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On a probability distribution of a binomial type generated by a mean

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1. Means and paths. In this note, we use operator means, in particular, the Kubo-Ando mean [6] plays a central role: A binary operation $\mathbf{m}$ on positive operators on a Hilbert space is called the ***Kubo-Ando (operator) mean*** if $\mathbf{m}$ satisfies the following axioms:

monotonicity:	$A \leq C,\ B \leq D \Rightarrow A \,\mathbf{m}\, B \leq C \,\mathbf{m}\, D.$
semicontinuity:	$A_n \downarrow A,\ B_n \downarrow B \Rightarrow A_n \,\mathbf{m}\, B_n \downarrow A \,\mathbf{m}\, B.$
transformer inequality:	$T^*(A \,\mathbf{m}\, B)T \leq T^*AT \,\mathbf{m}\, T^*BT.$
normalization:	$A \,\mathbf{m}\, A = A.$

By semicontinuity, we may assume positive operators are invertible. The ***representing function*** $f_{\mathbf{m}}(x) = 1 \,\mathbf{m}\, x$ for a Kubo-Ando mean $\mathbf{m}$ is operator monotone (concave) on $(0, \infty)$ and $\mathbf{m}$ is represented by

$$A \,\mathbf{m}\, B = A^{\frac{1}{2}} f_{\mathbf{m}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

A ***path*** $A \,\mathbf{m}_t B$ means parametrized operator means which is usually differentiable for t with $A \,\mathbf{m}_0 B = A$ and $A \,\mathbf{m}_1 B = 0$. A path is called ***symmetric*** if

$$A \,\mathbf{m}_t B = B \,\mathbf{m}_{1-t} A$$

holds for all $t \in [0,1]$. Typical example is (***quasi-arithmetic***) ***power means*** for $r \in [-1,1]$:

$$A \#_{r,t} B = A^{\frac{1}{2}} \left((1-t)I + t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r \right)^{\frac{1}{r}} A^{\frac{1}{2}},$$

which include important means:

arithmetic mean:	$A \nabla_t B = A \#_{1,t} B = (1-t)A + tB$
geometic mean:	$A \#_t B = A \#_{0,t} B \equiv \lim_{\varepsilon \to 0} A \#_{\varepsilon,t} B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$
harmonic mean:	$A \,!_t B = A \#_{-1,t} B = ((1-t)A^{-1} + tB^{-1})^{-1}.$

Moreover the above paths are *interpolational* in the sense that

$$(A\#_{r,p}B)\#_{r,t}(A\#_{r,q}B) = A\#_{r,(1-t)p+tq}B$$

for all $p, q, t \in [0,1]$.

2. Thompson metric. Let $\mathcal{A}^+$ be the positive invertible elements in a unital C*-algebra $\mathcal{A}$, which is discussed as differentiable manifold by Corach-Porta-Recht [3, ?]. Corach himself reformulated it in [4]. They showed the above manifold $\mathcal{A}^+$ is the Finsler space with a Finsler metric

$$L(X; A) = \|X\|_A = \|A^{-1/2}XA^{-1/2}\| :$$

Then the geodesic is the shortest path with respect to this metric: The length $\ell(\gamma)$ of path $\gamma(t)$ is defined by

$$\ell(\gamma) \equiv \int_0^1 L(\gamma'(t); \gamma(t))dt = \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|dt.$$

If $\gamma(t)$ is a path from A to B, then

$$\begin{aligned} d(A,B) &\equiv \inf_\gamma \ell(\gamma) = \ell(A\#_t B) = \|\log(A^{-1/2}BA^{-1/2})\| \\ &= \log\left(\max\{\|A^{-1/2}BA^{-1/2}\|, \|B^{-1/2}AB^{-1/2}\|\}\right) \\ &= \log\left(\max\{r(A^{-1}B), r(B^{-1}A)\}\right). \end{aligned}$$

Also the homogeneity of $\mathcal{A}^+$ implies

$$d(A,B) = d(X^*AX, X^*BX) = d(I, A^{-1/2}BA^{-1/2})$$

for invertible X. The metric d makes $\mathcal{A}^+$ a complete metric space and it is called the *Thompson (part) one* [12, 10].

3. Lawson-Lim's operator mean. Recently, Lawson-Lim [8, 9, 7] defines multivariable operator means parametrized by $t \in [0,1]$ which is an extension of Ando-Li-Mathius' geometric operator mean [1]: For a symmetric path $\mathbf{m}_t$ in Kubo-Ando means, it is defined inductively:

$$(n=2):\quad \mathbf{m}[2,t](A_1, A_2) = A_1\, \mathbf{m}_t\, A_2$$

$$(n+1):\quad \mathbf{m}[n+1,t](A_1, \cdots, A_{n+1}) = \lim_{r\to\infty} A_{\mathbf{m}}(r)_k \ \underline{\text{if the limit exists}}$$

$$\text{where} \quad \begin{cases} A_{\mathbf{m}}(r)_k = \mathbf{m}[n,t]((A_{\mathbf{m}}(r-1)_j)_{j\neq k}) \\ (A_{\mathbf{m}}(1)_k = A_k). \end{cases}$$

Then they showed that $\#[n,t](A_1,\cdots,A_n)$ always exists making use of the Thompson metric and that it coincides with Ando-Li-Mathius' one for $t = 1/2$. In [5], we pointed out that the arithmetic mean plays an essential part. In fact, it is expressed by the weight $\{t[n]_k\}$:

$$\nabla[n,t](A_1,\cdots,A_n) = \sum_{k=1}^{n} t[n]_k A_k.$$

Also the harmonic mean is

$$![n,t](A_1,\cdots,A_n) = \left(\sum_{k=1}^{n} t[n]_k A_k^{-1}\right)^{-1}.$$

If A_k are commuting, then the geometric mean is

$$\#[n,t](A_1,\cdots,A_n) = \prod_{k=1}^{n} A_k^{t[n]_k}.$$

Moreover we extend the convexity

$$d(A_1\#_t B_1, A_2\#_t B_2) \leqq d(A_1,B_1)\nabla_t d(A_2,B_2)$$

of the Thompson metric:

$$\begin{aligned} d(\#[n,t](A_1,\cdots,A_n), \#[n,t](B_1,\cdots,B_n) &\leqq \nabla[n,t](d(A_1,B_1),\cdots,d(A_n,B_n)) \\ &= \sum_{k=1}^{n} t[n]_k d(A_k,B_k), \end{aligned}$$

which shows the existence of the Lawson-Lim geometric mean.
Then we obtain the formulae for $t[n]_k$ in [5]:

Lemma.

$$t[n]_n = \frac{t}{1+(n-2)t}$$

$$t[n]_1 = \frac{1-t}{1+(n-2)(1-t)} = \frac{1-t}{(n-1)-(n-2)t)}$$

Theorem.

(i) $$t[n]_{n-m} = \frac{m(m+1)+2m(n-2m-2)t+\left(n^2-(4m+1)n+4m(m+1)\right)t^2}{(n-1)\,(m+(n-2m)t)\,(m+1+(n-2(m+1))t)}$$

(ii) $$\sum_{j>n-m-1} t[n]_j = t[n]_n + \cdots + t[n]_{n-m} = \frac{(m+1)\,(m+(n-2m-1)t)}{(n-1)\,(m+1+(n-2m-2)t)}.$$

Here we give another short proof of the above to show the probability distribution distribution function

$$F_n(k) = \sum_{j<k+1} t[n]_j = 1 - \frac{(n-k)\,(n-k-1+(2k-n+1)t)}{(n-1)\,(n-k+(2k-n)t)}.$$

Proof. Suppose the formula for $F_N(k)$ is valid for all k. Putting $v = F_N(k-1)$ and $w = F_N(k)$, we have

$$a_{n+1} = va_n + (1-v)b_n \quad \text{and} \quad b_{n+1} = wa_n + (1-w)b_n.$$

Thereby

$$a_{n+1} - b_{n+1} = (v-w)a_n + (w-v)b_n = (v-w)(a_n - b_n) = \cdots = (v-w)^n,$$

and hence $b_n = a_n - (v-w)^{n-1}$. Then we have $a_{n+1} - a_n = -(1-v)(v-w)^{n-1}$ and

$$a_{n+1} = a_1 - (1-v)\sum_{k=0}^{n-1}(v-w)^k \longrightarrow 1 - \frac{1-v}{1-v+w},$$

which coincides with $F_{N+1}(k)$. THerefore, the formulae $F_n(k)$ are valid by induction. Thus (ii) in Theorem is obtained by $1 - F_n(k)$ and (i) by $t[n]_k = F_n(k) - F_n(k-1)$. $\square$

Now we give the table for the density function $t[n]_k$:

$1-t$	t					
$\frac{1-t}{2-t}$	$\frac{1-t+t^2}{(2-t)(1+t)}$	$\frac{t}{1+t}$				
$\frac{1-t}{3-2t}$	$\frac{3-4t+2t^2}{3(3-2t)}$	$\frac{1+2t^2}{3(1+2t)}$	$\frac{t}{1+2t}$			
$\frac{1-t}{4-3t}$	$\frac{6-9t+4t^2}{2(4-3t)(3-t)}$	$\frac{3-2t+2t^2}{2(3-t)(2+t)}$	$\frac{1+t+4t^2}{2(2+t)(1+3t)}$	$\frac{t}{1+3t}$		
$\frac{1-t}{5-4t}$	$\frac{10-16t+7t^2}{5(5-4t)(2-t)}$	$\frac{2-2t+t^2}{5(2-t)}$	$\frac{1+t^2}{5(1+t)}$	$\frac{1+2t+7t^2}{5(1+t)(1+4t)}$	$\frac{t}{1+4t}$	
$\frac{1-t}{6-5t}$	$\frac{15-25t+11t^2}{3(5-3t)(6-5t)}$	$\frac{10-12t+5t^2}{3(4-t)(5-3t)}$	$\frac{2-t+t^2}{(4-t)(3+t)}$	$\frac{3+2t+5t^2}{3(3+t)(2+3t)}$	$\frac{1+3t+11t^2}{3(2+3t)(1+5t)}$	$\frac{t}{1+5t}$

The table for $t[n]_k$

Appendix : binomial mean $\mathbf{m}[n]_t$ for $\mathbf{m}_t$. From the viewpoint of probability distribution, a simple one-parameter extension of symmetric path can be defined inductively:

$$\begin{aligned}
\mathbf{m}[2]_t(A_1, A_2) &= A_1\,\mathbf{m}_t A_2 \\
\mathbf{m}[3]_t(A_1, A_2, A_3) &= (\,\mathbf{m}[2]_t(A_1, A_2))\,\mathbf{m}_t(\,\mathbf{m}[2]_t(A_2, A_3)) \\
\mathbf{m}[n+1]_t(A_1, \cdots, A_{n+1}) &= (\,\mathbf{m}[n]_t(A_1, \cdots, A_n))\,\mathbf{m}_t(\,\mathbf{m}[n]_t(A_2, \cdots, A_{n+1})).
\end{aligned}$$

This path is ***symmetric*** in the sense of

$$\mathbf{m}[n]_t(A_1, \cdots, A_n) = \mathbf{m}[n]_{1-t}(A_n, \cdots, A_1)$$

The binomial arithmetic mean is

$$\nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} {}_{n-1}C_{k-1}(1-t)^{n-k}t^{k-1}A_k,$$

and the barycenter is the usual arithmetic mean:

$$\int_0^1 \nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} {}_{n-1}C_{k-1}B(n-k+1, k)A_k = \frac{1}{n}\sum_{k=1}^{n} A_k$$

where $B(p,q)$ is the beta function. As in [11], a multivariable extension of ***logarithmic mean***

$$L[2](a, b) = \frac{b-a}{\log b - \log a}$$

is a fascinating one. Considering

$$L[2](A, B) = \int_0^1 A \#_t B\, dt$$

holds in Kubo-Ando means, we might define

$$L[n](A_1, \cdots, A_n) = \int_0^1 \#[n]_t(A_1, \cdots, A_n)dt.$$

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