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| Title | A new characterization of $ \ell_p$ by an $L_p$-function (Information and mathematics of non-additivity and non-extensivity : contacts with nonlinearity and non-commutativity) |
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| Citation | 数理解析研究所講究録 (2008), 1585: 30-37 |
| Issue Date | 2008-02 |
| URL | http://hdl.handle.net/2433/81521 |
| Right | |
| Type | Departmental Bulletin Paper |
| Textversion | publisher |

数理解析研究所講究録
第 1585 巻 2008 年 30-37



# A new characterization of $\ell_p$ by an $L_p$-function

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**Abstract**

In this talk, we shall show that the classical sequence space $\ell_p(1 < p < +\infty)$ is completely determined by one function $f(x)(\neq 0) \in L_p(\mathbb{R})$ which satisfies the $p$-integrability condition.

We introduce a new sequence space $\Lambda_p(f)$ defined by an $L_p$-function $f(\neq 0)$ for $1 \leq p < +\infty$ by

$$\Lambda_p(f) := \{\boldsymbol{a} \in \mathbf{R}^\infty : \Psi_p(\boldsymbol{a} : f) < +\infty\},$$

where

$$\Psi_p(\boldsymbol{a} : f) := \Big(\sum_n \int_{-\infty}^{+\infty} |f(x - a_n) - f(x)|^p \, dx\Big)^{\frac{1}{p}}.$$

We shall give a characterization for $\Lambda_p(f) = \ell_p$. We shall also discuss the linear and topological properties of $\Lambda_p(f)$.

# 1 Introduction

Let $f(\neq 0)$ be an $L_p$-function on the real line $\mathbf{R}$.
For $1 \leq p < +\infty$ and for a real sequence $\boldsymbol{a} = \{a_n\} \in \mathbf{R}^\infty$, we set

$$\Psi_p(\boldsymbol{a} : f) := \Big(\sum_n \int_{-\infty}^{+\infty} |f(x - a_n) - f(x)|^p \, dx\Big)^{\frac{1}{p}},$$

and define $\Lambda_p(f)$ by

$$\Lambda_p(f) := \{\boldsymbol{a} \in \mathbf{R}^\infty : \Psi_p(\boldsymbol{a} : f) < +\infty\}.$$

By the triangular inequality of $L_p$-norm and by the translation invariance of the Lebesgue measure, we have

$$\Psi_p(a-b:f) \le \Psi_p(a:f) + \Psi_p(b:f),$$

which implies that $\Lambda_p(f)$ is an additive subgroup of $\mathbf{R}^\infty$.

Define a metric on $\Lambda_p(f)$ by

$$d_p(a,b) := \Psi_p(a-b:f).$$

Then $(\Lambda_p(f), d_p(a,b))$ becomes a topological group. The space $\mathbb{R}_0^\infty$, the direct sum, is a dense subset of $(\Lambda_p(f), d_p(a,b))$.

# 2 Relations between $\Lambda_p(f)$ and $\ell_p$

We say $I_p(f) < +\infty$ if $f(x)$ is absolutely continuous on $\mathbb{R}$ and the *p-integral* defined by

$$I_p(f) := \int_{-\infty}^{+\infty} |f'(x)|^p \, dx$$

is finite. In particular $I_2(\sqrt{f})$, for probability density function $f(x)$, coincides with the Shepp's integral(Shepp[3]).

**Theorem 1** ([2]) Let $1 \le p < +\infty$ and let $f(\neq 0)$ be an $L_p$-function on $\mathbb{R}$. Then $\Lambda_p(f) \subset \ell_p$

*Proof.* Assume that $\Psi_p(a;f) < +\infty$ for $a = \{a_k\} \in \mathbb{R}^\infty$. Without loss of generality, we may assume $a_k \neq 0$ for every $k$.

First we shall prove $\{a_k\}$ is bounded. If there is a subsequence $\{a_{k'}\}$ such that $|a_{k'}| \to +\infty$, then $\Psi_p(a;f) < +\infty$ implies

$$0 = \lim_{k'} \left( \int_{-\infty}^{+\infty} \left| f(x-a_{k'}) - f(x) \right|^p dx \right)^{1/p} = 2^{\frac{1}{p}} ||f||_{L_p}$$

which contradicts to $||f||_{L_p} > 0$.

Next we shall prove that $\{a_k\}$ converges to 0. Assume that there is a subsequence $a_{k'}$ such that $a_{k'} \to a_0 \neq 0$. Then we have

$$0 = \lim_{k'} \int_{-\infty}^{+\infty} \left| f(x-a_{k'}) - f(x) \right|^p dx = \int_{-\infty}^{+\infty} \left| f(x-a_0) - f(x) \right|^p dx,$$

which implies $f(x-a_0) = f(x)$, *a.e.*$(dx)$. This contradicts to the integrability of $f(x)$.

Finally, we shall prove

$$\rho := \inf_k \int_{-\infty}^{+\infty} \left| \frac{f(x-a_k) - f(x)}{a_k} \right|^p dx > 0.$$

Assume that there exists a subsequence $a_{k'}$ such that

$$\int_{-\infty}^{+\infty} \left| \frac{f(x-a_{k'}) - f(x)}{a_{k'}} \right|^p dx \to 0$$

Then it follows that

$$\frac{f(x-a_{k'}) - f(x)}{a_{k'}} \to 0 \ in \ L_p(\mathbb{R}).$$

Consequently, $f(x)$ is absolutely continuous with $f'(x) = 0, a.e.(dx)$, that implies $f = 0$, which is a contradiction.

Therefore we have

$$+\infty > \sum_k \int_{-\infty}^{+\infty} \left| \frac{f(x-a_k) - f(x)}{a_k} \right|^p dx \ |a_k|^p \geq \rho \sum_k |a_k|^p,$$

which proves the theorem.

**Theorem 2** ([2]) Let $1 < p < +\infty$ and $f(\neq 0)$ be a non-negative integrable function on $\mathbb{R}$. Then $\Lambda_p(f) = \ell_p$ if and only if $I_p(f) < +\infty$.

*Proof.* Assume $\Psi_p(\boldsymbol{a}; f) < +\infty$ for every $\boldsymbol{a} = \{a_k\} \in \ell_p$. We set

$$\psi(a) := \int_{-\infty}^{+\infty} \left| f(x-a) - f(x) \right|^p dx,$$

$$u_n := 2^{-\frac{n}{p}},$$

and

$$F_n(x) := \frac{f(x-u_n) - f(x)}{u_n}.$$

Then we shall show

$$K := \sup_N 2^N \psi(u_N) = \sup_N \int_{-\infty}^{+\infty} \left| F_N(x) \right|^p dx < +\infty.$$

Assume, on the contrary, that for every $n$ there exists $N(n) > n$ satisfying

$$2^{N(n)} \psi(u_{N(n)}) > 2^n.$$

Then for the sequence

$$a_0 := (\overbrace{u_{N(1)}, \cdots, u_{N(1)}}^{2^{N(1)-1}}, \cdots, \overbrace{u_{N(n)}, \cdots, u_{N(n)}}^{2^{N(n)-n}}, \cdots),$$

we have $a_0 \in \ell_p$ and $\Psi_p(a_0; f) = +\infty$, which is a contradiction.

Since $L_p(\mathbb{R}, dx), 1 < p < +\infty$, is a separable reflexible Banach space, each bounded closed ball is compact and metrizable with respect to the weak topology. So that there exists a subsequence $\{F_{n_j}(x)\}$ and $h(x) \in L_p(\mathbb{R}, dx)$ such that $\{F_{n_j}(x)\}$ converges weakly to $h(x)$.

Consequently, $f(x)$ is absolutely continuous, $f'(x) = -h(x)$, $a.e.(dx)$, and we have

$$I_p(f) = \int_{-\infty}^{+\infty} \left|f'(x)\right|^p dx = \int_{-\infty}^{+\infty} |h(x)|^p dx < +\infty.$$

Conversely, assume $I_p(f) < +\infty$. Then by the mean value theorem and Fubini's theorem, we have

$$\int_{-\infty}^{+\infty} \left|f(x - a_k) - f(x)\right|^p dx = |a_k|^p \int_{-\infty}^{+\infty} \left|\int_0^1 f'(x - ta_k)dt\right|^p dx$$
$$\leq |a_k|^p \int_{-\infty}^{+\infty} dx \int_0^1 \left|f'(x - ta_k)\right|^p dt = |a_k|^p \int_{-\infty}^{+\infty} \left|f'(x)\right|^p dx = I_p(f)|a_k|^p,$$

which implies

$$\sum_k \int_{-\infty}^{+\infty} \left|f(x - a_k) - f(x)\right|^p dx \leq I_p(f) \sum_{k=1}^{+\infty} |a_k|^p < +\infty.$$

# 3 Linearity of $\Lambda_p(f)$

We say $f(x)$ is an N-modal function if there exist $a_n, n = 1, 2, \cdots, 2N + 1$ such that

$$-\infty = a_1 < a_2 < \cdots < a_{2N} < a_{2N+1} = +\infty,$$

$f(x)$ is non-decreasing on the interval $[a_{2k-1}, a_{2k}]$, and

$f(x)$ is non-increasing on the interval $[a_{2k}, a_{2k+1}]$.

Let $\alpha = \alpha(f) := \frac{1}{2} \min\{a_{k+1} - a_k\}$ if $N \geq 2$ and $\alpha := +\infty$ if $N = 1$.

**Lemma 3** Let $f(x) : [-2a, 2a] \to [0, +\infty)$ be a function such that $f(x)$ is non-decreasing on $[-2a, 0]$ and is non-increasing on $[0, 2a]$, where $a \geq 0$. Then for every $t \in [0, 1]$, we have

$$\int_0^a |f(x-ta)-f(x)|^p dx \leq \int_a^{2a} |f(x-a)-f(x)|^p dx + 3\int_0^a |f(x-a)-f(x)|^p dx.$$

Proof. Let $u$ be the $x$-coordinate of the cross point of $f(x)$ and of $f(x-ta)$ and $v$ be the $x$-coordinate of the cross point of $f(x-ta)$ and of $f(x-a)$. Then we have $0 \leq u \leq ta \leq v \leq a$. We have

$$\begin{aligned}
&\textstyle\int_0^{ta} |f(x-ta) - f(x)|^p dx = \left(\int_0^u + \int_u^{ta}\right)\\
&\leq \textstyle\int_0^u (f(x) - f(x-a))^p dx + \int_u^{ta} (f(x-ta) - f(x+a-ta))^p dx\\
&\leq \textstyle\int_0^{ta} |f(x-a) - f(x)|^p dx + \int_{u-ta}^{0} |f(x) - f(x+a)|^p dx\\
&= \textstyle\int_0^{ta} |f(x-a) - f(x)|^p dx + \int_{u-ta+a}^{a} |f(x-a) - f(x)|^p dx\\
&\leq 2\textstyle\int_0^a |f(x-a) - f(x)|^p dx,
\end{aligned}$$

where we have used the facts

$$f(x-a) \leq f(x-ta) \leq f(x) \text{ on } [0, u] \text{ and}$$

$$f(x+a-ta) \leq f(x) \leq f(x-ta) \text{ on } [u, ta].$$

On the other hand we have

$$\begin{aligned}
&\textstyle\int_{ta}^{a} |f(x-ta) - f(x)|^p dx = \left(\int_{ta}^{v} + \int_v^a\right)\\
&\leq \textstyle\int_{ta}^{v} (f(x-ta) - f(x+a-ta))^p dx + \int_v^a (f(x-a) - f(x))^p dx\\
&= \textstyle\int_0^{v-ta} (f(x) - f(x+a))^p dx + \int_v^a (f(x-a) - f(x))^p dx\\
&= \textstyle\int_a^{v-ta+a} (f(x-a) - f(x))^p dx + \int_v^a (f(x-a) - f(x))^p dx\\
&\leq \textstyle\int_a^{2a} |f(x-a) - f(x)|^p dx + \int_0^a |f(x-a) - f(x)|^p dx,
\end{aligned}$$

where we have used the facts

$$f(x+a-ta) \leq f(x) \leq f(x-ta) \text{ on } [ta, v], \text{ and}$$

$$f(x) \leq f(x-ta) \leq f(x-a) \text{ on } [v, a].$$

Consequently we have the inequality of Lemma 3.

**Lemma 4** Let $f(x) : [-2a, 2a] \to [0, +\infty)$ be a function such that $f(x)$ is non-increasing on $[-2a, 0]$ and is non-decreasing on $[0, 2a]$, where $a \geq 0$. Then for every $t \in [0, 1]$, we have

$$\int_0^a |f(x-ta)-f(x)|^p dx \leq \int_{-a}^0 |f(x-a)-f(x)|^p dx + 3\int_0^a |f(x-a)-f(x)|^p dx.$$

Proof. Let $u$ be the $x$-coordinate of the cross point of $f(x)$ and of $f(x-ta)$ and $v$ be the $x$-coordinate of the cross point of $f(x-ta)$ and of $f(x-a)$. Then we have $0 \leq u \leq ta \leq v \leq a$. We have

$$\begin{aligned}
&\textstyle\int_0^{ta} |f(x-ta) - f(x)|^p dx = \left(\int_0^u + \int_u^{ta}\right) \\
&\textstyle\leq \int_0^u (f(x-a) - f(x))^p dx + \int_u^{ta} (f(x-a-ta) - f(x-ta))^p dx \\
&\textstyle\leq \int_0^{ta} |f(x-a) - f(x)|^p dx + \int_{u-ta}^0 |f(x-a) - f(x)|^p dx \\
&\textstyle\leq \int_0^a |f(x-a) - f(x)|^p dx + \int_{-a}^0 |f(x-a) - f(x)|^p dx,
\end{aligned}$$

where we have used the facts

$$f(x) \leq f(x-ta) \leq f(x-a) \text{ on } [0, u] \text{ and}$$

$$f(x-ta) \leq f(x) \leq f(x-a-ta) \text{ on } [u, ta].$$

On the other hand we have

$$\begin{aligned}
&\textstyle\int_{ta}^a |f(x-ta) - f(x)|^p dx = \left(\int_{ta}^v + \int_v^a\right) \\
&\textstyle\leq \int_{ta}^v (f(x-a-ta) - f(x-ta))^p dx + \int_v^a (f(x) - f(x-a))^p dx \\
&\textstyle= \int_0^{v-ta} (f(x-a) - f(x))^p dx + \int_v^a (f(x) - f(x-a))^p dx \\
&\textstyle\leq 2\int_0^a |f(x-a) - f(x)|^p dx,
\end{aligned}$$

where we have used the facts

$$f(x-ta) \leq f(x) \leq f(x-a-ta) \text{ on } [ta, v], \text{ and}$$

$$f(x-a) \leq f(x-ta) \leq f(x) \text{ on } [v, a].$$

Consequently we have the inequality of Lemma 4.

**Theorem 5** Let $f(x)$ be a non-negative integrable N-modal function. Then for every $0 \leq a \leq \alpha$ and every $0 \leq t \leq 1$, we have

$$\int_{-\infty}^{+\infty} |f(x-ta) - f(x)|^p dx \leq 5 \int_{-\infty}^{+\infty} |f(x-a) - f(x)|^p dx.$$

Proof. On the subset

$$S := [a_1, a_2] \cup [a_2 + a, a_3] \cup [a_3 + a, a_4] \cup \cdots \cup [a_{2N} + a, a_{2N+1}]$$

we have

$$f(x-a) \leq f(x-ta) \leq f(x) \quad for\ x \in [a_1, a_2] \quad or\ x \in [a_{2k-1}, a_{2k}], \text{and}$$

$$f(x) \leq f(x-ta) \leq f(x-a) \quad for\ x \in [a_{2k}, a_{2k+1}] \quad or\ x \in [a_{2N} + a, a_{2N+1}],$$

which implies

$$\int_S |f(x-ta) - f(x)|^p dx \leq \int_S |f(x-a) - f(x)|^p dx.$$

By applying Lemma1 and Lemma2 for the function $g(x) = f(x + a_k)$, we have

$\int_{a_{2k}}^{a_{2k}+a} |f(x-ta) - f(x)|^p dx$

$\leq \int_{a_{2k}+a}^{a_{2k}+2a} |f(x-a) - f(x)|^p dx + 3 \int_{a_{2k}}^{a_{2k}+a} |f(x-a) - f(x)|^p dx$, and

$\int_{a_{2k+1}}^{a_{2k+1}+a} |f(x-ta) - f(x)|^p dx$

$\leq \int_{a_{2k+1}-a}^{a_{2k+1}} |f(x-a) - f(x)|^p dx + 3 \int_{a_{2k+1}}^{a_{2k+1}+a} |f(x-a) - f(x)|^p dx.$

Consequently we have the inequality.

**Theorem 6** Let $f(x)$ be a non-negative integrable N-modal function. Then $\Lambda_p(f)$ is a linear space.

Proof. Let $\{a_n\} \in \Lambda_p(f)$. We shall show that $t\{a_n\} \in \Lambda_p(f)$ for every $0 \leq t \leq 1$. Without loss of generality, we may assume $a_n \geq 0$. Since $\Lambda_p(f) \subset \ell_p$, there exists $K$ such that $a_n \leq \alpha$ for every $n \geq K$. The finite sequence $t(a_1, \cdots, a_{K-1}, 0, 0, \cdots)$ belongs to $\Lambda_p(f)$ and the sequence $t(0, 0. \cdots, 0, a_K, a_{K+!}, \cdots)$ belongs to $\Lambda_p(f)$ by Theorem 1, so that $t\{a_n\} \in \Lambda_p(f)$.

# 4 Completeness of $\Lambda_p(f)$

**Theorem 7** ([1]) Let $f(\neq 0)$ be an $L_p$-function. Then $\Lambda_p(f)$ is complete with respect to $d_p$ for $1 \leq p < +\infty$.

*Proof.* Let $a^{(k)} \in \Lambda_p(f), k = 1, 2, \ldots$, be a Cauchy sequence in $d_p$. Then for every $\varepsilon > 0$, there exists $N$ such that

$$(*) \qquad \sum_n \int_{-\infty}^{+\infty} \left| f(x - a_n^{(k)} + a_n^{(l)}) - f(x) \right|^p dx \leq \varepsilon^p.$$

for $k, l \geq N$. For any fixed $n$, we have

$$\int_{-\infty}^{+\infty} \left| f(x - a_n^{(k)} + a_n^{(l)}) - f(x) \right|^p dx \to 0, \ \textit{as}\ k, l \to +\infty.$$

Then it follows that $a_n^{(k)} - a_n^{(l)} \to 0$ as $k, l \to +\infty$, that is, $\{a_n^{(k)}\}$ is a Cauchy sequence(see the proof of Theorem 2.)

Let $a_n^{(0)} := \lim_k a_n^{(k)}$. Then we shall show $a^{(k)} \to a^{(0)} := \{a_n^{(0)} \mid n = 1, 2, \ldots\}$ in $d_p$. In the inequality $(*)$, taking $\liminf_{l \to +\infty}$, by the Fatou's Lemma, we have

$$\begin{aligned} \varepsilon^p &\geq \sum_n \liminf_{l \to +\infty} \int_{-\infty}^{+\infty} \left| f(x - a_n^{(k)}) - f(x - a_n^{(l)}) \right|^p dx \\ &= \sum_n \int_{-\infty}^{+\infty} \left| f(x - a_n^{(k)}) - f(x - a_n^{(0)}) \right|^p dx = d_p(a^{(k)}, a^{(0)})^p, \end{aligned}$$

which shows $a^{(k)} \to a^{(0)}$ with respect to $d_p$.

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