric resolvents. Finally, we prove a strong convergence theorem for finding a solution of an equilibrium problem in a Banach space by using the shrinking projection method.

1 Introduction

Let $E$ be a real Banach space and let $E^*$ be a dual space of $E$. Let $C$ be a closed convex subset of $E$ and let $f$ be a bifunction from $C \times C$ to $R$, where $R$ is the set of real numbers. The equilibrium problem is formulated as follows: Find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \quad \text{for all } y \in C.$$ 

In this case, such a point $\hat{x} \in C$ is called a solution of the problem. The set of such solutions $\hat{x}$ is denoted by $EP(f)$. Many problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Blum and Oettli [5] widely discussed the existence of solutions of such an equilibrium problem. Combettes-Hirstoaga [9], Tada and Takahashi [46], and Takahashi and Takahashi [48] proposed some methods for approximation of solutions of the equilibrium problem in a Hilbert space. However, the problem of approximating solutions of the equilibrium problem in a Banach space is difficult. We also know the problem of finding a point $u \in E$ satisfying

$$0 \in Au,$$

where $A$ is a maximal monotone operator from $E$ to $E^*$. Such a problem contains numerous problems in physics, optimization and economics. A well-known method to solve this problem is called the proximal point algorithm: $x_1 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \ n = 1, 2, \ldots,$$

where $J_{r_n}$ is the resolvent of $r_n A$.
where \( \{r_n\} \subset (0, \infty) \) and \( J_{r_n} \) are the resolvents of \( A \). Many researchers have studied this algorithm in a Hilbert space, see, for instance, [11, 18, 27, 42, 45] and in a Banach space, see, for instance, [17, 19, 20, 33]. A mapping \( S \) of \( C \) into \( E \) is called nonexpansive if

\[
\|Sx - Sy\| \leq \|x - y\|
\]

for all \( x, y \in C \). We denote by \( F(S) \) the set of fixed points of \( S \). There are some methods for approximation of fixed points of a nonexpansive mapping; see, for instance, [12, 26, 36, 43, 64]. In particular, in 2003 Nakajo–Takahashi [32] proved the following strong convergence theorem by using the hybrid method:

**Theorem 1.1 (Nakajo and Takahashi [32]).** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and let \( T \) be a nonexpansive mapping of \( C \) into itself such that \( F(T) \neq \emptyset \). Suppose \( x_1 = x \in C \) and \( \{x_n\} \) is given by

\[
\begin{align*}
y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
\{u_{n+1} = P_{C_n \cap Q_n}x_n, \ n \in N,\end{align*}
\]

where \( P_{C_n \cap Q_n} \) is the metric projection from \( C \) onto \( C_n \cap Q_n \) and \( \{\alpha_n\} \) is chosen so that \( 0 \leq \alpha_n \leq a < 1 \). Then, \( \{x_n\} \) converges strongly to \( P_{F(T)}x \), where \( P_{F(T)} \) is the metric projection from \( C \) onto \( F(T) \).

Let us call the hybrid method in Theorem 1.1 the normal hybrid method. Very recently, Takahashi, Takeuchi and Kubota [61] proved the following theorem by using another hybrid method called the shrinking projection method.

**Theorem 1.2 (Takahashi, Takeuchi and Kubota [61]).** Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( T \) be a nonexpansive mapping of \( C \) into itself such that \( F(T) \neq \emptyset \) and let \( x_0 \in H \). For \( C_1 = C \) and \( u_1 = P_{C_1}x_0 \), define a sequence \( \{u_n\} \) of \( C \) as follows:

\[
\begin{align*}
y_n &= \alpha_n u_n + (1 - \alpha_n)Tu_n, \\
C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\
u_{n+1} &= P_{C_{n+1}}x_0, \ n \in N,
\end{align*}
\]

where \( 0 \leq \alpha_n \leq a < 1 \) for all \( n \in N \). Then, \( \{u_n\} \) converges strongly to \( x_0 = P_{F(T)}x_0 \).

In this article, we prove strong and weak convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Next, using the normal hybrid method and a new hybrid method called the shrinking projection method, we study two strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space. Further, we obtain a necessary and sufficient condition for the existence of solutions of the equilibrium problem by using the metric resolvents. Finally, we prove a strong convergence theorem for finding a solution of an equilibrium problem in a Banach space by using the shrinking projection method.
2 Preliminaries

Throughout this paper, we denote by $N$ and $R$ the sets of positive integers and real numbers, respectively. Let $E$ be a Banach space and let $E^*$ be the topological dual of $E$. For all $x \in E$ and $x^* \in E^*$, we denote the value of $x^*$ at $x$ by $\langle x, x^* \rangle$. Then, the duality mapping $J$ on $E$ is defined by

$$J(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \}$$

for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty; see [51] for more details. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to $x$ in $E$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x^*_n\}$ to $x^*$ in $E^*$ by $x^*_n \rightharpoonup x^*$. A Banach space $E$ is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2)$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \epsilon$. The space $E$ is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E) = \{ x \in E : \|x\| = 1 \}$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$. We know that if $E$ is smooth, strictly convex and reflexive, then the duality mapping $J$ is single-valued, one-to-one and onto; see [51, 52] for more details.

Let $E$ be a smooth Banach space and define the real valued function $\phi$ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $y, x \in E$. Then, we have that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) - 2\langle x - z, Jz - Jy \rangle$$

for all $x, y, z \in E$. Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed convex subset of $E$. Following Alber [1], the generalized projection $\Pi_C$ from $E$ onto $C$ is defined by

$$\Pi_C(x) = \arg \min_{y \in C} \phi(y, x)$$

for all $x \in E$. If $E$ is a Hilbert space, then $\phi(y, x) = \|y-x\|^2$ and $\Pi_C$ is the metric projection of $H$ onto $C$. We know the following lemmas for generalized projections.

**Lemma 2.1 (Alber [1], Kamimura and Takahashi [20]).** Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \text{for all} \ x \in C \text{ and } y \in E.$$

**Lemma 2.2 (Alber [1], Kamimura and Takahashi [20]).** Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space, let $x \in E$ and let $z \in C$. Then

$$z = \Pi_C x \iff \langle y - z, Jx - Jz \rangle \leq 0 \quad \text{for all} \ y \in C.$$
Let $E$ be a smooth, strictly convex and reflexive Banach space, and let $A$ be a set-valued mapping from $E$ to $E^*$ with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$, domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$. We denote a set-valued operator $A$ from $E$ to $E^*$ by $A \subset E \times E^*$. $A$ is said to be monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone operator. We know that if $A$ is a maximal monotone operator, then $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$ is closed and convex; see $[51, 52]$ for more details. The following theorem is well-known.

**Theorem 2.3 (Rockafellar [41]).** Let $E$ be a smooth, strictly convex and reflexive Banach space and let $A \subset E \times E^*$ be a monotone operator. Then $A$ is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.

Let $E$ be a smooth, strictly convex and reflexive Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1}(\bigcap_{r>0}R(J + rA)).$$

Then we can define the resolvent $J_r : C \to D(A)$ of $A$ by

$$J_rx = \{z \in D(A) : Jz \in Jz + rAz\}$$

for all $x \in C$. We know that $J_rx$ consists of one point. For all $r > 0$, the Yosida approximation $A_r : C \to E^*$ is defined by $A_r x = \frac{x - Jrx}{r}$ for all $x \in C$. We also know the following lemma; see, for instance, [24].

**Lemma 2.4.** Let $E$ be a smooth, strictly convex and reflexive Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1}(\bigcap_{r>0}R(J + rA)).$$

Let $r > 0$ and let $J_r$ and $A_r$ be the resolvent and the Yosida approximation of $A$, respectively. Then, the following hold:

1. $\phi(u, J_rx) + \phi(Jr, x, x) \leq \phi(u, x)$ for all $x \in C$ and $u \in A^{-1}0$;
2. $(J_rx, A_rx) \in A$ for all $x \in C$;
3. $F(J_r) = A^{-1}0$.

Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $T$ be a mapping from $C$ into itself. We denoted by $F(T)$ the set of fixed points of $T$. A point $p \in C$ is said to be an asymptotic fixed point of $T$ if there exists $\{x_n\}$ in $C$ which converges weakly to $p$ and $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of $T$ by $\hat{F}(T)$. Following Matsushita and Takahashi [29], a mapping $T : C \to C$ is said to be relatively nonexpansive if the following conditions are satisfied:

1. $F(T)$ is nonempty;
2. $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$;
3. $\hat{F}(T) = F(T)$.

The following lemma is due to Matsushita and Takahashi [28].
Lemma 2.5 (Matsushita and Takahashi [28]). Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, and let $T$ be a relatively nonexpansive mapping from $C$ into itself. Then $F(T)$ is closed and convex.

We also know the following lemma.

Lemma 2.6 (Kanimura and Takahashi [20]). Let $E$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_n \phi(x_n, y_n) = 0$, then $\lim_n \|x_n - y_n\| = 0$.

3 Equilibrium Problems and Relatively Nonexpansive Mappings

In this section, we prove strong and weak convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

Let $E$ be a Banach space and let $C$ be a nonempty closed convex subset of $E$. A function $f : C \times C \to R$ is said to be maximal monotone with respect to $C$ if, for every $x \in C$ and $x^* \in E^*$,

$$f(x, y) + (y - x, x^*) \geq 0$$

for all $y \in C$, whenever $(z - x, x^*) \geq f(z, x)$ for all $z \in C$.

In this article, we assume that a bifunction $f$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) $f$ is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous;

(A4) $\limsup_{t \downarrow 0} f(tx + (1 - t)x, y) \leq f(x, y)$ for all $x, y, z \in C$.

Assume that $f$ satisfies (A1)–(A4). Then, $f$ is maximal monotone. In fact, for every $x \in C$ and $x^* \in E^*$, suppose that

$$\langle z - x, x^* \rangle \geq f(z, x)$$

for all $z \in C$. Putting $z_t = (1 - t)x + ty$ with $y \in C$ and $t \in (0, 1)$, we have

$$0 = f(z_t, z_t)$$

$$\leq (1 - t)f(z_t, x) + tf(z_t, y)$$

$$\leq (1 - t)(z_t - x, x^*) + tf(z_t, y)$$

$$\leq t(1 - t)(y - x, x^*) + tf(z_t, y).$$

Hence, we have

$$0 \leq (1 - t)(y - x, x^*) + f(z_t, y).$$

Since $f$ is upper hemicontinuous, we have

$$0 \leq \langle y - x, x^* \rangle + f(x, y).$$

Hence, $f$ is maximal monotone. The following result is in Blum and Oettl [5]. See [2] for the proof.

Lemma 3.1 (Blum and Oettl [5]). Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $R$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that

$$f(z, y) + \frac{1}{r}(y - z, Jz - Jx) \geq 0$$

for all $y \in C$. 

Motivated by Combettes and Hirstoaga [9] in a Hilbert space, we obtain the following lemma.

**Lemma 3.2.** Let $C$ be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space $E$, and let $f$ be a bifunction from $C \times C$ to $R$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \rightarrow C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z,y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0 \text{ for all } y \in C \right\}$$

for all $x \in E$. Then, the following hold:

1. $T_r$ is single-valued;
2. $T_r$ is a firmly nonexpansive-type mapping [24], i.e., for all $x,y \in E$,
   \[ \langle T_r x - T_r y, J T_r x - J T_r y \rangle \leq \langle T_r x - T_r y, J x - J y \rangle; \]
3. $F(T_r) = EP(f)$;
4. $EP(f)$ is closed and convex.

We claim that $T_r$ is single-valued. Indeed, for $x \in C$ and $r > 0$, let $z_1, z_2 \in T_r x$. Then,

$$f(z_1, z_2) + \frac{1}{r} \langle z_2 - z_1, Jz_1 - Jx \rangle \geq 0$$

and

$$f(z_2, z_1) + \frac{1}{r} \langle z_1 - z_2, Jz_2 - Jx \rangle \geq 0.$$

Adding two inequalities, we have

$$f(z_1, z_2) + f(z_2, z_1) + \frac{1}{r} \langle z_2 - z_1, Jz_1 - Jz_2 \rangle \geq 0.$$

From (A2) and $r > 0$, we have

$$\langle z_2 - z_1, Jz_1 - Jz_2 \rangle \geq 0.$$

Since $E$ is strictly convex, we have $z_1 = z_2$.

Next, we claim that $T_r$ is a firmly nonexpansive-type mapping. Indeed, for $x,y \in C$, we have

$$f(T_r x, T_r y) + \frac{1}{r} \langle T_r y - T_r x, J T_r x - Jx \rangle \geq 0,$$

and

$$f(T_r y, T_r x) + \frac{1}{r} \langle T_r x - T_r y, J T_r y - Jy \rangle \geq 0.$$

Adding two inequalities, we have

$$f(T_r x, T_r y) + f(T_r y, T_r x) + \frac{1}{r} \langle T_r y - T_r x, J T_r x - J T_r y - Jx + Jy \rangle \geq 0.$$

From (A2) and $r > 0$, we have

$$\langle T_r y - T_r x, J T_r x - J T_r y - Jx + Jy \rangle \geq 0.$$

Therefore, we have

$$\langle T_r x - T_r y, J T_r x - J T_r y \rangle \leq \langle T_r x - T_r y, J x - J y \rangle.$$

We call such $T_r$ the relative resolvent of $f$ for $r > 0$. Using Lemma 3.2, we have the following result.
Lemma 3.3. Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $R$ satisfying (A1)–(A4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Proof. From Lemma 3.2 (2), we have, for all $x, y \in E$,

$$\phi(T_r x, T_r y) + \phi(T_r y, T_r x) \leq \phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y).$$

Letting $y = q \in F(T_r)$, we have

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

This completes the proof. □

Now, we prove a strong convergence theorem for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

Theorem 3.4 (Takahashi and Zembayashi [63]). Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $R$ satisfying (A1)–(A4) and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases}
  x_0 = x \in C, \\
  y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) J S x_n), \\
  u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\
  H_n = \{x \in C : \phi(x, u_n) \leq \phi(x, x_n)\}, \\
  W_n = \{x \in C : \langle x_n - x, Jx - Jx_n \rangle \geq 0\}, \\
  x_{n+1} = \Pi_{H_n \cap W_n} x
\end{cases}$$

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap EP(f)} x$, where $\Pi_{F(S) \cap EP(f)}$ is the generalized projection of $E$ onto $F(S) \cap EP(f)$.

Further, we prove a weak convergence theorem for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Before proving the theorem, we need the following proposition.

Proposition 3.5. Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $R$ satisfying (A1)–(A4) and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u_1 \in E$,

$$\begin{cases}
  x_n \in C \text{ such that } f(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \forall y \in C, \\
  u_{n+1} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) J S x_n)
\end{cases}$$

for every $n \in N$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$ and $\{r_n\} \subset [0, \infty)$. Then, $\{\Pi_{F(S) \cap EP(f)} x_n\}$ converges strongly to $z \in F(S) \cap EP(f)$, where $\Pi_{F(S) \cap EP(f)}$ is the generalized projection of $E$ onto $F(S) \cap EP(f)$.
Using Proposition 3.5, we can prove the following theorem.

**Theorem 3.6 (Takahashi and Zembayashi [63]).** Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequences generated by \(u_1 \in E\),

\[
\begin{align*}
    x_n & \in C \text{ such that } f(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Jx_n \rangle \geq 0, \forall y \in C, \\
    u_{n+1} & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n)
\end{align*}
\]

for every $n \in N$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $J$ is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in F(S) \cap EP(f)$, where $z = \lim_{n \to \infty} \Pi_{F(S) \cap EP(f)} x_n$.

4 Maximal Monotone Operators and Relatively Nonexpansive Mappings

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the normal hybrid method.

**Theorem 4.1 (Inoue, Takahashi and Zembayashi [25]).** Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying

\[D(A) \subset C \subset J^{-1}(\cap_{r > 0} R(J + rA))\]

and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

\[
\begin{align*}
    u_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSJ_r x_n), \\
    H_n & = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
    W_n & = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
    x_{n+1} & = \Pi_{H \cap W_n} x
\end{align*}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n \to \infty} (1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap A^{-1}0} x$, where $\Pi_{F(S) \cap A^{-1}0}$ is the generalized projection of $E$ onto $F(S) \cap A^{-1}0$.

As direct consequences of Theorem 4.1, we can obtain the following corollaries.

**Corollary 4.2.** Let $E$ be a uniformly smooth and uniformly convex Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all
$r > 0$. Let \{x_n\} be a sequence generated by $x_0 = x \in C$ and

\[
\begin{aligned}
&u_n = J_{r_n}x_n, \\
&H_n = \{z \in E : \phi(z,u_n) \leq \phi(z,x_n)\}, \\
&W_n = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
&x_{n+1} = \Pi_{H_n \cap W_n}x
\end{aligned}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$ and \{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, \{x_n\} converges strongly to $\Pi_{A^{-1}0}x$, where $\Pi_{A^{-1}0}$ is the generalized projection of $E$ onto $A^{-1}0$.

**Proof.** Putting $S = I$, $C = E$ and $\alpha_n = 0$ in Theorem 4.1, we obtain Corollary 4.2. □

**Corollary 4.3 (Matsushita and Takahashi [29]).** Let $E$ be a uniformly smooth and uniformly convex Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \neq \emptyset$. Let \{x_n\} be a sequence generated by $x_0 = x \in C$ and

\[
\begin{aligned}
&u_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSJ_{r_n}x_n), \\
&H_n = \{z \in C : \phi(z,u_n) \leq \phi(z,x_n)\}, \\
&W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
&x_{n+1} = \Pi_{H_n \cap W_n}x
\end{aligned}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, \{\alpha_n\} \subset [0,1]$ satisfies $\lim\inf_{n \to \infty}(1 - \alpha_n) > 0$. Then, \{x_n\} converges strongly to $\Pi_{F(S)}x$, where $\Pi_{F(S)}$ is the generalized projection of $E$ onto $F(S)$.

**Proof.** Set $A = \partial i_C$ in Theorem 4.1, where $i_C$ is the indicator function of $C$ and $\partial i_C$ is the subdifferential of $i_C$. Then, we have that $A$ is a maximal monotone operator and $J_r = \Pi_C$, where $J_r$ is the resolvent of $A = \partial i_C$ for $r > 0$. So, from Theorem 4.1, we obtain Corollary 4.3. □

Using an idea of [61], we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the shrinking projection method.

**Theorem 4.4 (Inoue, Takahashi and Zembayashi [25]).** Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying

\[
D(A) \subset C \subset J^{-1}(\cap_{r>0}R(J + rA))
\]

and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let \{x_n\} be a sequence generated by $x_0 = x \in C$, $H_0 = C$ and

\[
\begin{aligned}
&u_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSJ_{r_n}x_n), \\
&H_{n+1} = \{z \in H_n : \phi(z,u_n) \leq \phi(z,x_n)\}, \\
&x_{n+1} = \Pi_{H_{n+1}}x
\end{aligned}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, \{\alpha_n\} \subset [0,1]$ satisfies $\lim\inf_{n \to \infty}(1 - \alpha_n) > 0$ and \{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, \{x_n\} converges strongly to $\Pi_{F(S) \cap A^{-1}0}x$, where $\Pi_{F(S) \cap A^{-1}0}$ is the generalized projection of $E$ onto $F(S) \cap A^{-1}0$. 
As direct consequences of Theorem 4.4, we can obtain the following corollaries.

**Corollary 4.5.** Let $E$ be a uniformly smooth and uniformly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in E$, $H_0 = E$ and

$$
\begin{align*}
  u_n &= J_{r_n}x_n, \\
  H_{n+1} &= \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
  x_{n+1} &= \Pi_{H_{n+1}}x
\end{align*}
$$

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x$.

**Proof.** Putting $S = I$, $C = H_0 = E$ and $\alpha_n = 0$ in Theorem 4.4, we obtain Corollary 4.5. □

**Corollary 4.6.** Let $E$ be a uniformly smooth and uniformly convex Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

$$
\begin{align*}
  u_n &= J^{-1}(\alpha_nJx_n + (1-\alpha_n)JSx_n), \\
  H_{n+1} &= \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
  x_{n+1} &= \Pi_{H_{n+1}}x
\end{align*}
$$

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0, 1)$ satisfies $\inf_{n \to \infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)}x$, where $\Pi_{F(S)}$ is the generalized projection of $E$ onto $F(S)$.

**Proof.** Putting $A = \partial i_C$ in Theorem 4.4, we obtain Corollary 4.6. □

### 5 Equilibrium Problems and Metric Resolvents

In this section, we prove a strong convergence theorem for finding a solution of the equilibrium problem by using the metric resolvents. Using Lemma 3.1, we first obtain the following result.

**Lemma 5.1.** Let $C$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $R$ satisfying $(A1)-(A4)$, let $r > 0$ and let $x \in E$. Then, there exists a unique $z_r \in C$ such that

$$
f(z_r, y) + \frac{1}{r}(y - z_r, J(z_r - x)) \geq 0 \quad \text{for all } y \in C.
$$

**Proof.** Fix $x \in C$. Then, we define $g : (C - x) \times (C - x) \to R$ as follows:

$$
g(z, y) = f(z + x, y + x) + \frac{1}{r}(y - z, Jx). \quad (5.1)
$$

From the properties of $f$, it is easy to prove that $g$ satisfies the following conditions:

(A1) $g(z, z) = 0$ for all $z \in C - x$;
(A2) \( g \) is monotone with respect to \( C - x \);
(A3) for all \( z \in C - x \), \( g(z, \cdot) \) is convex and lower semicontinuous;
(A4) \( g \) is upper hemicontinuous with respect to \( C - x \).

Hence, from Lemma 3.1 there exists a unique element \( z_r \) such that
\[
f(z_r + x, y + x) + \frac{1}{r} \langle y - z_r, Jx \rangle + \frac{1}{r} \langle y - z_r, Jz_r - Jx \rangle \geq 0
\]
for all \( y \in C - x \). This implies that
\[
f(z_r + x, y + x) + \frac{1}{r} \langle y - z_r, Jz_r \rangle \geq 0
\]
for all \( y \in C - x \). Putting \( u_r = z_r + x \) and \( v = y + x \), we have
\[
f(u_r, v) + \frac{1}{r} \langle v - u_r, J(u_r - x) \rangle \geq 0
\]
for all \( v \in C \). This completes the proof. \( \square \)

Under the conditions in Theorem 5.1, for every \( r > 0 \) we may define a single-valued mapping \( F_r : E \to C \) by
\[
F_r x = \{ z \in C : 0 \leq f(z, y) + \frac{1}{r} \langle y - z, J(z - x) \rangle, y \in C \} \quad (5.2)
\]
for \( x \in E \), which is called the metric resolvent of \( f \) for \( r > 0 \). Also, we can define the Yosida approximation as follows:
\[
A_r x = \frac{1}{r} J(x - F_r x). \quad (5.3)
\]

As in Takahashi[52, pp.163-165], we can prove the following theorem for Yosida approximations. Before proving it, we need the following lemma; see, for instance, Takahashi[52, Problem 4.5.4].

**Lemma 5.2.** Let \( E \) be a Banach space. Assume that \( u_n \rightharpoonup v \), \( v_n \rightharpoonup v^* \) and
\[
\lim_{m,n \to \infty} \left\langle u_n - u_m, v_n - v_m \right\rangle = 0.
\]
Then, \( \lim_{n \to \infty} \langle u_n, v_n \rangle = \langle u, v^* \rangle \).

**Lemma 5.3.** Assume \( r > 0 \). Then, \( A_r : E \to E^* \) is monotone and demicontinuous. Further, if \( D \subset E \) is bounded, then \( A_r D \subset E^* \) is bounded.

To show a necessary and sufficient condition for the existence of solutions of the equilibrium problem, we need the following lemma [51, Theorem 7.1.8]; see also [4].

**Lemma 5.4** ([51, 4]). Let \( E \) be a reflexive Banach space and let \( K \) be a bounded closed convex subset of \( E \). Suppose \( A \) is a monotone and demicontinuous operator. Then there exists \( u_0 \in K \) such that
\[
\langle y - u_0, Au_0 \rangle \geq 0 \quad \text{for all } y \in K.
\]

Using Lemma 5.4, we obtain the following lemma.
Lemma 5.5. Let $E$ be a smooth and uniformly convex Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a bifunction $C \times C$ to $R$ satisfying (A1)-(A4). For $C_1 = C$ and $x_1 = x \in E$, define the sequence $\{x_n\}$ as follows:

$$
\begin{align*}
&y_n = F_{r_n}x_n, \\
&C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\}, \\
&x_{n+1} = P_{C_{n+1}}(x_1),
\end{align*}
$$

where $0 < r_n < \infty$ and $P_{C_n}$ is the metric projection of $E$ onto $C_n$. Then $\{x_n\}$ is well-defined.

We also have the following lemma.

Lemma 5.6. If $EP(f) \neq \emptyset$, then $EP(f) \subset C_n$ for all $n \in \mathbb{N}$.

Proof. It is obvious that $EP(f) \subset C_1 = C$. Suppose $EP(f) \subset C_n$ for some $n \in \mathbb{N}$. Let $z \in EP(f)$. From $y_n = F_{r_n}x_n$ and the monotonicity of $f$, we have

$$
\langle y_n - y, \frac{1}{r_n}J(x_n - y_n) \rangle \geq f(y, y_n)
$$

for all $y \in C$. Put $y = z$. Then we have

$$
\langle y_n - z, \frac{1}{r_n}J(x_n - y_n) \rangle \geq f(z, y_n) \geq 0.
$$

Therefore, $z \in C_{n+1}$. By the mathematical induction, we obtain $z \in C_n$ for all $n \in \mathbb{N}$. $\square$

Now, we obtain a necessary and sufficient condition for the existence of solutions of the equilibrium problem in a Banach space.

Theorem 5.7 (Takahashi and Takahashi [47]). Let $E$ be a smooth and uniformly convex Banach space and let $f$ be a bifunction $C \times C$ to $R$ satisfying (A1)-(A4). For $C_1 = C$ and $x_1 = x \in E$, define the sequence $\{x_n\}$ as follows:

$$
\begin{align*}
&y_n = F_{r_n}x_n, \\
&C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\}, \\
&x_{n+1} = P_{C_{n+1}}(x_1),
\end{align*}
$$

where $\liminf_{n \to \infty} r_n > 0$ and $P_{C_n}$ is the metric projection of $E$ onto $C_n$. Then $\{x_n\}$ is bounded if and only if $EP(f) \neq \emptyset$.

Finally, we can prove a strong convergence theorem for finding a solution of the equilibrium problem by using the shrinking projection method.

Theorem 5.8 (Takahashi and Takahashi [47]). Let $E$ be a smooth and uniformly convex Banach space and let $C$ a nonempty closed convex subset of $E$. Let $f$ be a bifunction $C \times C$ to $R$ satisfying (A1)-(A4). For $C_1 = C$ and $x_1 = x \in E$, define the sequence $\{x_n\}$ as follows:

$$
\begin{align*}
&y_n = F_{r_n}x_n, \\
&C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\}, \\
&x_{n+1} = P_{C_{n+1}}(x_1),
\end{align*}
$$

where $\liminf_{n \to \infty} r_n > 0$ and $P_{C_n}$ is the metric projection of $E$ onto $C_n$. If $EP(f) \neq \emptyset$, then $\{x_n\}$ converges strongly to the element $P_{EP(f)}(x_1)$, where $P_{EP(f)}$ is the metric projection of $E$ onto $EP(f)$.
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