On a probability distribution of a binomial type generated by a mean (Information and mathematics of non-additivity and non-extensivity: contacts with nonlinearity and non-commutativity)

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On a probability distribution of a binomial type generated by a mean

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1. Means and paths. In this note, we use operator means, in particular, the Kubo-Ando mean [6] plays a central role: A binary operation $m$ on positive operators on a Hilbert space is called the Kubo-Ando (operator) mean if $m$ satisfies the following axioms:

- monotonicity: $A \leq C,$ $B \leq D \Rightarrow A \circ B \leq C \circ D.$
- semicontinuity: $A_n \downarrow A,$ $B_n \downarrow B \Rightarrow A_n \circ B_n \downarrow A \circ B.
- transformer inequality: $T^*(A \circ B)T \leq T^*AT \circ T^*BT.$
- normalization: $A \circ A = A.$

By semicontinuity, we may assume positive operators are invertible. The representing function $f_m(x) = 1 \circ x$ for a Kubo-Ando mean $m$ is operator monotone (concave) on $(0, \infty)$ and $m$ is represented by

$$A \circ B = A^{\frac{1}{2}} f_m(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$  

A path $A \circ_t B$ means parametrized operator means which is usually differentiable for $t$ with $A \circ_0 B = A$ and $A \circ_1 B = 0.$ A path is called symmetric if

$$A \circ_t B = B \circ_{1-t} A$$

holds for all $t \in [0,1].$ Typical example is (quasi-arithmetic) power means for $r \in [-1,1]:$

$$A^{\#_r,t} B = A^{\frac{1}{2}} (I + t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r)^{\frac{1}{2}} A^{\frac{1}{2}},$$

which include important means:

- arithmetic mean: $A \triangledown_t B = A^{\#_{1,t}} B = (1 - t) A + t B$
- geometric mean: $A^{\#_0,t} B = \lim_{\epsilon \to 0} A^{\#_{\epsilon,t}} B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$
- harmonic mean: $A \div_t B = A^{\#_{-1,t}} B = ((1 - t) A^{-1} + t B^{-1})^{-1}.$
Moreover the above paths are interpolational in the sense that
\[(A\#_{p}B)\#_{t}(A\#_{q}B) = A\#_{r_{t}(1-t)p+sq}B\]
for all \(p, q, t \in [0, 1]\).

2. Thompson metric. Let \(A^+\) be the positive invertible elements in a unital C*-algebra \(A\), which is discussed as differentiable manifold by Corach-Porta-Recht [3, 4]. Corach himself reformulated it in [4]. They showed the above manifold \(A^+\) is the Finsler space with a Finsler metric
\[L(X; A) = \|X\|_A = \|A^{-1/2}XA^{-1/2}\|:\]
Then the geodesic is the shortest path with respect to this metric: The length \(\ell(\gamma)\) of path \(\gamma(t)\) is defined by
\[\ell(\gamma) \equiv \int_{0}^{1} L(\gamma'(t); \gamma(t))dt = \int_{0}^{1} \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|dt.\]
If \(\gamma(t)\) is a path from \(A\) to \(B\), then
\[d(A, B) \equiv \inf_{\gamma} \ell(\gamma) = \ell(A\#_{t}B) = \|\log(A^{-1/2}BA^{-1/2})\|
= \log(\max\{\|A^{-1/2}BA^{-1/2}\|, \|B^{-1/2}AB^{-1/2}\|\})
= \log(\max\{r(A^{-1}B), r(B^{-1}A)\}).\]
Also the homogeneity of \(A^+\) implies
\[d(A, B) = d(X^*AX, X^*BX) = d(I, A^{-1/2}BA^{-1/2})\]
for invertible \(X\). The metric \(d\) makes \(A^+\) a complete metric space and it is called the Thompson (part) one [12, 10].

3. Lawson-Lim’s operator mean. Recently, Lawson-Lim [8, 9, 7] defines multivariable operator means parametrized by \(t \in [0, 1]\) which is an extension of Ando-Li-Mathius’ geometric operator mean [1]: For a symmetric path \(m_t\) in Kubo-Ando means, it is defined inductively:
\[(n = 2): \quad m[2, t](A_1, A_2) = A_1 m_t A_2\]
\[(n + 1): \quad m[n + 1, t](A_1, \cdots, A_{n+1}) = \lim_{r \rightarrow \infty} A_m(r)_{k} \text{ iff the limit exists,}
\text{where}\]
\[
\begin{cases}
A_m(r)_k = m[n, t][(A_m(r-1)_j)]_{j \neq k} \\
(A_m(1)_k = A_k).
\end{cases}
\]
Then they showed that \( \#[n, t](A_1, \cdots, A_n) \) always exists making use of the Thompson metric and that it coincides with Ando-Li-Mathius' one for \( t = 1/2 \). In [5], we pointed out that the arithmetic mean plays an essential part. In fact, it is expressed by the weight \( \{t[n]_k\} \):

\[
\nabla[n, t](A_1, \cdots, A_n) = \sum_{k=1}^{n} t[n]_k A_k.
\]

Also the harmonic mean is

\[
!\[n, t](A_1, \cdots, A_n) = \left( \sum_{k=1}^{n} t[n]_k A_k^{-1} \right)^{-1}.
\]

If \( A_k \) are commuting, then the geometric mean is

\[
\#[n, t](A_1, \cdots, A_n) = \prod_{k=1}^{n} A_k^{t[n]_k}.
\]

Moreover we extend the convexity

\[
d(A_1 \# B_1, A_2 \# B_2) \leq d(A_1, B_1) \nabla_t d(A_2, B_2)
\]

of the Thompson metric:

\[
d([n, t](A_1, \cdots, A_n), [n, t](B_1, \cdots, B_n)) \leq \nabla[t, n](d(A_1, B_1), \cdots, d(A_n, B_n)) = \sum_{k=1}^{n} t[n]_k d(A_k, B_k),
\]

which shows the existence of the Lawson-Lim geometric mean.

Then we obtain the formulae for \( t[n]_k \) in [5]:

**Lemma.**

\[
t[n]_n = \frac{t}{1 + (n-2)t},
\]

\[
t[n]_1 = \frac{1-t}{1 + (n-2)(1-t)} = \frac{1-t}{(n-1)-(n-2)t}.
\]

**Theorem.**

(i) \( t[n]_{n-m} = \frac{m(m+1) + 2m(n-2m-2)t + (n^2 - (4m+1)n + 4m(m+1)) t^2}{(n-1)(m+1) + (n-2m+1)t} \)

(ii) \( \sum_{j>n-m-1} t[n]_j = t[n]_n + \cdots + t[n]_{n-m} = \frac{(m+1)(m+(n-2m-1)t)}{(n-1)(m+1+(n-2m-2)t)} \).
Here we give another short proof of the above to show the probability distribution distribution function

$$F_n(k) = \sum_{j<k+1} t[n]_j = 1 - \frac{(n-k)(n-k-1 + (2k-n+1)t)}{(n-1)(n-k + (2k-n)t)}.$$  

Proof. Suppose the formula for $F_N(k)$ is valid for all $k$. Putting $v = F_N(k-1)$ and $w = F_N(k)$, we have

$$a_{n+1} = va_n + (1-v)b_n \quad \text{and} \quad b_{n+1} = wa_n + (1-w)b_n.$$  

Thereby

$$a_{n+1} - b_{n+1} = (v-w)a_n + (w-v)b_n = (v-w)(a_n - b_n) = \cdots = (v-w)^n,$$

and hence $b_n = a_n - (v-w)^n-1$. Then we have $a_{n+1} - a_n = -(1-v)(v-w)^n-1$ and

$$a_{n+1} = a_1 - (1-v) \sum_{k=0}^{n-1} (v-w)^k \longrightarrow 1 - \frac{1-v}{1-v+w},$$

which coincides with $F_{N+1}(k)$. Therefore, the formulae $F_n(k)$ are valid by induction. Thus (ii) in Theorem is obtained by $1 - F_n(k)$ and (i) by $t[n]_k = F_n(k) - F_n(k-1)$. □

Now we give the table for the density function $t[n]_k$:

<table>
<thead>
<tr>
<th>$1-t$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1-t}{2-t}$</td>
<td>$\frac{1-t^2}{(2-t)(1+t)}$</td>
</tr>
<tr>
<td>$\frac{1-t}{3-2t}$</td>
<td>$\frac{3-4t+2t^2}{3(3-2t)}$</td>
</tr>
<tr>
<td>$\frac{1-t}{4-3t}$</td>
<td>$\frac{6-9t+4t^2}{2(4-3t)(3-t)}$</td>
</tr>
<tr>
<td>$\frac{1-t}{5-4t}$</td>
<td>$\frac{10-16t+7t^2}{5(5-4t)(2-t)}$</td>
</tr>
<tr>
<td>$\frac{1-t}{6-5t}$</td>
<td>$\frac{15-25t+11t^2}{3(6-5t)(6-5t)}$</td>
</tr>
<tr>
<td>$\frac{1-t}{7-6t}$</td>
<td>$\frac{10-12t+5t^2}{3(4-t)(5-3t)}$</td>
</tr>
<tr>
<td>$\frac{1-t}{8-7t}$</td>
<td>$\frac{1+3t+11t^2}{3(2+3t)(1+5t)}$</td>
</tr>
</tbody>
</table>

The table for $t[n]_k$
Appendix: binomial mean $m[n]_t$ for $m_t$. From the viewpoint of probability distribution, a simple one-parameter extension of symmetric path can be defined inductively:

$$m[2]_t(A_1, A_2) = A_1 m_t A_2$$
$$m[3]_t(A_1, A_2, A_3) = (m[2]_t(A_1, A_2)) m_t (m[2]_t(A_2, A_3))$$
$$m[n+1]_t(A_1, \cdots, A_{n+1}) = (m[n]_t(A_1, \cdots, A_n)) m_t (m[n]_t(A_2, \cdots, A_{n+1})).$$

This path is symmetric in the sense of

$$m[n]_t(A_1, \cdots, A_n) = m[n]_{1-t}(A_n, \cdots, A_1)$$

The binomial arithmetic mean is

$$\nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} \binom{n-1}{k-1} (1-t)^{n-k} t^{k-1} A_k,$$

and the barycenter is the usual arithmetic mean:

$$\int_0^1 \nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n-1} \binom{n-1}{k-1} B(n - k + 1, k) A_k = \frac{1}{n} \sum_{k=1}^{n} A_k$$

where $B(p, q)$ is the beta function. As in [11], a multivariable extension of logarithmic mean

$$L[2](a, b) = \frac{b - a}{\log b - \log a}$$

is a fascinating one. Considering

$$L[2](A, B) = \int_0^1 A \#_t B \, dt$$

holds in Kubo-Ando means, we might define

$$L[n](A_1, \cdots, A_n) = \int_0^1 \#[n]_t(A_1, \cdots, A_n) dt.$$
Reference


