Title

On a probability distribution of a binomial type generated by a mean (Information and mathematics of non-additivity and non-extensivity : contacts with nonlinearity and non-commutativity)

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Citation

数理解析研究所講究録 (2008), 1585: 24-29

Issue Date

2008-02

URL

http://hdl.handle.net/2433/81522

Right

Type

Departmental Bulletin Paper

Textversion

publisher

Kyoto University
On a probability distribution of a binomial type generated by a mean

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1. Means and paths. In this note, we use operator means, in particular, the Kubo-Ando mean [6] plays a central role: A binary operation \( m \) on positive operators on a Hilbert space is called the Kubo-Ando (operator) mean if \( m \) satisfies the following axioms:

- **monotonicity:** \( A \leq C, B \leq D \Rightarrow A \ m B \leq C \ m D \).
- **semicontinuity:** \( A_n \downarrow A, B_n \downarrow B \Rightarrow A_n \ m B_n \downarrow A \ m B \).
- **transformer inequality:** \( T^*(A \ m B)T \leq T^*AT \ m T^*BT \).
- **normalization:** \( A \ m A = A \).

By semicontinuity, we may assume positive operators are invertible. The representing function \( f_m(x) = 1 \ m x \) for a Kubo-Ando mean \( m \) is operator monotone (concave) on \((0, \infty)\) and \( m \) is represented by

\[
A \ m B = A^{\frac{1}{2}} f_m(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.
\]

A path \( A \ m_t B \) means parametrized operator means which is usually differentiable for \( t \) with \( A \ m_0 B = A \) and \( A \ m_1 B = 0 \). A path is called symmetric if

\[
A \ m_t B = B \ m_{1-t} A
\]

holds for all \( t \in [0, 1] \). Typical example is (quasi-arithmetic) power means for \( r \in [-1, 1] \):

\[
A \#_{r,t} B = A^{\frac{1}{2}} \left( (1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r \right)^{\frac{1}{2}} A^{\frac{1}{2}},
\]

which include important means:

- **arithmetic mean:** \( A \nabla_t B = A \#_{1,t} B = (1-t)A + tB \)
- **geometric mean:** \( A \#_{0,t} B \equiv \lim_{\epsilon \to 0} A \#_{\epsilon,t} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} \)
- **harmonic mean:** \( A \!_t B = A \#_{-1,t} B = ((1-t)A^{-1} + tB^{-1})^{-1} \).
Moreover the above paths are *interpolational* in the sense that

$$(A\#_{r,p}B)\#_{r,t}(A\#_{r,q}B) = A\#_{r,(1-t)p+ tq}B$$

for all $p, q, t \in [0, 1]$.

2. **Thompson metric.** Let $A^+$ be the positive invertible elements in a unital C*-algebra $A$, which is discussed as differentiable manifold by Corai-Porta-Recht [3, 7]. Corach himself reformulated it in [4]. They showed the above manifold $A^+$ is the Finsler space with a Finsler metric

$$L(X; A) = \|X\|_A = \|A^{-1/2}XA^{-1/2}\| :$$

Then the geodesic is the shortest path with respect to this metric: The length $\ell(\gamma)$ of path $\gamma(t)$ is defined by

$$\ell(\gamma) \equiv \int_0^1 L(\gamma'(t); \gamma(t))\,dt = \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|\,dt.$$ 

If $\gamma(t)$ is a path from $A$ to $B$, then

$$d(A, B) \equiv \inf_{\gamma} \ell(\gamma) = \ell(A\# tB) = \|\log(A^{-1/2}BA^{-1/2})\|$$

$$= \log \left( \max\{\|A^{-1/2}BA^{-1/2}\|, \|B^{-1/2}AB^{-1/2}\|\} \right)$$

$$= \log \left( \max\{r(A^{-1}B), r(B^{-1}A)\} \right).$$

Also the homogeneity of $A^+$ implies

$$d(A, B) = d(X^*AX, X^*BX) = d(I, A^{-1/2}BA^{-1/2})$$

for invertible $X$. The metric $d$ makes $A^+$ a complete metric space and it is called the *Thompson (part) one* [12, 10].

3. **Lawson-Lim’s operator mean.** Recently, Lawson-Lim [8, 9, 7] defines multivariable operator means parametrized by $t \in [0, 1]$ which is an extension of Ando-Li-Mathius’ geometric operator mean [1]: For a symmetric path $m_t$ in Kubo-Ando means, it is defined inductively:

$$(n = 2): \quad m[2, t](A_1, A_2) = A_1 \, m_t \, A_2$$

$$(n + 1): \quad m[n + 1, t](A_1, \ldots, A_{n+1}) = \lim_{r \to \infty} A_m(r)_k \text{ if the limit exists}$$

where

$$\begin{cases} 
A_m(r)_k = m[n, t]((A_m(r-1))_{j\neq k}) \\
(A_m(1)_k = A_k).
\end{cases}$$
Then they showed that $\#[n, t](A_1, \cdots, A_n)$ always exists making use of the Thompson metric and that it coincides with Ando-Li-Mathius' one for $t = 1/2$. In [5], we pointed out that the arithmetic mean plays an essential part. In fact, it is expressed by the weight $\{t[n]_k\}$:

$$\nabla[n, t](A_1, \cdots, A_n) = \sum_{k=1}^{n} t[n]_k A_k.$$

Also the harmonic mean is

$$!/[n, t](A_1, \cdots, A_n) = \left(\sum_{k=1}^{n} t[n]_k A_k^{-1}\right)^{-1}.$$

If $A_k$ are commuting, then the geometric mean is

$$\#[n, t](A_1, \cdots, A_n) = \prod_{k=1}^{n} A_k^{t[n]_k}.$$

Moreover we extend the convexity

$$d(A_1\# B_1, A_2\# B_2) \leq d(A_1, B_1) \nabla_t d(A_2, B_2)$$

of the Thompson metric:

$$d([n, t](A_1, \cdots, A_n), \#[n, t](B_1, \cdots, B_n)) \leq \nabla[n, t](d(A_1, B_1), \cdots, d(A_n, B_n)) = \sum_{k=1}^{n} t[n]_k d(A_k, B_k),$$

which shows the existence of the Lawson-Lim geometric mean.

Then we obtain the formulae for $t[n]_k$ in [5]:

**Lemma.**

$$t[n]_n = \frac{t}{1 + (n-2)t},$$

$$t[n]_1 = \frac{1-t}{1+(n-2)(1-t)} = \frac{1-t}{(n-1)-(n-2)t}.$$

**Theorem.**

(i) $t[n]_{n-m} = \frac{m(m+1) + 2m(n-2m-2)t + (n^2 - 4m+1)n + 4m(m+1)(n-1)(m+1)(m+2(m+1)t)}{(n-1)(m+m-2m)t(m+1+(n-2m+1)t)} t^2$

(ii) $\sum_{j>n-m-1} t[n]_j = \sum_{j=n-m-1}^{n} t[n]_j = \frac{(m+1)(m+(n-2m-1)t)}{(n-1)(m+1+(n-2m-2)t)}.$
Here we give another short proof of the above to show the probability distribution distribution function

\[ F_n(k) = \sum_{j<k+1} t[n]_j = 1 - \frac{(n-k)(n-k-1 + (2k-n+1)t)}{(n-1)(n-k + (2k-n)t)}. \]

**Proof.** Suppose the formula for \( F_N(k) \) is valid for all \( k \). Putting \( v = F_N(k-1) \) and \( w = F_N(k) \), we have

\[ a_{n+1} = va_n + (1-v)b_n \quad \text{and} \quad b_{n+1} = wa_n + (1-w)b_n. \]

Thereby

\[ a_{n+1} - b_{n+1} = (v-w)a_n + (w-v)b_n = (v-w)(a_n - b_n) = \cdots = (v-w)^n, \]

and hence \( b_n = a_n - (v-w)^{n-1} \). Then we have \( a_{n+1} - a_n = -(1-v)(v-w)^{n-1} \) and

\[ a_{n+1} = a_1 - (1-v) \sum_{k=0}^{n-1} (v-w)^k \longrightarrow 1 - \frac{1-v}{1-v+w}, \]

which coincides with \( F_{N+1}(k) \). Therefore, the formulae \( F_n(k) \) are valid by induction. Thus (ii) in Theorem is obtained by \( 1 - F_n(k) \) and (i) by \( t[n]_k = F_n(k) - F_n(k-1) \). \( \square \)

Now we give the table for the density function \( t[n]_k \):

<table>
<thead>
<tr>
<th>( 1-t )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1-t}{2-t} )</td>
<td>( \frac{1-t+t^2}{(2-t)(1+t)} )</td>
</tr>
<tr>
<td>( \frac{1-t}{3-2t} )</td>
<td>( \frac{3-4t+2t^2}{3(3-2t)} )</td>
</tr>
<tr>
<td>( \frac{1-t}{4-3t} )</td>
<td>( \frac{6-9t+4t^2}{2(4-3t)(3-t)} )</td>
</tr>
<tr>
<td>( \frac{1-t}{5-4t} )</td>
<td>( \frac{10-16t+7t^2}{5(5-4t)(2-t)} )</td>
</tr>
<tr>
<td>( \frac{1-t}{6-5t} )</td>
<td>( \frac{15-25t+11t^2}{3(6-5t)(6-5t)} )</td>
</tr>
</tbody>
</table>

The table for \( t[n]_k \)
Appendix: binomial mean $m[n]_{t}$ for $m_{t}$. From the viewpoint of probability distribution, a simple one-parameter extension of symmetric path can be defined inductively:

$$m[2]_{t}(A_1, A_2) = A_1 m_{t} A_2$$

$$m[3]_{t}(A_1, A_2, A_3) = (m[2]_{t}(A_1, A_2)) m_{t} (m[2]_{t}(A_2, A_3))$$

$$m[n + 1]_{t}(A_1, \cdots, A_{n+1}) = (m[n]_{t}(A_1, \cdots, A_n)) m_{t} (m[n]_{t}(A_2, \cdots, A_{n+1})).$$

This path is symmetric in the sense of

$$m[n]_{t}(A_1, \cdots, A_n) = m[n]_{1-t}(A_n, \cdots, A_1)$$

The binomial arithmetic mean is

$$\nabla[n]_{t}(A_1, \cdots, A_n) = \sum_{k=1}^{n} \binom{n-1}{k-1}(1-t)^{n-k} t^{k-1} A_k,$$

and the barycenter is the usual arithmetic mean:

$$\int_{0}^{1} \nabla[n]_{t}(A_1, \cdots, A_n) = \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k+1, k) A_k = \frac{1}{n} \sum_{k=1}^{n} A_k$$

where $B(p, q)$ is the beta function. As in [11], a multivariable extension of logarithmic mean

$$L[2](a, b) = \frac{b - a}{\log b - \log a}$$

is a fascinating one. Considering

$$L[2](A, B) = \int_{0}^{1} A \#_{t} B dt$$

holds in Kubo-Ando means, we might define

$$L[n](A_1, \cdots, A_n) = \int_{0}^{1} \#[n]_{t}(A_1, \cdots, A_n) dt.$$


