Title	On the existence of bivariate kernel with given marginal kernels (Information and mathematics of non-additivity and non-extensivity : contacts with nonlinearity and non-commutativity)
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Citation	数理解析研究所講究録 (2008), 1585: 47-50
Issue Date	2008-02
URL	http://hdl.handle.net/2433/81519
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1585 巻 2008 年 47-50

On the existence of bivariate kernel with given marginal kernels

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Abstract

We raise a question on jointly Markovian sample paths given marginal Markov chains, and prove that such a bivariate Markov chain exists when the state space is a compact Polish space.

1 Introduction

Strassen [4] proved the existence of a probability measure λ to realize a pair (X, X') of random variables whose marginals, say p and p', are given. Kamae, Krengel and O'Brien [3] investigated extensively the realization of an ordered pair $X \leq X'$, and associate Strassen's result with stochastic ordering when the underlying space S is equipped with a partial ordering $\leq$. A probability measure p on S is said to be *stochastically smaller* than p', denoted by $p \preceq p'$, if $\int f(s)\, p(ds) \leq \int f(s)\, p'(ds)$ for every real-valued increasing function f on S. Then $p \preceq p'$ is a necessary and sufficient condition for the existence of probability measure λ whose marginals are p and p', and whose support lies on the set $\Delta = \{(s, s') \in S \times S : s \leq s'\}$ (the Nachbin-Strassen theorem; see Theorem 1 of [3]). This existence theorem was immediately applied to that of Markov chains. A Markov transition kernel k is said to be *stochastically cross-monotone* to a kernel k' (or, k *stochastically dominates* k'), if $k(r, \cdot) \preceq k'(r', \cdot)$ whenever $r \leq r'$. Assuming the cross-monotonicity between k and k', the respective Markov chain sample paths

$$X = (X_0, X_1, \ldots) \text{ and } X' = (X'_0, X'_1, \ldots) \tag{1.1}$$

can be realized so as to maintain the pairwise order $X'_n \leq X'_n$ for all $n \geq 0$ if the initial distribution π_0 for X_0 is also stochastically smaller than π'_0 for X'_0 (Theorem 2 of [3]).

The paired sample path $(X_n, X'_n)_{n=0,1,\ldots}$ in (1.1) is not necessarily Markovian. But if so, there is a bivariate kernel K on Δ satisfying the marginal conditions

$$k(r, E) = K((r, r'), E \times S) \text{ and } k'(r', E') = K((r, r'), S \times E') \tag{1.2}$$

for $(r, r') \in \Delta$ and measurable sets E and E'. Note in (1.2) that we view the measure $K((r, r'), \cdot)$ as if it lies on $S \times S$ and has its support on Δ. Such a bivariate kernel exists via the Nachbin-Strassen theorem when S is discrete (finite or countable). A probability measure $\lambda^{(r,r')}(\cdot)$ on Δ exists for each pair $(r, r') \in \Delta$ so that it has marginals $k(r, \cdot)$ and $k'(r', \cdot)$. Then $\lambda^{(r,r')}(\cdot)$ can be collectively viewed as a kernel $K((r, r'), \cdot)$. When S is continuous (typically referred to a Polish space), however, the measurability of $\lambda^{(r,r')}$ with respect to (r, r') has to be taken into account. This raises a question on whether $\lambda^{(r,r')}$ can be selected to ensure the measurability, and this expository paper discusses our investigation on a compact Polish space.

2 Measure space and selections

Let S be a compact Polish space, and let $\mathcal{C}$ be the space of real-valued continuous functions on the product space $S \times S$. The space $\mathcal{C}$ becomes a Banach space with the norm $\|f\| = \sup |f(S \times S)|$. A *Radon measure* λ is a continuous linear functional on $\mathcal{C}$, and the functional has an integral form $\lambda(f) = \int f(r)\lambda(dr)$. The space $\mathcal{M}$ of Radon measures is a complete lattice, and the positive cone $\mathcal{M}^+$ consists of positive Radon measures (Theorem 11.2 of Choquet [2]). The space $\mathcal{M}$ is equipped with weak∗ topology, and the cone $\mathcal{M}^+$ is metrizable and separable (Theorem 12.10 of [2]). Let $\mathcal{D}$ be a countable dense subset of $\mathcal{C}$. The family of the semi-norms, $|\lambda(f)|$ for $f \in \mathcal{D}$, introduces the topology on $\mathcal{M}$, and it coincides with the weak∗ topology on the cone $\mathcal{M}^+$. Then we can form a countable subbase via

$$U_{f,q} := \{\lambda \in \mathcal{M}^+ : \lambda(f) > q\}, \quad f \in \mathcal{D}, q \in \mathbb{Q},$$

where $\mathbb{Q}$ denotes the set of all rational numbers.

Let Λ be a closed set-valued map from Δ to $\mathcal{M}^+$. If Λ is measurable, there exists a selection function $\lambda^{(r,r')} \in \Lambda(r, r')$ such that the map $\lambda^{(r,r')}$ is measurable from Δ to $\mathcal{M}^+$ (Theorem 8.1.3 of Aubin and Frankowska [1]). In particular, $\lambda^{(r,r')}(f)$ becomes a measurable function on Δ for each $f \in \mathcal{C}$. To see whether Λ is measurable, it suffices to show that

$$\Lambda^{-1}(U_{f,q}) = \{(r, r') \in \Delta : \Lambda(r, r') \cap U_{f,q} \neq \emptyset\}$$

is a Borel measurable subset for every $f \in \mathcal{D}$, and $q \in \mathbb{Q}$ (Definition 8.1.1 of [1]). It is easily observed that $\Lambda(r, r') \cap U_{f,q} \neq \emptyset$ is equivalent to

$$q < H_f(r, r') = \sup_{\lambda \in \Lambda(r,r')} \lambda(f).$$

Thus, the verification of measurability of the set-valued map Λ is reduced to that of the function $H_f(r, r')$.

3 Validation of measurability

Let $\mathcal{C}_S$ be the space of real-valued continuous functions on S. We write the direct sum $(f_1 \oplus f_2)(s,s') = f_1(s) + f_2(s')$ for $f_1, f_2 \in \mathcal{C}_S$, and the subspace $\mathcal{C}_S \oplus \mathcal{C}_S = \{f_1 \oplus f_2 : f_1, f_2 \in \mathcal{C}_S\}$ on $\mathcal{C}$. A probability measure p is stochastically smaller than p' if and only if $p(f_1) + p(f_2) \le \sup(f_1 \oplus f_2)(\Delta)$ for any $f_1, f_2 \in \mathcal{C}_S$. The Nachbin-Strassen theorem can be similarly stated on a par with this form of stochastic inequality. If $p \preceq p'$ then there exists $\lambda \in \mathcal{M}^+$ satisfying (i) $\lambda(f_1 \oplus f_2) = p(f_1) + p(f_2)$ for any $f_1, f_2 \in \mathcal{C}_S$, and (ii) $\lambda(f) \le \sup f(\Delta)$ for any $f \in \mathcal{C}$. The above conditions clearly imply that (i) λ has the marginals p and p', and (ii) it has a support on Δ.

A Markov transition kernel k on S is a collection of positive Radon measures $k(s,\cdot)$ on S such that $k(s,\cdot)$ is a probability measure for each $s \in S$ and

$$\langle k, f\rangle = \int f(s)\, k(r, ds)$$

is a measurable function of r for every $f \in \mathcal{C}_S$. Suppose that a Markov transition kernel k is cross-monotone to k'. Then the cross-monotonicity is equivalently stated as

$$(3.1) \qquad (\langle k, f_1\rangle \oplus \langle k', f_2\rangle)(r,r') \le \sup(f_1 \oplus f_2)(\Delta)$$

for any $f_1, f_2 \in \mathcal{C}_S$. For each $(r,r') \in \Delta$ we define the subset $\Lambda(r,r')$ consisting of $\lambda^{(r,r')} \in \mathcal{M}^+$ which satisfies the following two conditions.

$$(3.2) \qquad \lambda^{(r,r')}(f_1 \oplus f_2) = (\langle k, f_1\rangle \oplus \langle k', f_2\rangle)(r,r') \quad \text{for } f_1, f_2 \in \mathcal{C}_S;$$

$$(3.3) \qquad \lambda^{(r,r')}(f) \le \sup f(\Delta) \quad \text{for } f \in \mathcal{C}.$$

It is easily observed that $\Lambda(r,r')$ is closed and that it is nonempty via the Nachbin-Strassen theorem. Let

$$(3.4) \qquad H_f(r,r') = \inf_{f_1, f_2 \in \mathcal{C}_S} [\sup(f_1 \oplus f_2 + f)(\Delta) - (\langle k, f_1\rangle \oplus \langle k', f_2\rangle)(r,r')]$$

for each $(r,r') \in \Delta$ and $f \in \mathcal{C}$. Then we have

Proposition 3.1. $\sup_{\lambda \in \Lambda(r,r')} \lambda(f) = H_f(r,r')$.

Proof. Let $(r,r') \in \Delta$ and $f \in \mathcal{C}$ be fixed. By replacing f with $f_1 \oplus f_2 + f$ in (3.3), we can immediately observe that

$$(3.5) \qquad \lambda(f) \le H_f(r,r')$$

for every $\lambda \in \Lambda(r,r')$. By (3.2) the equality attains in (3.5) if $f \in \mathcal{C}_S \oplus \mathcal{C}_S$; thus, it is assumed that $f \notin \mathcal{C}_S \oplus \mathcal{C}_S$. Let $\ell(f_1 \oplus f_2) = (\langle k, f_1\rangle \oplus \langle k', f_2\rangle)(r,r')$ for

$f_1, f_2 \in \mathcal{C}_S$. Then ℓ is a well-defined linear functional on $\mathcal{C}_S \oplus \mathcal{C}_S$. By applying (3.1) and (3.2) together, we can observe that

$$-\sup(-f-g)(\Delta) - \ell(g) \leq \sup(f+g')(\Delta) - \ell(g')$$

for any $g, g' \in \mathcal{C}_S \oplus \mathcal{C}_S$, and therefore, that

$$\kappa_f = \inf_{g \in \mathcal{C}_S \oplus \mathcal{C}_S} (\sup(f+g)(\Delta) - \ell(g))$$

has a finite value. We can extend the subspace $\tilde{E} = \{g + tf : g \in \mathcal{C}_S \oplus \mathcal{C}_S, t \in \mathbb{R}\}$ by adding the element f, and define

$$\tilde{\ell}(g + tf) := \ell(g) + t\kappa_f$$

for $g \in \mathcal{C}_S \oplus \mathcal{C}_S$ and $t \in \mathbb{R}$. The map $\tilde{\ell}$ is a well-defined linear functional on $\tilde{E}$ and satisfies (3.2) and (3.3) with $\tilde{\ell}$ and $\tilde{E}$ in place of $\lambda^{(r,r')}$ and $\mathcal{C}$. The same argument is essentially recycled to show that $\tilde{\ell}$ on $\tilde{E}$ is extended to $\lambda^{(r,r')} \in \Lambda(r,r')$ via Zorn's lemma. This particular $\lambda^{(r,r')}$ will achieve the equality in (3.5). □

Observe that the space $\mathcal{C}_S$ in the infimum of (3.4) can be replaced by a countable dense set, and consequently $H_f(r,r')$ is a measurable function of (r,r') for each $f \in \mathcal{C}$. Therefore, Λ is a measurable set-valued map from Δ to $\mathcal{M}^+$, and there exists a measurable selection $\lambda^{(r,r')} \in \Lambda(r,r')$; thus, $K((r,r'),\cdot) = \lambda^{(r,r')}(\cdot)$ becomes a desired bivariate kernel on Δ satisfying (1.2).

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