<table>
<thead>
<tr>
<th>Title</th>
<th>On the existence of bivariate kernel with given marginal kernels (Information and mathematics of non-additivity and non-extensivity: contacts with nonlinearity and non-commutativity)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Machida, Motoya; Shibakov, Alexander</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2008), 1585: 47-50</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/81519">http://hdl.handle.net/2433/81519</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On the existence of bivariate kernel with given marginal kernels

Motoya Machida and Alexander Shibakov

Tennessee Technological University

Abstract

We raise a question on jointly Markovian sample paths given marginal Markov chains, and prove that such a bivariate Markov chain exists when the state space is a compact Polish space.

1 Introduction

Strassen [4] proved the existence of a probability measure $\lambda$ to realize a pair $(X, X')$ of random variables whose marginals, say $p$ and $p'$, are given. Kamae, Krengel and O'Brien [3] investigated extensively the realization of an ordered pair $X \leq X'$, and associate Strassen's result with stochastic ordering when the underlying space $S$ is equipped with a partial ordering $\leq$. A probability measure $p$ on $S$ is said to be stochastically smaller than $p'$, denoted by $p \leq p'$, if $\int f(s) p(ds) \leq \int f(s) p'(ds)$ for every real-valued increasing function $f$ on $S$. Then $p \leq p'$ is a necessary and sufficient condition for the existence of probability measure $\lambda$ whose marginals are $p$ and $p'$, and whose support lies on the set $\Delta = \{(s, s') \in S \times S : s \leq s'\}$ (the Nachbin-Strassen theorem; see Theorem 1 of [3]). This existence theorem was immediately applied to that of Markov chains. A Markov transition kernel $k$ is said to be stochastically cross-monotone to a kernel $k'$ (or, $k$ stochastically dominates $k'$), if $k(r, \cdot) \preceq k'(r', \cdot)$ whenever $r \leq r'$. Assuming the cross-monotonicity between $k$ and $k'$, the respective Markov chain sample paths

\begin{equation}
X = (X_0, X_1, \ldots) \quad \text{and} \quad X' = (X'_0, X'_1, \ldots)
\end{equation}

can be realized so as to maintain the pairwise order $X'_n \leq X'_n$ for all $n \geq 0$ if the initial distribution $\pi_0$ for $X_0$ is also stochastically smaller than $\pi'_0$ for $X'_0$ (Theorem 2 of [3]).

The paired sample path $(X_n, X'_n)_{n=0,1,\ldots}$ in (1.1) is not necessarily Markovian. But if so, there is a bivariate kernel $K$ on $\Delta$ satisfying the marginal conditions

\begin{equation}
k(r, E) = K((r, r'), E \times S) \quad \text{and} \quad k'(r', E') = K((r, r'), S \times E')
\end{equation}
for \((r, r') \in \Delta\) and measurable sets \(E\) and \(E'\). Note in (1.2) that we view the measure \(K((r, r'), \cdot)\) as if it lies on \(S \times S\) and has its support on \(\Delta\). Such a bivariate kernel exists via the Nachbin-Strassen theorem when \(S\) is discrete (finite or countable). A probability measure \(\lambda^{(r,r')}(\cdot)\) on \(\Delta\) exists for each pair \((r, r') \in \Delta\) so that it has marginals \(k(r, \cdot)\) and \(k'(r', \cdot)\). Then \(\lambda^{(r,r')}(\cdot)\) can be collectively viewed as a kernel \(K((r, r'), \cdot)\). When \(S\) is continuous (typically referred to a Polish space), however, the measurability of \(\lambda^{(r,r')}\) with respect to \((r, r')\) has to be taken into account. This raises a question on whether \(\lambda^{(r,r')}\) can be selected to ensure the measurability, and this expository paper discusses our investigation on a compact Polish space.

2 Measure space and selections

Let \(S\) be a compact Polish space, and let \(C\) be the space of real-valued continuous functions on the product space \(S \times S\). The space \(C\) becomes a Banach space with the norm \(\|f\| = \sup |f(S \times S)|\). A Radon measure \(\lambda\) is a continuous linear functional on \(C\), and the functional has an integral form \(\lambda(f) = \int f(r)\lambda(\text{d}r)\). The space \(\mathcal{M}\) of Radon measures is a complete lattice, and the positive cone \(\mathcal{M}^+\) consists of positive Radon measures (Theorem 11.2 of Choquet [2]). The space \(\mathcal{M}\) is equipped with weak* topology, and the cone \(\mathcal{M}^+\) is metrizable and separable (Theorem 12.10 of [2]). Let \(\mathcal{D}\) be a countable dense subset of \(C\). The family of the semi-norms, \(|\lambda(f)|\) for \(f \in \mathcal{D}\), introduces the topology on \(\mathcal{M}\), and it coincides with the weak* topology on the cone \(\mathcal{M}^+\). Then we can form a countable subbase via

\[
U_{f,q} := \{\lambda \in \mathcal{M}^+: \lambda(f) > q\}, \quad f \in \mathcal{D}, q \in \mathbb{Q},
\]

where \(\mathbb{Q}\) denotes the set of all rational numbers.

Let \(\Lambda\) be a closed set-valued map from \(\Delta\) to \(\mathcal{M}^+\). If \(\Lambda\) is measurable, there exists a selection function \(\lambda^{(r,r')} \in \Lambda(r, r')\) such that the map \(\lambda^{(r,r')}\) is measurable from \(\Delta\) to \(\mathcal{M}^+\) (Theorem 8.1.3 of Aubin and Frankowska [1]). In particular, \(\lambda^{(r,r')}(f)\) becomes a measurable function on \(\Delta\) for each \(f \in C\). To see whether \(\Lambda\) is measurable, it suffices to show that

\[
\Lambda^{-1}(U_{f,q}) = \{(r, r') \in \Delta : \Lambda(r, r') \cap U_{f,q} \neq \emptyset\}
\]

is a Borel measurable subset for every \(f \in \mathcal{D}\), and \(q \in \mathbb{Q}\) (Definition 8.1.1 of [1]). It is easily observed that \(\Lambda(r, r') \cap U_{f,q} \neq \emptyset\) is equivalent to

\[
q < H_f(r, r') = \sup_{\lambda \in \Lambda(r, r')} \lambda(f).
\]

Thus, the verification of measurability of the set-valued map \(\Lambda\) is reduced to that of the function \(H_f(r, r')\).
3 Validation of measurability

Let $C_S$ be the space of real-valued continuous functions on $S$. We write the direct sum $(f_1 \oplus f_2)(s, s') = f_1(s) + f_2(s')$ for $f_1, f_2 \in C_S$, and the subspace $C_S \oplus C_S = \{f_1 \oplus f_2 : f_1, f_2 \in C_S\}$ on $C$. A probability measure $p$ is stochastically smaller than $p'$ if and only if $p(f_1) + p(f_2) \leq \sup(f_1 \oplus f_2)(\Delta)$ for any $f_1, f_2 \in C_S$. The Nachbin-Strassen theorem can be similarly stated on a par with this form of stochastic inequality.

A probability measure $p$ is stochastically smaller than $p'$ if and only if $p(f_1) + p(f_2) \leq \sup(f_1 \oplus f_2)(\Delta)$ for any $f_1, f_2 \in C_S$.

The above conditions clearly imply that (i) $\lambda$ has the marginals $p$ and $p'$, and (ii) it has a support on $\Delta$.

A Markov transition kernel $k$ on $S$ is a collection of positive Radon measures $k(s, \cdot)$ on $S$ such that $k(s, \cdot)$ is a probability measure for each $s \in S$ and

$$\langle k, f \rangle = \int f(s) k(r, ds)$$

is a measurable function of $r$ for every $f \in C_S$. Suppose that a Markov transition kernel $k$ is cross-monotone to $k'$. Then the cross-monotonicity is equivalently stated as

$$\langle k, f_1 \rangle \oplus \langle k', f_2 \rangle)(r, r') \leq \sup(f_1 \oplus f_2)(\Delta)$$

for any $f_1, f_2 \in C_S$. For each $(r, r') \in \Delta$ we define the subset $\Lambda(r, r')$ consisting of $\lambda^{(r,r')} \in \mathcal{M}^+$ which satisfies the following two conditions.

$$\lambda^{(r,r')}(f_1 \oplus f_2) = (\langle k, f_1 \rangle \oplus \langle k', f_2 \rangle)(r, r') \quad \text{for } f_1, f_2 \in C_S;$$

$$\lambda^{(r,r')}(f) \leq \sup f(\Delta) \quad \text{for } f \in C.$$

It is easily observed that $\Lambda(r, r')$ is closed and that it is nonempty via the Nachbin-Strassen theorem. Let

$$H_f(r, r') = \inf_{f_1, f_2 \in C_S} [\sup(f_1 \oplus f_2 + f)(\Delta) - (\langle k, f_1 \rangle \oplus \langle k', f_2 \rangle)(r, r')]$$

for each $(r, r') \in \Delta$ and $f \in C$. Then we have

**Proposition 3.1.** $\sup_{\lambda \in \Lambda(r, r')} \lambda(f) = H_f(r, r').$

**Proof.** Let $(r, r') \in \Delta$ and $f \in C$ be fixed. By replacing $f$ with $f_1 \oplus f_2 + f$ in (3.3), we can immediately observe that

$$\lambda(f) \leq H_f(r, r')$$

for every $\lambda \in \Lambda(r, r')$. By (3.2) the equality attains in (3.5) if $f \in C_S \oplus C_S$; thus, it is assumed that $f \not\in C_S \oplus C_S$. Let $\ell(f_1 \oplus f_2) = (\langle k, f_1 \rangle \oplus \langle k', f_2 \rangle)(r, r')$ for
$f_1, f_2 \in C_S$. Then $\ell$ is a well-defined linear functional on $C_S \oplus C_S$. By applying (3.1) and (3.2) together, we can observe that

$$-\sup(-f - g)(\Delta) - \ell(g) \leq \sup(f + g')(\Delta) - \ell(g')$$

for any $g, g' \in C_S \oplus C_S$, and therefore, that

$$\kappa_f = \inf_{g \in C_S \oplus C_S} \left( \sup(f + g)(\Delta) - \ell(g) \right)$$

has a finite value. We can extend the subspace $\tilde{E} = \{g + tf : g \in C_S \oplus C_S, t \in \mathbb{R}\}$ by adding the element $f$, and define

$$\tilde{\ell}(g + tf) := \ell(g) + t\kappa_f$$

for $g \in C_S \oplus C_S$ and $t \in \mathbb{R}$. The map $\tilde{\ell}$ is a well-defined linear functional on $\tilde{E}$ and satisfies (3.2) and (3.3) with $\tilde{\ell}$ and $\tilde{E}$ in place of $\lambda^{(r,r')}$ and $C$. The same argument is essentially recycled to show that $\tilde{\ell}$ on $\tilde{E}$ is extended to $\lambda^{(r,r')} \in \Lambda(r, r')$ via Zorn's lemma. This particular $\lambda^{(r,r')}$ will achieve the equality in (3.5). \hfill \square

Observe that the space $C_S$ in the infimum of (3.4) can be replaced by a countable dense set, and consequently $H_f(r, r')$ is a measurable function of $(r, r')$ for each $f \in C$. Therefore, $\Lambda$ is a measurable set-valued map from $\Delta$ to $\mathcal{M}^+$, and there exists a measurable selection $\lambda^{(r,r')} \in \Lambda(r, r')$; thus, $K((r, r'), \cdot) = \lambda^{(r,r')}(\cdot)$ becomes a desired bivariate kernel on $\Delta$ satisfying (1.2).

### References


