数理解析研究所講究録
第 1585 巻 2008 年 24-29

On a probability distribution of a binomial type generated by a mean

大阪教育大 藤井 淳一 (Jun Ichi Fujii)
Departments of Arts and Sciences (Information Science)
Osaka Kyoiku University

1. Means and paths. In this note, we use operator means, in particular, the Kubo-Ando mean [6] plays a central role: A binary operation $\mathbf{m}$ on positive operators on a Hilbert space is called the ***Kubo-Ando (operator) mean*** if $\mathbf{m}$ satisfies the following axioms:

monotonicity: $A \le C,\ B \le D \Rightarrow A\,\mathbf{m}\,B \le C\,\mathbf{m}\,D.$

semicontinuity: $A_n \downarrow A,\ B_n \downarrow B \Rightarrow A_n\,\mathbf{m}\,B_n \downarrow A\,\mathbf{m}\,B.$

transformer inequality: $T^*(A\,\mathbf{m}\,B)T \le T^*AT\,\mathbf{m}\,T^*BT.$

normalization: $A\,\mathbf{m}\,A = A.$

By semicontinuity, we may assume positive operators are invertible. The ***representing function*** $f_{\mathbf{m}}(x) = 1\,\mathbf{m}\,x$ for a Kubo-Ando mean $\mathbf{m}$ is operator monotone (concave) on $(0,\infty)$ and $\mathbf{m}$ is represented by

$$A\,\mathbf{m}\,B = A^{\frac{1}{2}} f_{\mathbf{m}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

A ***path*** $A\,\mathbf{m}_t B$ means parametrized operator means which is usually differentiable for t with $A\,\mathbf{m}_0 B = A$ and $A\,\mathbf{m}_1 B = 0$. A path is called ***symmetric*** if

$$A\,\mathbf{m}_t B = B\,\mathbf{m}_{1-t} A$$

holds for all $t \in [0,1]$. Typical example is (***quasi-arithmetic***) ***power means*** for $r \in [-1,1]$:

$$A\#_{r,t}B = A^{\frac{1}{2}}\left((1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right)^{\frac{1}{r}} A^{\frac{1}{2}},$$

which include important means:

arithmetic mean: $A\nabla_t B = A\#_{1,t}B = (1-t)A + tB$

geometic mean: $A\#_t B = A\#_{0,t}B \equiv \lim_{\varepsilon\to 0} A\#_{\varepsilon,t}B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$

harmonic mean: $A\,!_t B = A\#_{-1,t}B = ((1-t)A^{-1} + tB^{-1})^{-1}.$

Moreover the above paths are *interpolational* in the sense that

$$(A\#_{r,p}B)\#_{r,t}(A\#_{r,q}B) = A\#_{r,(1-t)p+tq}B$$

for all $p, q, t \in [0,1]$.

2. Thompson metric. Let $\mathcal{A}^+$ be the positive invertible elements in a unital C*-algebra $\mathcal{A}$, which is discussed as differentiable manifold by Corach-Porta-Recht [3, ?]. Corach himself reformulated it in [4]. They showed the above manifold $\mathcal{A}^+$ is the Finsler space with a Finsler metric

$$L(X;A) = \|X\|_A = \|A^{-1/2}XA^{-1/2}\| :$$

Then the geodesic is the shortest path with respect to this metric: The length $\ell(\gamma)$ of path $\gamma(t)$ is defined by

$$\ell(\gamma) \equiv \int_0^1 L(\gamma'(t);\gamma(t))dt = \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|dt.$$

If $\gamma(t)$ is a path from A to B, then

$$\begin{aligned} d(A,B) &\equiv \inf_{\gamma} \ell(\gamma) = \ell(A\#_t B) = \|\log(A^{-1/2}BA^{-1/2})\| \\ &= \log\left(\max\{\|A^{-1/2}BA^{-1/2}\|,\ \|B^{-1/2}AB^{-1/2}\|\}\right) \\ &= \log\left(\max\{r(A^{-1}B),\ r(B^{-1}A)\}\right). \end{aligned}$$

Also the homogeneity of $\mathcal{A}^+$ implies

$$d(A,B) = d(X^*AX, X^*BX) = d(I, A^{-1/2}BA^{-1/2})$$

for invertible X. The metric d makes $\mathcal{A}^+$ a complete metric space and it is called the *Thompson (part) one* [12, 10].

3. Lawson-Lim's operator mean. Recently, Lawson-Lim [8, 9, 7] defines multivariable operator means parametrized by $t \in [0,1]$ which is an extension of Ando-Li-Mathius' geometric operator mean [1]: For a symmetric path $\mathbf{m}_t$ in Kubo-Ando means, it is defined inductively:

$$(n=2):\quad \mathbf{m}[2,t](A_1,A_2) = A_1\,\mathbf{m}_t\,A_2$$

$$(n+1):\quad \mathbf{m}[n+1,t](A_1,\cdots,A_{n+1}) = \lim_{r\to\infty} A_{\mathbf{m}}(r)_k \ \text{if the limit exists}$$

$$\text{where}\quad \begin{cases} A_{\mathbf{m}}(r)_k = \mathbf{m}[n,t]((A_{\mathbf{m}}(r-1)_j)_{j\neq k}) \\ (A_{\mathbf{m}}(1)_k = A_k). \end{cases}$$

Then they showed that $\#[n,t](A_1,\cdots,A_n)$ always exists making use of the Thompson metric and that it coincides with Ando-Li-Mathius' one for $t=1/2$. In [5], we pointed out that the arithmetic mean plays an essential part. In fact, it is expressed by the weight $\{t[n]_k\}$:

$$\nabla[n,t](A_1,\cdots,A_n)=\sum_{k=1}^{n} t[n]_k A_k.$$

Also the harmonic mean is

$$![n,t](A_1,\cdots,A_n)=\left(\sum_{k=1}^{n} t[n]_k A_k^{-1}\right)^{-1}.$$

If A_k are commuting, then the geometric mean is

$$\#[n,t](A_1,\cdots,A_n)=\prod_{k=1}^{n} A_k^{t[n]_k}.$$

Moreover we extend the convexity

$$d(A_1\#_t B_1, A_2\#_t B_2)\leqq d(A_1,B_1)\nabla_t d(A_2,B_2)$$

of the Thompson metric:

$$\begin{aligned} d(\#[n,t](A_1,\cdots,A_n),\#[n,t](B_1,\cdots,B_n)) &\leqq \nabla[n,t](d(A_1,B_1),\cdots,d(A_n,B_n)) \\ &= \sum_{k=1}^{n} t[n]_k d(A_k,B_k), \end{aligned}$$

which shows the existence of the Lawson-Lim geometric mean.
Then we obtain the formulae for $t[n]_k$ in [5]:

Lemma.

$$t[n]_n=\frac{t}{1+(n-2)t}$$

$$t[n]_1=\frac{1-t}{1+(n-2)(1-t)}=\frac{1-t}{(n-1)-(n-2)t)}$$

Theorem.

(i) $$t[n]_{n-m}=\frac{m(m+1)+2m(n-2m-2)t+\left(n^2-(4m+1)n+4m(m+1)\right)t^2}{(n-1)\left(m+(n-2m)t\right)\left(m+1+(n-2(m+1))t\right)}$$

(ii) $$\sum_{j>n-m-1} t[n]_j=t[n]_n+\cdots+t[n]_{n-m}=\frac{(m+1)\left(m+(n-2m-1)t\right)}{(n-1)\left(m+1+(n-2m-2)t\right)}.$$

Here we give another short proof of the above to show the probability distribution distribution function

$$F_n(k) = \sum_{j<k+1} t[n]_j = 1 - \frac{(n-k)\,(n-k-1+(2k-n+1)t)}{(n-1)\,(n-k+(2k-n)t)}.$$

Proof. Suppose the formula for $F_N(k)$ is valid for all k. Putting $v = F_N(k-1)$ and $w = F_N(k)$, we have

$$a_{n+1} = va_n + (1-v)b_n \quad \text{and} \quad b_{n+1} = wa_n + (1-w)b_n.$$

Thereby

$$a_{n+1} - b_{n+1} = (v-w)a_n + (w-v)b_n = (v-w)(a_n - b_n) = \cdots = (v-w)^n,$$

and hence $b_n = a_n - (v-w)^{n-1}$. Then we have $a_{n+1} - a_n = -(1-v)(v-w)^{n-1}$ and

$$a_{n+1} = a_1 - (1-v)\sum_{k=0}^{n-1}(v-w)^k \longrightarrow 1 - \frac{1-v}{1-v+w},$$

which coincides with $F_{N+1}(k)$. THerefore, the formulae $F_n(k)$ are valid by induction. Thus (ii) in Theorem is obtained by $1 - F_n(k)$ and (i) by $t[n]_k = F_n(k) - F_n(k-1)$. □

Now we give the table for the density function $t[n]_k$:

$1-t$	t					
$\frac{1-t}{2-t}$	$\frac{1-t+t^2}{(2-t)(1+t)}$	$\frac{t}{1+t}$				
$\frac{1-t}{3-2t}$	$\frac{3-4t+2t^2}{3(3-2t)}$	$\frac{1+2t^2}{3(1+2t)}$	$\frac{t}{1+2t}$			
$\frac{1-t}{4-3t}$	$\frac{6-9t+4t^2}{2(4-3t)(3-t)}$	$\frac{3-2t+2t^2}{2(3-t)(2+t)}$	$\frac{1+t+4t^2}{2(2+t)(1+3t)}$	$\frac{t}{1+3t}$		
$\frac{1-t}{5-4t}$	$\frac{10-16t+7t^2}{5(5-4t)(2-t)}$	$\frac{2-2t+t^2}{5(2-t)}$	$\frac{1+t^2}{5(1+t)}$	$\frac{1+2t+7t^2}{5(1+t)(1+4t)}$	$\frac{t}{1+4t}$	
$\frac{1-t}{6-5t}$	$\frac{15-25t+11t^2}{3(5-3t)(6-5t)}$	$\frac{10-12t+5t^2}{3(4-t)(5-3t)}$	$\frac{2-t+t^2}{(4-t)(3+t)}$	$\frac{3+2t+5t^2}{3(3+t)(2+3t)}$	$\frac{1+3t+11t^2}{3(2+3t)(1+5t)}$	$\frac{t}{1+5t}$

The table for $t[n]_k$

Appendix : binomial mean $\mathbf{m}[n]_t$ for $\mathbf{m}_t$. From the viewpoint of probability distribution, a simple one-parameter extension of symmetric path can be defined inductively:

$$\mathbf{m}[2]_t(A_1, A_2) = A_1 \,\mathbf{m}_t A_2$$
$$\mathbf{m}[3]_t(A_1, A_2, A_3) = (\,\mathbf{m}[2]_t(A_1, A_2))\,\mathbf{m}_t(\,\mathbf{m}[2]_t(A_2, A_3))$$
$$\mathbf{m}[n+1]_t(A_1, \cdots, A_{n+1}) = (\,\mathbf{m}[n]_t(A_1, \cdots, A_n))\,\mathbf{m}_t(\,\mathbf{m}[n]_t(A_2, \cdots, A_{n+1})).$$

This path is ***symmetric*** in the sense of

$$\mathbf{m}[n]_t(A_1, \cdots, A_n) = \mathbf{m}[n]_{1-t}(A_n, \cdots, A_1)$$

The binomial arithmetic mean is

$$\nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} {}_{n-1}C_{k-1}(1-t)^{n-k}t^{k-1}A_k,$$

and the barycenter is the usual arithmetic mean:

$$\int_0^1 \nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} {}_{n-1}C_{k-1}B(n-k+1, k)A_k = \frac{1}{n}\sum_{k=1}^{n} A_k$$

where $B(p, q)$ is the beta function. As in [11], a multivariable extension of ***logarithmic mean***

$$L[2](a, b) = \frac{b-a}{\log b - \log a}$$

is a fascinating one. Considering

$$L[2](A, B) = \int_0^1 A\#_t B \; dt$$

holds in Kubo-Ando means, we might define

$$L[n](A_1, \cdots, A_n) = \int_0^1 \#[n]_t(A_1, \cdots, A_n) dt.$$

参考文献

[1] T.Ando, C.-K.Li and R.Mathias: Geometric means, Linear Alg. Appl., **385**(2004), 305–334.

[2] E.Andruchow, G.Corach and D.Stojanoff: Geometrical significance of Lowner-Heinz inequality, Proc. Amer. Math. Soc., **128**(2000), 1031–1037.

[3] G.Corach, H.Porta and L.Recht: Geodesics and operator means in the space of positive operators. Internat. J. Math. **4**(1993), no. 2, 193–202.

[4] G.Corach and A.L.Maestripieri: Differential and metrical structure of positive operators, Positivity, **3**(1999), 297–315.
http://www.iam.conicet.gov.ar/PUBINV-DVI-PDF/MAESTRIPIERI/CM6.pdf

[5] J.I.Fujii, M.Fujii, M.Nakamura, J.Pečarić and Y.Seo: A reverse inequality for the weighted geometric mean due to Lawson-Lim, to appear in Linear Alg. Appl..

[6] F.Kubo and T.Ando: Means of positive linear operators, Math. Ann., **246**(1980), 205–224.

[7] S.Kim and Y.Lim: A converse inequality of higher-order weighted arithmetic and geometric means of positive definite operators, preprint.

[8] J.Lawson and Y.Lim: A general framework for extending means to higher orders, preprint.
http://arxiv.org/PS_cache/math/pdf/0612/0612293v1.pdf

[9] J.Lawson and Y.Lim: Higher order weighted matrix means and related matrix inequalities, preprint.

[10] R.D.Nussbaum: Hilbert's projective metric and iterated nonlinear maps, Mem. Amer. Math. Soc., **75**(1988), no.391.

[11] D.Petz: Means of positive matrices: Geometry and a conjecture, Annales Mathematicae et Informaticae, **32**(2005), 129–139.
http://www.renyi.hu/%7Epetz/pdf/109publ.pdf

[12] A.C.Thompson: On certain contraction mappings in a partially ordered vector space, Proc. Amer. Mat. Soc., **14**(1963), 438–443.