Title: Strong and Weak Convergence Theorems for Equilibrium Problems and Nonlinear Mappings in Banach Spaces

Author(s): Takahashi, Wataru

Citation: 数理解析研究所講究録 (2008), 1585: 91-105

Issue Date: 2008-02

URL: http://hdl.handle.net/2433/81515

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
Strong and Weak Convergence Theorems for Equilibrium Problems and Nonlinear Mappings in Banach Spaces

Wataru Takahashi
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology

Abstract. In this article, we prove strong and weak convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Next, we prove two strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the normal hybrid method and a new hybrid method called the shrinking projection method. Further, we obtain a necessary and sufficient condition for the existence of solutions of the equilibrium problem by using the metric resolvents. Finally, we prove a strong convergence theorem for finding a solution of an equilibrium problem in a Banach space by using the shrinking projection method.

1 Introduction

Let $E$ be a real Banach space and let $E^*$ be a dual space of $E$. Let $C$ be a closed convex subset of $E$ and let $f$ be a bifunction from $C \times C$ to $R$, where $R$ is the set of real numbers. The equilibrium problem is formulated as follows: Find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \quad \text{for all } y \in C.$$ 

In this case, such a point $\hat{x} \in C$ is called a solution of the problem. The set of such solutions $\hat{x}$ is denoted by $EP(f)$. Many problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Blum and Oettli [5] widely discussed the existence of solutions of such an equilibrium problem. Combettes-Hirstoaga [9], Tada and Takahashi [46], and Takahashi and Takahashi [48] proposed some methods for approximation of solutions of the equilibrium problem in a Hilbert space. However, the problem of approximating solutions of the equilibrium problem in a Banach space is difficult. We also know the problem of finding a point $u \in E$ satisfying

$$0 \in Au,$$

where $A$ is a maximal monotone operator from $E$ to $E^*$. Such a problem contains numerous problems in physics, optimization and economics. A well-known method to solve this problem is called the proximal point algorithm: $x_1 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \ldots,$$
where \( \{r_n\} \subset (0, \infty) \) and \( J_{r_n} \) are the resolvents of \( A \). Many researchers have studied this algorithm in a Hilbert space, see, for instance, [11, 18, 27, 42, 45] and in a Banach space, see, for instance, [17, 19, 20, 33]. A mapping \( S \) of \( C \) into \( E \) is called nonexpansive if

\[
\|Sx - Sy\| \leq \|x - y\|
\]

for all \( x, y \in C \). We denote by \( F(S) \) the set of fixed points of \( S \). There are some methods for approximation of fixed points of a nonexpansive mapping; see, for instance, [12, 26, 36, 43, 64]. In particular, in 2003 Nakajo–Takahashi [32] proved the following strong convergence theorem by using the hybrid method:

**Theorem 1.1 (Nakajo and Takahashi [32]).** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and let \( T \) be a nonexpansive mapping of \( C \) into itself such that \( F(T) \neq \emptyset \). Suppose \( x_1 = x \in C \) and \( \{x_n\} \) is given by

\[
\begin{align*}
y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
u_{n+1} &= P_{C_n \cap Q_n}x, \quad n \in \mathbb{N},
\end{align*}
\]

where \( P_{C_n \cap Q_n} \) is the metric projection from \( C \) onto \( C_n \cap Q_n \) and \( \{\alpha_n\} \) is chosen so that \( 0 \leq \alpha_n \leq a < 1 \). Then, \( \{x_n\} \) converges strongly to \( P_{F(T)}x \), where \( P_{F(T)} \) is the metric projection from \( C \) onto \( F(T) \).

Let us call the hybrid method in Theorem 1.1 the normal hybrid method. Very recently, Takahashi, Takeuchi and Kubota [61] proved the following theorem by using another hybrid method called the shrinking projection method.

**Theorem 1.2 (Takahashi, Takeuchi and Kubota [61]).** Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( T \) be a nonexpansive mapping of \( C \) into itself such that \( F(T) \neq \emptyset \) and let \( x_0 \in H \). For \( C_1 = C \) and \( u_1 = P_{C_1}x_0 \), define a sequence \( \{u_n\} \) of \( C \) as follows:

\[
\begin{align*}
y_n &= \alpha_n u_n + (1 - \alpha_n)Tu_n, \\
C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\
u_{n+1} &= P_{C_{n+1}}x_0, \quad n \in \mathbb{N},
\end{align*}
\]

where \( 0 \leq \alpha_n \leq a < 1 \) for all \( n \in \mathbb{N} \). Then, \( \{u_n\} \) converges strongly to \( z_0 = P_{F(T)}x_0 \).

In this article, we prove strong and weak convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Next, using the normal hybrid method and a new hybrid method called the shrinking projection method, we study two strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space. Further, we obtain a necessary and sufficient condition for the existence of solutions of the equilibrium problem by using the metric resolvents. Finally, we prove a strong convergence theorem for finding a solution of an equilibrium problem in a Banach space by using the shrinking projection method.
2 Preliminaries

Throughout this paper, we denote by $N$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let $E$ be a Banach space and let $E^*$ be the topological dual of $E$. For all $x \in E$ and $x^* \in E^*$, we denote the value of $x^*$ at $x$ by $\langle x, x^* \rangle$. Then, the duality mapping $J$ on $E$ is defined by
\[ J(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \} \]
for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty; see [51] for more details. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to $x$ in $E$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to $x^*$ in $E^*$ by $x_n^* \rightharpoonup^{*} x^*$. A Banach space $E$ is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \epsilon$. The space $E$ is said to be smooth if the limit
\[ \lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \]
exists for all $x, y \in S(E) = \{ x \in E : \|x\| = 1 \}$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$. We know that if $E$ is smooth, strictly convex and reflexive, then the duality mapping $J$ is single-valued, one-to-one and onto; see [51, 52] for more details.

Let $E$ be a smooth Banach space and define the real valued function $\phi$ by
\[ \phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 \]
for all $y, x \in E$. Then, we have that
\[ \phi(y, x) = \phi(x, x) + \phi(z, y) - 2\langle x - z, Jz - Jy \rangle \]
for all $x, y, z \in E$. Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed convex subset of $E$. Following Alber [1], the generalized projection $\Pi_C$ from $E$ onto $C$ is defined by
\[ \Pi_C(x) = \arg \min_{y \in C} \phi(y, x) \]
for all $x \in E$. If $E$ is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and $\Pi_C$ is the metric projection of $H$ onto $C$. We know the following lemmas for generalized projections.

**Lemma 2.1 (Alber [1], Kamimura and Takahashi [20]).** Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then
\[ \phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \text{for all } x \in C \text{ and } y \in E. \]

**Lemma 2.2 (Alber [1], Kamimura and Takahashi [20]).** Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space, let $x \in E$ and let $z \in C$. Then
\[ z = \Pi_C x \iff \langle y - z, Jx - Jz \rangle \leq 0 \quad \text{for all } y \in C. \]
Let $E$ be a smooth, strictly convex and reflexive Banach space, and let $A$ be a set-valued mapping from $E$ to $E^*$ with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$, domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$. We denote a set-valued operator $A$ from $E$ to $E^*$ by $A \subset E \times E^*$. $A$ is said to be monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone operator. We know that if $A$ is a maximal monotone operator, then $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$ is closed and convex; see [51, 52] for more details. The following theorem is well-known.

**Theorem 2.3 (Rockafellar [41]).** Let $E$ be a smooth, strictly convex and reflexive Banach space and let $A \subset E \times E^*$ be a monotone operator. Then $A$ is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.

Let $E$ be a smooth, strictly convex and reflexive Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1} (\cap_{r>0} R(J + rA)).$$

Then we can define the resolvent $J_r : C \to D(A)$ of $A$ by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz\}$$

for all $x \in C$. We know that $J_r x$ consists of one point. For all $r > 0$, the Yosida approximation $A_r : C \to E^*$ is defined by $A_r x = \frac{Jx - J_Jx}{r}$ for all $x \in C$. We also know the following lemma; see, for instance, [24].

**Lemma 2.4.** Let $E$ be a smooth, strictly convex and reflexive Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1} (\cap_{r>0} R(J + rA)).$$

Let $r > 0$ and let $J_r$ and $A_r$ be the resolvent and the Yosida approximation of $A$, respectively. Then, the following hold:

1. $\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x)$ for all $x \in C$ and $u \in A^{-1}0$;
2. $(J_r x, A_r x) \in A$ for all $x \in C$;
3. $F(J_r) = A^{-1}0$.

Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $T$ be a mapping from $C$ into itself. We denoted by $F(T)$ the set of fixed points of $T$. A point $p \in C$ is said to be an asymptotic fixed point of $T$ if there exists $\{x_n\}$ in $C$ which converges weakly to $p$ and $\lim_{n \to \infty} \|x_n - T x_n\| = 0$. We denote the set of all asymptotic fixed points of $T$ by $\hat{F}(T)$. Following Matsushita and Takahashi [29], a mapping $T : C \to C$ is said to be relatively nonexpansive if the following conditions are satisfied:

1. $F(T)$ is nonempty;
2. $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$;
3. $\hat{F}(T) = F(T)$.

The following lemma is due to Matsushita and Takahashi [28].
Lemma 2.5 (Matsushita and Takahashi [28]). Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, and let $T$ be a relatively nonexpansive mapping from $C$ into itself. Then $F(T)$ is closed and convex.

We also know the following lemma.

Lemma 2.6 (Kamimura and Takahashi [20]). Let $E$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_n \phi(x_n, y_n) = 0$, then $\lim_n \|x_n - y_n\| = 0$.

3 Equilibrium Problems and Relatively Nonexpansive Mappings

In this section, we prove strong and weak convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

Let $E$ be a Banach space and let $C$ be a nonempty closed convex subset of $E$. A function $f : C \times C \to R$ is said to be maximal monotone with respect to $C$ if, for every $x \in C$ and $x^* \in E^*$,

$$f(x, y) + (y - x, x^*) \geq 0$$

for all $y \in C$, whenever $(z - x, x^*) \geq f(z, x)$ for all $z \in C$.

In this article, we assume that a bifunction $f$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) $f$ is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous;

(A4) $\limsup_{t \downarrow 0} f(tx + (1 - t)x, y) \leq f(x, y)$ for all $x, y, z \in C$.

Assume that $f$ satisfies (A1)–(A4). Then, $f$ is maximal monotone. In fact, for every $x \in C$ and $x^* \in E^*$, suppose that

$$\langle z - x, x^* \rangle \geq f(z, x)$$

for all $z \in C$. Putting $z_t = (1 - t)x + ty$ with $y \in C$ and $t \in (0, 1)$, we have

$$0 = f(z_t, z_t)$$

$$\leq (1 - t)f(z_t, x) + tf(z_t, y)$$

$$\leq (1 - t)(z_t - x, x^*) + tf(z_t, y)$$

$$\leq t(1 - t)(y - x, x^*) + tf(z_t, y).$$

Hence, we have $0 \leq (1 - t)(y - x, x^*) + f(z_t, y)$. Since $f$ is upper hemicontinuous, we have

$$0 \leq \langle y - x, x^* \rangle + f(z, y).$$

Hence, $f$ is maximal monotone. The following result is in Blum and Oettli [5]. See [2] for the proof.

Lemma 3.1 (Blum and Oettli [5]). Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $R$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that

$$f(z, x) + \frac{1}{r}(y - z, Jz - Jx) \geq 0$$

for all $y \in C$. 
Motivated by Combettes and Hirstoaga [9] in a Hilbert space, we obtain the following lemma.

**Lemma 3.2.** Let $C$ be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space $E$, and let $f$ be a bifunction from $C \times C$ to $R$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0 \text{ for all } y \in C \right\}$$

for all $x \in E$. Then, the following hold:

1. $T_r$ is single-valued;
2. $T_r$ is a firmly nonexpansive-type mapping [24], i.e., for all $x, y \in E$,
   $$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$
3. $F(T_r) = EP(f)$;
4. $EP(f)$ is closed and convex.

We claim that $T_r$ is single-valued. Indeed, for $x \in C$ and $r > 0$, let $z_1, z_2 \in T_r x$. Then,

$$f(z_1, z_2) + \frac{1}{r} \langle z_2 - z_1, Jz_1 - Jx \rangle \geq 0$$

and

$$f(z_2, z_1) + \frac{1}{r} \langle z_1 - z_2, Jz_2 - Jx \rangle \geq 0.$$

Adding two inequalities, we have

$$f(z_1, z_2) + f(z_2, z_1) + \frac{1}{r} \langle z_2 - z_1, Jz_1 - Jz_2 \rangle \geq 0.$$

From (A2) and $r > 0$, we have

$$\langle z_2 - z_1, Jz_1 - Jz_2 \rangle \geq 0.$$

Since $E$ is strictly convex, we have $z_1 = z_2$.

Next, we claim that $T_r$ is a firmly nonexpansive-type mapping. Indeed, for $x, y \in C$, we have

$$f(T_r x, T_r y) + \frac{1}{r} \langle T_r y - T_r x, JT_r x - Jx \rangle \geq 0,$$

and

$$f(T_r y, T_r x) + \frac{1}{r} \langle T_r x - T_r y, JT_r y - Jy \rangle \geq 0.$$

Adding two inequalities, we have

$$f(T_r x, T_r y) + f(T_r y, T_r x) + \frac{1}{r} \langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy \rangle \geq 0.$$

From (A2) and $r > 0$, we have

$$\langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy \rangle \geq 0.$$

Therefore, we have

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle.$$

We call such $T_r$ the relative resolvent of $f$ for $r > 0$. Using Lemma 3.2, we have the following result.
Lemma 3.3. Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $R$ satisfying $(A1)$–$(A4)$, and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,
\[
\phi(q,T_r x) + \phi(T_r x, x) \leq \phi(q, x).
\]

Proof. From Lemma 3.2 (2), we have, for all $x, y \in E$,
\[
\phi(T_r x, T_r y) + \phi(T_r y, T_r x) \leq \phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y).
\]
Letting $y = q \in F(T_r)$, we have
\[
\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).
\]
This completes the proof.

Now, we prove a strong convergence theorem for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

Theorem 3.4 (Takahashi and Zembayashi [63]). Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $R$ satisfying $(A1)$–$(A4)$ and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

\[
\begin{cases}
x_0 = x \in C, \\
y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \forall y \in C, \\
H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
W_n = \{z \in C : \langle x_n - z, J x - J x_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{H_n \cap W_n} x
\end{cases}
\]
for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap EP(f)} x$, where $\Pi_{F(S) \cap EP(f)}$ is the generalized projection of $E$ onto $F(S) \cap EP(f)$.

Further, we prove a weak convergence theorem for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Before proving the theorem, we need the following proposition.

Proposition 3.5. Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $R$ satisfying $(A1)$–$(A4)$ and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u_1 \in E$,

\[
\begin{cases}
x_n \in C \text{ such that } f(x_n, y) + \frac{1}{r_n} \langle y - x_n, J x_n - J u_n \rangle \geq 0, \forall y \in C, \\
u_{n+1} = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n)
\end{cases}
\]
for every $n \in N$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$ and $\{r_n\} \subset [0, \infty)$. Then, $\{\Pi_{F(S) \cap EP(f)} x_n\}$ converges strongly to $z \in F(S) \cap EP(f)$, where $\Pi_{F(S) \cap EP(f)}$ is the generalized projection of $E$ onto $F(S) \cap EP(f)$.
Using Proposition 3.5, we can prove the following theorem.

**Theorem 3.6 (Takahashi and Zembayashi [63]).** Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a bifunction from $C \times C$ to $R$ satisfying $(A1)-(A4)$ and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u_1 \in E$,

\[
\begin{align*}
x_n & \in C \text{ such that } f(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \forall y \in C, \\
u_{n+1} & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n)
\end{align*}
\]

for every $n \in N$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $J$ is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in F(S) \cap EP(f)$, where $z = \lim_{n \to \infty} \Pi_{F(S) \cap EP(f)}x_n$.

\section{Maximal Monotone Operators and Relatively Nonexpansive Mappings}

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the normal hybrid method.

**Theorem 4.1 (Inoue, Takahashi and Zembayashi [25]).** Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying

\[D(A) \subset C \subset J^{-1}((\cap_{r>0}R(J+rA)))\]

and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

\[
\begin{align*}
&u_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSJ_{r_n}x_n), \\
&H_n = \{z \in C : \phi(z,u_n) \leq \phi(z,x_n)\}, \\
&W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
&x_{n+1} = \Pi_{H_n \cap W_n}x
\end{align*}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n \to \infty}(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap A^{-1}0}x$, where $\Pi_{F(S) \cap A^{-1}0}$ is the generalized projection of $E$ onto $F(S) \cap A^{-1}0$.

As direct consequences of Theorem 4.1, we can obtain the following corollaries.

**Corollary 4.2.** Let $E$ be a uniformly smooth and uniformly convex Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all
Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

\[
\begin{align*}
  u_n &= J_{r_n}x_n, \\
  H_n &= \{z \in E : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
  W_n &= \{z \in E : (x_n - z, Jz - Jx_n) \geq 0\}, \\
  x_{n+1} &= \Pi_{H_n \cap W_n} x
\end{align*}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x$, where $\Pi_{A^{-1}0}$ is the generalized projection of $E$ onto $A^{-1}0$.

**Proof.** Putting $S = I$, $C = E$ and $\alpha_n = 0$ in Theorem 4.1, we obtain Corollary 4.2. \qed

**Corollary 4.3 (Matsushita and Takahashi [29]).** Let $E$ be a uniformly smooth and uniformly convex Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

\[
\begin{align*}
  u_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSJ_{r_n}x_n), \\
  H_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
  W_n &= \{z \in C : (x_n - z, Jz - Jx_n) \geq 0\}, \\
  x_{n+1} &= \Pi_{H_n \cap W_n} x
\end{align*}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0, 1)$ satisfies $\liminf_{n \to \infty}(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)}x$, where $\Pi_{F(S)}$ is the generalized projection of $E$ onto $F(S)$.

**Proof.** Set $A = \partial i_C$ in Theorem 4.1, where $i_C$ is the indicator function of $C$ and $\partial i_C$ is the subdifferential of $i_C$. Then, we have that $A$ is a maximal monotone operator and $J_r = \Pi_C$, where $J_r$ is the resolvent of $A = \partial i_C$ for $r > 0$. So, from Theorem 4.1, we obtain Corollary 4.3. \qed

Using an idea of [61], we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the shrinking projection method.

**Theorem 4.4 (Inoue, Takahashi and Zembayashi [25]).** Let $E$ be a uniformly smooth and uniformly convex Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying

\[
D(A) \subset C \subset J^{-1}(\cap_{r > 0} R(J + rA))
\]

and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $H_0 = C$ and

\[
\begin{align*}
  u_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSJ_r x_n), \\
  H_{n+1} &= \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
  x_{n+1} &= \Pi_{H_{n+1}} x
\end{align*}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0, 1)$ satisfies $\liminf_{n \to \infty}(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap A^{-1}0}x$, where $\Pi_{F(S) \cap A^{-1}0}$ is the generalized projection of $E$ onto $F(S) \cap A^{-1}0$. 

As direct consequences of Theorem 4.4, we can obtain the following corollaries.

**Corollary 4.5.** Let $E$ be a uniformly smooth and uniformly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in E$, $H_0 = E$ and

\[
\begin{aligned}
  u_n &= J_{r_n}x_n, \\
  H_{n+1} &= \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
  x_{n+1} &= \Pi_{H_{n+1}}x
\end{aligned}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}$. 

**Proof.** Putting $S = I$, $C = H_0 = E$ and $\alpha_n = 0$ in Theorem 4.4, we obtain Corollary 4.5. \[\square\]

**Corollary 4.6.** Let $E$ be a uniformly smooth and uniformly convex Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $S$ be a relatively nonexpansive mapping from $C$ into itself such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

\[
\begin{aligned}
  u_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\
  H_{n+1} &= \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
  x_{n+1} &= \Pi_{H_{n+1}}x
\end{aligned}
\]

for every $n \in N \cup \{0\}$, where $J$ is the duality mapping on $E$, $\{\alpha_n\} \subset [0, 1)$ satisfies $\liminf_{n \to \infty}(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)}x$, where $\Pi_{F(S)}$ is the generalized projection of $E$ onto $F(S)$.

**Proof.** Putting $A = \partial i_C$ in Theorem 4.4, we obtain Corollary 4.6. \[\square\]

### 5 Equilibrium Problems and Metric Resolvents

In this section, we prove a strong convergence theorem for finding a solution of the equilibrium problem by using the metric resolvents. Using Lemma 3.1, we first obtain the following result.

**Lemma 5.1.** Let $C$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $R$ satisfying $(A1)-(A4)$, let $r > 0$ and let $x \in E$. Then, there exists a unique $z_r \in C$ such that

$$f(z_r, y) + \frac{1}{r} \langle y - z_r, J(z_r - x) \rangle \geq 0 \quad \text{for all} \quad y \in C.$$  

**Proof.** Fix $x \in C$. Then, we define $g : (C - x) \times (C - x) \to R$ as follows:

$$g(z, y) = f(z + x, y + x) + \frac{1}{r} \langle y - z, Jx \rangle. \quad (5.1)$$

From the properties of $f$, it is easy to prove that $g$ satisfies the following conditions;

$(A1) \quad g(z, z) = 0$ for all $z \in C - x$;
(A2) $g$ is monotone with respect to $C - x$;
(A3) for all $z \in C - x$, $g(z, \cdot)$ is convex and lower semicontinuous;
(A4) $g$ is upper hemicontinuous with respect to $C - x$.

Hence, from Lemma 3.1 there exists a unique element $z_r$ such that
\[ f(z_r + x, y + x) + \frac{1}{r} \langle y - z_r, Jx \rangle + \frac{1}{r} \langle y - z_r, Jz_r - Jx \rangle \geq 0 \]
for all $y \in C - x$. This implies that
\[ f(z_r + x, y + x) + \frac{1}{r} \langle y - z_r, Jz_r \rangle \geq 0 \]
for all $y \in C - x$. Putting $u_r = z_r + x$ and $v = y + x$, we have
\[ f(u_r, v) + \frac{1}{r} \langle v - u_r, J(u_r - x) \rangle \geq 0 \]
for all $v \in C$. This completes the proof. \[\square\]

Under the conditions in Theorem 5.1, for every $r > 0$ we may define a single-valued mapping $F_r : E \to C$ by
\[ F_r x = \{ z \in C : 0 \leq f(z, y) + \frac{1}{r} \langle y - z, J(z - x) \rangle, y \in C \} \tag{5.2} \]
for $x \in E$, which is called the metric resolvent of $f$ for $r > 0$. Also, we can define the Yosida approximation as follows:
\[ A_r x = \frac{1}{r} J(x - F_r x). \tag{5.3} \]

As in Takahashi[52, pp.163-165], we can prove the following theorem for Yosida approximations. Before proving it, we need the following lemma; see, for instance, Takahashi[52, Problem 4.5.4].

**Lemma 5.2.** Let $E$ be a Banach space. Assume that $u_n \rightharpoonup v$, $v_n \rightharpoonup v^*$ and
\[ \lim_{m,n \to \infty} \langle u_n - u_m, v_n - v_m \rangle = 0. \]
Then, \[ \lim_{n \to \infty} \langle u_n, v_n \rangle = \langle u, v^* \rangle. \]

**Lemma 5.3.** Assume $r > 0$. Then, $A_r : E \to E^*$ is monotone and demicontinuous. Further, if $D \subset E$ is bounded, then $A_r D \subset E^*$ is bounded.

To show a necessary and sufficient condition for the existence of solutions of the equilibrium problem, we need the following lemma [51, Theorem 7.1.8]; see also [4].

**Lemma 5.4** ([51, 4]). Let $E$ be a reflexive Banach space and let $K$ be a bounded closed convex subset of $E$. Suppose $A$ is a monotone and demicontinuous operator. Then there exists $u_0 \in K$ such that
\[ \langle y - u_0, Au_0 \rangle \geq 0 \quad \text{for all} \ y \in K. \]

Using Lemma 5.4, we obtain the following lemma.
Lemma 5.5. Let $E$ be a smooth and uniformly convex Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $f$ be a bifunction $C \times C$ to $R$ satisfying (A1)–(A4). For $C_1 = C$ and $x_1 = x \in E$, define the sequence $\{x_n\}$ as follows:

$$
\begin{align*}
  y_n &= F_{r_n} x_n; \\
  C_{n+1} &= \{z \in C_n : (y_n - z, J(x_n - y_n)) \geq 0\}, \\
  x_{n+1} &= P_{C_{n+1}}(x_1),
\end{align*}
$$

where $0 < r_n < \infty$ and $P_{C_n}$ is the metric projection of $E$ onto $C_n$. Then $\{x_n\}$ is well-defined.

We also have the following lemma.

Lemma 5.6. If $EP(f) \neq \emptyset$, then $EP(f) \subset C_n$ for all $n \in N$.

Proof. It is obvious that $EP(f) \subset C_1 = C$. Suppose $EP(f) \subset C_n$ for some $n \in N$. Let $z \in EP(f)$. From $y_n = F_{r_n} x_n$ and the monotonicity of $f$, we have

$$
\langle y_n - y, \frac{1}{r_n} J(x_n - y_n) \rangle \geq f(y, y_n)
$$

for all $y \in C$. Put $y = z$. Then we have

$$
\langle y_n - z, \frac{1}{r_n} J(x_n - y_n) \rangle \geq f(z, y_n) \geq 0.
$$

Therefore, $z \in C_{n+1}$. By the mathematical induction, we obtain $z \in C_n$ for all $n \in N$. \qed

Now, we obtain a necessary and sufficient condition for the existence of solutions of the equilibrium problem in a Banach space.

Theorem 5.7 (Takahashi and Takahashi [47]). Let $E$ be a smooth and uniformly convex Banach space and let $f$ be a bifunction $C \times C$ to $R$ satisfying (A1)–(A4). For $C_1 = C$ and $x_1 = x \in E$, define the sequence $\{x_n\}$ as follows:

$$
\begin{align*}
  y_n &= F_{r_n} x_n; \\
  C_{n+1} &= \{z \in C_n : (y_n - z, J(x_n - y_n)) \geq 0\}, \\
  x_{n+1} &= P_{C_{n+1}}(x_1),
\end{align*}
$$

where $\liminf_{n \to \infty} r_n > 0$ and $P_{C_n}$ is the metric projection of $E$ onto $C_n$. Then $\{x_n\}$ is bounded if and only if $EP(f) \neq \emptyset$.

Finally, we can prove a strong convergence theorem for finding a solution of the equilibrium problem by using the shrinking projection method.

Theorem 5.8 (Takahashi and Takahashi [47]). Let $E$ be a smooth and uniformly convex Banach space and let $C$ a nonempty closed convex subset of $E$. Let $f$ be a bifunction $C \times C$ to $R$ satisfying (A1)–(A4). For $C_1 = C$ and $x_1 = x \in E$, define the sequence $\{x_n\}$ as follows:

$$
\begin{align*}
  y_n &= F_{r_n} x_n; \\
  C_{n+1} &= \{z \in C_n : (y_n - z, J(x_n - y_n)) \geq 0\}, \\
  x_{n+1} &= P_{C_{n+1}}(x_1),
\end{align*}
$$

where $\liminf_{n \to \infty} r_n > 0$ and $P_{C_n}$ is the metric projection of $E$ onto $C_n$. If $EP(f) \neq \emptyset$, then $\{x_n\}$ converges strongly to the element $P_{EP(f)}(x_1)$, where $P_{EP(f)}$ is the metric projection of $E$ onto $EP(f)$. 

References


[21] M. Kikkawa and W. Takahashi, *Strong convergence theorems by the viscosity approxi-


[43] N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpans-