A new characterization of $\ell_p$ by an $L_p$-function

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Abstract

In this talk, we shall show that the classical sequence space $\ell_p(1 < p < +\infty)$ is completely determined by one function $f(x)(\neq 0) \in L_p(\mathbb{R})$ which satisfies the $p$-integrability condition.

We introduce a new sequence space $\Lambda_p(f)$ defined by an $L_p$-function $f(\neq 0)$ for $1 \leq p < +\infty$ by

$$\Lambda_p(f) := \{a \in \mathbb{R}^\infty : \Psi_p(a : f) < +\infty\},$$

where

$$\Psi_p(a : f) := \left( \sum_n \int_{-\infty}^{+\infty} |f(x - a_n) - f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

We shall give a characterization for $\Lambda_p(f) = \ell_p$. We shall also discuss the linear and topological properties of $\Lambda_p(f)$.

1 Introduction

Let $f(\neq 0)$ be an $L_p$-function on the real line $\mathbb{R}$. For $1 \leq p < +\infty$ and for a real sequence $a = \{a_n\} \in \mathbb{R}^\infty$, we set

$$\Psi_p(a : f) := \left( \sum_n \int_{-\infty}^{+\infty} |f(x - a_n) - f(x)|^p \, dx \right)^{\frac{1}{p}},$$

and define $\Lambda_p(f)$ by

$$\Lambda_p(f) := \{a \in \mathbb{R}^\infty : \Psi_p(a : f) < +\infty\}.$$
By the triangular inequality of $L_p$-norm and by the translation invariance of the Lebesgue measure, we have

$$\Psi_p(a - b : f) \leq \Psi_p(a : f) + \Psi_p(b : f),$$

which implies that $\Lambda_p(f)$ is an additive subgroup of $\mathbb{R}^\infty$.

Define a metric on $\Lambda_p(f)$ by

$$d_p(a, b) := \Psi_p(a - b : f).$$

Then $(\Lambda_p(f), d_p(a, b))$ becomes a topological group. The space $\mathbb{R}_0^\infty$, the direct sum, is a dense subset of $(\Lambda_p(f), d_p(a, b))$.

2 Relations between $\Lambda_p(f)$ and $\ell_p$

We say $I_p(f) < +\infty$ if $f(x)$ is absolutely continuous on $\mathbb{R}$ and the $p$-integral defined by

$$I_p(f) := \int_{-\infty}^{+\infty} |f'(x)|^p \, dx$$

is finite. In particular $I_2(\sqrt{f})$, for probability density function $f(x)$, coincides with the Shepp's integral (Shepp[3]).

**Theorem 1** ([2]) Let $1 \leq p < +\infty$ and let $f(\neq 0)$ be an $L_p$-function on $\mathbb{R}$. Then $\Lambda_p(f) \subset \ell_p$

**Proof.** Assume that $\Psi_p(a; f) < +\infty$ for $a = \{a_k\} \in \mathbb{R}^\infty$. Without loss of generality, we may assume $a_k \neq 0$ for every $k$.

First we shall prove $\{a_k\}$ is bounded. If there is a subsequence $\{a_k'\}$ such that $|a_k'| \to +\infty$, then $\Psi_p(a; f) < +\infty$ implies

$$0 = \lim_{k'} \left( \int_{-\infty}^{+\infty} \left| f(x - a_k') - f(x) \right|^p \, dx \right)^{1/p} = 2^\frac{1}{p} \|f\|_{L_p},$$

which contradicts to $\|f\|_{L_p} > 0$.

Next we shall prove that $\{a_k\}$ converges to 0. Assume that there is a subsequence $a_{k'}$ such that $a_{k'} \to a_0 \neq 0$. Then we have

$$0 = \lim_{k'} \int_{-\infty}^{+\infty} \left| f(x - a_{k'}) - f(x) \right|^p \, dx = \int_{-\infty}^{+\infty} \left| f(x - a_0) - f(x) \right|^p \, dx,$$

which implies $f(x - a_0) = f(x)$, a.e. $(dx)$. This contradicts to the integrability of $f(x)$. 

Finally, we shall prove

\[ \rho := \inf_k \int_{-\infty}^{+\infty} \left| f(x-a_k) - f(x) \right|^p \, dx > 0. \]

Assume that there exists a subsequence \( a_{k'} \) such that

\[ \int_{-\infty}^{+\infty} \left| f(x-a_{k'}) - f(x) \right|^p \, dx \to 0 \]

Then it follows that

\[ f(x-a_{k'}) - f(x) \to 0 \text{ in } L_p(\mathbb{R}). \]

Consequently, \( f(x) \) is absolutely continuous with \( f'(x) = 0, \text{a.e.}(dx) \), that implies \( f = 0 \), which is a contradiction.

Therefore we have

\[ +\infty > \sum_{k} \int_{-\infty}^{+\infty} \left| f(x-a_k) - f(x) \right|^p \, dx \, |a_k|^p \geq \rho \sum_{k} |a_k|^p, \]

which proves the theorem.

**Theorem 2** ([2]) Let \( 1 < p < +\infty \) and \( f(\neq 0) \) be a non-negative integrable function on \( \mathbb{R} \). Then \( \Lambda_p(f) = \ell_p \) if and only if \( I_p(f) < +\infty \).

**Proof.** Assume \( \Psi_p(a; f) < +\infty \) for every \( a = \{a_k\} \in \ell_p \). We set

\[ \psi(a) := \int_{-\infty}^{+\infty} \left| f(x-a) - f(x) \right|^p \, dx, \]

\[ u_n := 2^{-n}, \]

and

\[ F_n(x) := f(x-u_n) - f(x). \]

Then we shall show

\[ K := \sup_N 2^N \psi(u_N) = \sup_N \int_{-\infty}^{+\infty} \left| F_N(x) \right|^p \, dx < +\infty. \]

Assume, on the contrary, that for every \( n \) there exists \( N(n) > n \) satisfying

\[ 2^{N(n)} \psi(u_{N(n)}) > 2^n. \]
Then for the sequence

\[ a_0 := (\hat{u}_{N(1)}, \cdots, \hat{u}_{N(1)}, \cdots, \hat{u}_{N(n)}, \cdots, \hat{u}_{N(n)}, \cdots), \]

we have \( a_0 \in \ell_p \) and \( \Psi_p(a_0; f) = +\infty \), which is a contradiction.

Since \( L_p(\mathbb{R}, dx), 1 < p < +\infty \), is a separable reflexible Banach space, each bounded closed ball is compact and metrizable with respect to the weak topology. So that there exists a subsequence \( \{F_{n_j}(x)\} \) and \( h(x) \in L_p(\mathbb{R}, dx) \) such that \( \{F_{n_j}(x)\} \) converges weakly to \( h(x) \).

Consequently, \( f(x) \) is absolutely continuous, \( f'(x) = -h(x), \ a.e.(dx) \), and we have

\[ I_p(f) = \int_{-\infty}^{+\infty} |f'(x)|^p dx = \int_{-\infty}^{+\infty} |h(x)|^p dx < +\infty. \]

Conversely, assume \( I_p(f) < +\infty \). Then by the mean value theorem and Fubini’s theorem, we have

\[ \int_{-\infty}^{+\infty} |f(x-a_k) - f(x)|^p dx = \left| a_k \right|^p \left[ \int_{-\infty}^{+\infty} \left| \int_0^1 f'(x - ta_k) dt \right|^p dx \right] \leq |a_k|^p \int_{-\infty}^{+\infty} dx \int_0^1 \left| f'(x - ta_k) \right|^p dt = |a_k|^p \int_{-\infty}^{+\infty} \left| f'(x) \right|^p dx = I_p(f) |a_k|^p, \]

which implies

\[ \sum_k \int_{-\infty}^{+\infty} |f(x-a_k) - f(x)|^p dx \leq I_p(f) \sum_{k=1}^{+\infty} |a_k|^p < +\infty. \]

3 Linearity of \( \Lambda_p(f) \)

We say \( f(x) \) is an \( N \)-modal function if there exist \( a_n, n = 1, 2, \cdots, 2N + 1 \) such that

\[ -\infty = a_1 < a_2 < \cdots < a_{2N} < a_{2N+1} = +\infty, \]

\( f(x) \) is non-decreasing on the interval \([a_{2k-1}, a_{2k}]\), and

\( f(x) \) is non-increasing on the interval \([a_{2k}, a_{2k+1}]\).

Let \( \alpha = \alpha(f) := \frac{1}{2} \min\{a_{k+1} - a_k\} \) if \( N \geq 2 \) and \( \alpha := +\infty \) if \( N = 1 \).
**Lemma 3** Let $f(x) : [-2a, 2a] \to [0, +\infty)$ be a function such that $f(x)$ is non-decreasing on $[-2a, 0]$ and is non-increasing on $[0, 2a]$, where $a \geq 0$. Then for every $t \in [0, 1]$, we have

$$\int_0^a |f(x-ta) - f(x)|^p \, dx \leq \int_a^{2a} |f(x-a) - f(x)|^p \, dx + 3 \int_0^a |f(x-a) - f(x)|^p \, dx.$$  

**Proof.** Let $u$ be the $x$-coordinate of the cross point of $f(x)$ and of $f(x-ta)$ and $v$ be the $x$-coordinate of the cross point of $f(x-ta)$ and of $f(x-a)$. Then we have $0 \leq u \leq ta \leq v \leq a$. We have

$$\int_0^{ta} |f(x-ta) - f(x)|^p \, dx = \left( \int_0^u + \int_u^{ta} \right)$$

$$\leq \int_0^u (f(x) - f(x-a))^p \, dx + \int_u^{ta} (f(x-ta) - f(x+a-ta))^p \, dx$$

$$\leq \int_0^{ta} |f(x-a) - f(x)|^p \, dx + \int_0^u |f(x) - f(x+a)|^p \, dx$$

$$= \int_0^{ta} |f(x-a) - f(x)|^p \, dx + \int_{u-ta}^a |f(x-a) - f(x)|^p \, dx$$

$$\leq 2 \int_0^a |f(x-a) - f(x)|^p \, dx,$$

where we have used the facts

$$f(x-a) \leq f(x-ta) \leq f(x) \text{ on } [0, u] \text{ and }$$

$$f(x-a) \leq f(x) \leq f(x-ta) \text{ on } [u, ta].$$

On the other hand we have

$$\int_{ta}^a |f(x-ta) - f(x)|^p \, dx = \left( \int_{ta}^v + \int_v^a \right)$$

$$\leq \int_{ta}^v (f(x-ta) - f(x+a-ta))^p \, dx + \int_v^a (f(x-a) - f(x))^p \, dx$$

$$= \int_{ta}^{u-ta} (f(x) - f(x+a))^p \, dx + \int_{u-ta}^a (f(x-a) - f(x))^p \, dx$$

$$= \int_{ta}^{u-ta} (f(x-a) - f(x))^p \, dx + \int_{v}^a (f(x-a) - f(x))^p \, dx$$

$$\leq \int_a^{2a} |f(x-a) - f(x)|^p \, dx + \int_0^a |f(x-a) - f(x)|^p \, dx,$$

where we have used the facts

$$f(x+a-ta) \leq f(x) \leq f(x-ta) \text{ on } [ta, v], \text{ and }$$

$$f(x) \leq f(x-ta) \leq f(x-a) \text{ on } [v, a].$$

Consequently we have the inequality of Lemma 3.
Lemma 4 Let $f(x) : [-2a, 2a] \rightarrow [0, +\infty)$ be a function such that $f(x)$ is non-increasing on $[-2a, 0]$ and is non-decreasing on $[0, 2a]$, where $a \geq 0$. Then for every $t \in [0, 1]$, we have

$$\int_{0}^{a} |f(x-ta)-f(x)|^{p} dx \leq \int_{-a}^{0} |f(x-a)-f(x)|^{p} dx + 3 \int_{0}^{a} |f(x-a)-f(x)|^{p} dx.$$

Proof. Let $u$ be the $x$-coordinate of the cross point of $f(x)$ and of $f(x-ta)$ and $v$ be the $x$-coordinate of the cross point of $f(x-ta)$ and of $f(x-a)$. Then we have $0 \leq u \leq ta \leq v \leq a$. We have

$$\int_{0}^{ta} |f(x-ta) - f(x)|^{p} dx = \left( \int_{0}^{u} + \int_{u}^{ta} \right)$$

$$\leq \int_{0}^{u} (f(x-a) - f(x))^{p} dx + \int_{u}^{ta} (f(x-a-ta) - f(x-ta))^{p} dx$$

$$\leq \int_{0}^{ta} |f(x-a) - f(x)|^{p} dx + \int_{v-ta}^{0} |f(x-a) - f(x)|^{p} dx$$

$$\leq \int_{0}^{a} |f(x-a) - f(x)|^{p} dx + \int_{-a}^{0} |f(x-a) - f(x)|^{p} dx,$$

where we have used the facts

$f(x) \leq f(x-ta) \leq f(x-a)$ on $[0, u]$ and

$f(x-ta) \leq f(x) \leq f(x-a-ta)$ on $[u, ta]$.

On the other hand we have

$$\int_{ta}^{a} |f(x-ta) - f(x)|^{p} dx = \left( \int_{ta}^{u} + \int_{u}^{a} \right)$$

$$\leq \int_{ta}^{u} (f(x-a-ta) - f(x-ta))^{p} dx + \int_{u}^{a} (f(x) - f(x-a))^{p} dx$$

$$= \int_{0}^{u-ta} (f(x-a) - f(x))^{p} dx + \int_{u}^{a} (f(x) - f(x-a))^{p} dx$$

$$\leq 2 \int_{0}^{a} |f(x-a) - f(x)|^{p} dx,$$

where we have used the facts

$f(x-ta) \leq f(x) \leq f(x-a-ta)$ on $[ta, v]$, and

$f(x-a) \leq f(x-ta) \leq f(x)$ on $[v, a]$.

Consequently we have the inequality of Lemma 4.

Theorem 5 Let $f(x)$ be a non-negative integrable N-modal function. Then for every $0 \leq a \leq \alpha$ and every $0 \leq t \leq 1$, we have
\[
\int_{-\infty}^{+\infty} |f(x - ta) - f(x)|^p dx \leq 5 \int_{-\infty}^{+\infty} |f(x - a) - f(x)|^p dx.
\]

Proof. On the subset

\[ S := [a_1, a_2] \cup [a_2 + a, a_3] \cup [a_3 + a, a_4] \cup \cdots \cup [a_{2N} + a, a_{2N+1}] \]

we have

\[ f(x - a) \leq f(x - ta) \leq f(x) \text{ for } x \in [a_1, a_2] \text{ or } x \in [a_{2k-1}, a_{2k}], \]

and

\[ f(x) \leq f(x - ta) \leq f(x - a) \text{ for } x \in [a_{2k}, a_{2k+1}] \text{ or } x \in [a_{2N} + a, a_{2N+1}], \]

which implies

\[
\int_S |f(x - ta) - f(x)|^p dx \leq \int_S |f(x - a) - f(x)|^p dx.
\]

By applying Lemma 1 and Lemma 2 for the function \( g(x) = f(x + a_k) \), we have

\[
\int_{a_{2k}}^{a_{2k}+a} |f(x - ta) - f(x)|^p dx
\]

\[
\leq \int_{a_{2k}+a}^{a_{2k}+2a} |f(x - a) - f(x)|^p dx + 3 \int_{a_{2k}}^{a_{2k}+a} |f(x - a) - f(x)|^p dx, \text{ and}
\]

\[
\int_{a_{2k+1}}^{a_{2k+1}+a} |f(x - ta) - f(x)|^p dx
\]

\[
\leq \int_{a_{2k+1}-a}^{a_{2k+1}} |f(x - a) - f(x)|^p dx + 3 \int_{a_{2k+1}}^{a_{2k+1}+a} |f(x - a) - f(x)|^p dx.
\]

Consequently we have the inequality.

**Theorem 6** Let \( f(x) \) be a non-negative integrable N-modal function. Then \( \Lambda_p(f) \) is a linear space.

Proof. Let \( \{a_n\} \in \Lambda_p(f) \). We shall show that \( t\{a_n\} \in \Lambda_p(f) \) for every \( 0 \leq t \leq 1 \). Without loss of generality, we may assume \( a_n \geq 0 \). Since \( \Lambda_p(f) \subset \ell_p \), there exists \( K \) such that \( a_n \leq \alpha \) for every \( n \geq K \). The finite sequence \( t(a_1, \cdots, a_{K-1}, 0, 0, \cdots) \) belongs to \( \Lambda_p(f) \) and the sequence \( t(0, 0, \cdots, 0, a_K, a_{K+1}, \cdots) \) belongs to \( \Lambda_p(f) \) by Theorem 1, so that \( t\{a_n\} \in \Lambda_p(f) \).
4 Completeness of $\Lambda_p(f)$

Theorem 7 ([1]) Let $f(\neq 0)$ be an $L_p$-function. Then $\Lambda_p(f)$ is complete with respect to $d_p$ for $1 \leq p < +\infty$.

Proof. Let $a^{(k)} \in \Lambda_p(f), k = 1, 2, \ldots$, be a Cauchy sequence in $d_p$. Then for every $\epsilon > 0$, there exists $N$ such that

\[ \sum_n \int_{-\infty}^{+\infty} |f(x - a^{(k)}_n + a^{(l)}_n) - f(x)|^p dx \leq \epsilon^p. \]

for $k, l \geq N$. For any fixed $n$, we have

\[ \int_{-\infty}^{+\infty} |f(x - a^{(k)}_n + a^{(l)}_n) - f(x)|^p dx \rightarrow 0, \text{ as } k, l \rightarrow +\infty. \]

Then it follows that $a^{(k)}_n - a^{(l)}_n \rightarrow 0$ as $k, l \rightarrow +\infty$, that is, $\{a^{(k)}_n\}$ is a Cauchy sequence (see the proof of Theorem 2.)

Let $a^{(0)}_n := \lim_k a^{(k)}_n$. Then we shall show $a^{(k)} \rightarrow a^{(0)} := \{a^{(0)}_n | n = 1, 2, \ldots\}$ in $d_p$. In the inequality $(\ast)$, taking $\lim \inf_{l \rightarrow +\infty}$, by the Fatou's Lemma, we have

\[ \epsilon^p \geq \sum_n \lim \inf_{l \rightarrow +\infty} \int_{-\infty}^{+\infty} |f(x - a^{(k)}_n) - f(x - a^{(l)}_n)|^p dx \]

\[ = \sum_n \int_{-\infty}^{+\infty} |f(x - a^{(k)}_n) - f(x - a^{(0)}_n)|^p dx = d_p(a^{(k)}, a^{(0)})^p, \]

which shows $a^{(k)} \rightarrow a^{(0)}$ with respect to $d_p$.

References

