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Fredholm determinant of complex transfer operators for complex dynamical systems and Artin-Mazur zeta function

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§0. Introduction

In this note, we consider a transfer operator induced by complex dynamical system on the Riemann sphere, operating on the space of "hyperfunctions" supported on the Julia set and its dual operator operating on the dual space.

Our transfer operator operates on the space of functions and the space of differential forms. The alternating product of the Fredholm determinants coincides with the Artin-Mazur dynamical zeta function.

§1. Setting and transfer operator

Let us begin with a simplest case. Let $R:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ be a hyperbolic rational map of degree D. We assume that the infinity is an attractive periodic point. Let $F=F(R)$ denote the Fatou set of R and let $J=J(R)$ denote the Julia set of R. In the following considerations, we treat integrations along paths surrounding the Julia set. In most cases, we integrate functions which are holomorphic in a neighborhood of J except on J. Suppose U is an open neighborhood of J and $f: U\setminus J \longrightarrow \mathbb{C}$ is holomorphic in the deleted neighborhood $U\setminus J$. We denote by γ_J the "ideal" path of integration around J, seeing Julia set on the left hand side. We consider γ_J running near ∂U, i.e. γ_J play the role of Cauchy's integration path for points near J.

We denote by γ_F the path of integration along "∂F" seeing the Fatou set on the left hand side. Note that γ_J and $-\gamma_F$ are homologous in $U \setminus J$. We consider γ_F as a Cauchy's integration path for points in F.

Let $\mathcal{O}(J)$ denote the space of continuous functions $f: J \to \mathbb{C}$ which can be extended to a holomorphic function in a neighborhood of J. Let U be an open neighborhood of J satisfying $R^{-1}(U) \subset U$ and contains no critical point. As we assumed that R is hyperbolic, $R^{-1}(\partial U)$ is homologous to ∂U in $F(R) \cap U$. And the integration path γ_F and γ_J are homologous to ∂U or $-\partial U$ depending on its orientation. For each $f \in \mathcal{O}(J)$, there exists an open neighborhood of J such that f extends holomorphically to it. By taking this open neighborhood of J appropriately, it can be chosen to be backward invariant $R^{-1}(U) \subset U$.

The simplest transfer operator $\mathcal{L}_0 : \mathcal{O}(J) \to \mathcal{O}(J)$ is defined as follows.

<u>Definition</u> 1.1. Let $f: U \to \mathbb{C}$ be a holomorphic function defined in a neighborhood U of Julia set J of R. We define $\mathcal{L}_0 f$ by

$$(\mathcal{L}_0 f)(z) = \sum_{y \in R^{-1}(z)} f(y).$$

In our situation with hyperbolic rational R and $\infty \in F$, this operator can be rewritten in an integral operator form.

$$(\mathcal{L}_0 f)(z) = \frac{1}{2\pi i} \int_{\gamma_J} \frac{f(\zeta)\, R'(\zeta)}{R(\zeta) - z}\, d\zeta,$$

where the path of integration γ_J is described above. In this integration, we consider $z \in U$ and γ_J is along ∂U, so that $\zeta \in \gamma_J$ and the denominator $R(\zeta) - z$ does not vanish.

As $J \subset U$ and $R^{-1}(z) \subset R^{-1}(U) \subset U$, the residue formula gives the equality

$$(\mathcal{L}_0 f)(z) = \frac{1}{2\pi i}\int_{\gamma_J} \frac{f(\zeta)\, R'(\zeta)}{R(\zeta) - z}\, d\zeta = \sum_{y \in R^{-1}(z)} f(y).$$

Here, we used the assumption that R is hyperbolic and no critical point in the neighborhood of J. Let $B(U)$ denote the Banach space of bounded holomorphic functions on U, equiped with the supremum norm. Then, the transfer operator $\mathcal{L}_0$ restricted to $B(U)$ maps $B(U)$ into itself. Since $\mathcal{L}_0$ restricted to $B(U)$ can be expressed as an integral operator, it is continuous and defines a compact operator. Obviously, the operator norm is $\|\mathcal{L}_0\| = D$, where D is the degree of R. D is an eigenvalue of $\mathcal{L}_0$ and constant function $\mathbb{1}$ is an eigenfunction of $\mathcal{L}_0$. We have the following situation.

Fix an open neighborhood U of J which does not contain critical points of R and satisfies $R^{-1}(U) \subset\subset U$. $B(U)$ denotes the Banach space of bounded holomorphic functions with supremum norm $\|f\| = \sup_{x \in U} |f(x)|$. Our transfer operator $\mathcal{L}_0 : B(U) \to B(U)$ is a completely continuous linear operator. Let us briefly recall the Fredholm theory. We consider the eigenvalue problem in the form

$$f = \lambda \mathcal{L}_0 f \quad , \quad \lambda \in \mathbb{C}, \quad f \in B(U).$$

λ is called a singular value of $\mathcal{L}_0$ and f is an eigenfunction. The Fredholm theory saids the following.

<u>Theorem</u> 1.2. The Fredholm determinant $|I - \lambda \mathcal{L}_0|$ is an entire function. The multiplicity of each singular value as zero of the Fredholm determinant is same as the dimension of the eigenspace to the singular value.

Specially useful tool from Fredholm theory is the following trace formula.

$$|I - \lambda \mathcal{L}| = \exp\left(-\sum_{m=1}^{\infty} \frac{\lambda^m}{m} \operatorname{tr} \mathcal{L}_0^m\right)$$

holds for λ with sufficiently small $|\lambda|$.

§2. Dual space and dual operator

Let $\mathcal{O}^*(J)$ denote the space of holomorphic and continuous linear functionals of $\mathcal{O}(J)$. Here, we consider holomorphic dual instead of usual dual. The dual operator $\hat{\mathcal{L}}_0^*: \mathcal{O}^*(J) \to \mathcal{O}^*(J)$ is defined as usual. We reserve notation $\mathcal{L}_0^*$ to be used later. For $h^* \in \mathcal{O}^*(J)$ and $f \in \mathcal{O}(J)$, we define $\hat{\mathcal{L}}_0^* h^*$ by

$$(\hat{\mathcal{L}}_0^* h^*)[f] = h^*[\mathcal{L}_0 f].$$

For $\zeta \in F(R)$, we define the unit pole $\chi_\zeta \in \mathcal{O}(J)$ by

$$\chi_\zeta(z) = \frac{1}{z-\zeta} \qquad \text{for } z \in U.$$

For $h^* \in \mathcal{O}^*(J)$, define $h \in \mathcal{O}_0(F)$ by

$$h(\zeta) = h^*[-\chi_\zeta]$$

here, $\mathcal{O}_0(F(R))$ denotes the space of holomorphic functions in $F(R)$ vanishing at the infinity. This correspondence $\mathcal{O}^*(J) \to \mathcal{O}_0(F)$ is called the Cauchy transformation. For $g \in \mathcal{O}(J)$, there exists an open neighbohood U of J such that g can be extended holomorphically to U. Then take a multi curve γ_J for the path of integration running along ∂U and express $g(\zeta)$ for $\zeta \in U$ in the Cauchy's integration form

$$g(\zeta) = \frac{1}{2\pi i}\int_{\gamma_J} \frac{g(z)}{z-\zeta}\, dz = \frac{1}{2\pi i}\int_{\gamma_J} g(z)(-\chi_z(\zeta))\, dz.$$

We regard this formula as representing $g(\zeta)$ as a "linear combination" of unit poles.

For $h^* \in \mathcal{O}^*(J)$, the value $h^*[g]$ is given by

$$\begin{aligned} h^*[g] &= h^*\left[\frac{1}{2\pi i}\int_{\gamma_J} g(z)(-\chi_z(\zeta))\, dz\right] \\ &= \frac{1}{2\pi i}\int_{\gamma_J} g(z)\, h^*[-\chi_z]\, dz \\ &= \frac{1}{2\pi i}\int_{\gamma_J} g(z)\, h(z)\, dz. \end{aligned}$$

Hence $h^* \in \mathcal{O}^*(J)$ can be represented by $h \in \mathcal{O}_0(F)$. On the other hand, any $h \in \mathcal{O}(F)$ defines a holomorphic linear functional h^* by the above formula.

Thus the Cauchy transformation $\mathcal{O}^*(J) \to \mathcal{O}_0(F)$ is an isomorphism. This fact is a very special case of Köthe spaces. The dual operator $\tilde{\mathcal{L}}_0^*$ induces an operator from $\mathcal{O}_0(F)$ into itself, which we denote by $\mathcal{L}_0^*$. We have the following diagram

$$\begin{tikzcd} \mathcal{O}^*(J) \arrow[r, "\tilde{\mathcal{L}}_0^*"] \arrow[d, "\text{Cauchy tr.}"'] & \mathcal{O}^*(J) \arrow[d, "\text{Cauchy transformation}"] \\ \mathcal{O}_0(F) \arrow[r, "\mathcal{L}_0^*"] & \mathcal{O}_0(F) \end{tikzcd}$$

Note that our space $\mathcal{O}_0(F)$ should be understood as a space of hyperfunctions.

$$\mathcal{O}_0(F) \simeq \mathcal{H}(J)/\mathcal{O}(J)$$

where $\mathcal{H}(J)$ is the space of germs of functions defined in a deleted neighborhood $U \setminus J$ of Julia set and holomorphic in $U \setminus J$, and modulo $\mathcal{O}(J)$ means the equivalence class modulo local holomorphic functions defined and holomorphic in an open neighborhood of J. Thus, an element of $\mathcal{O}_0(F)$ represents singularities along J.

The dual operator $\mathcal{L}_0^*$ represented as $\mathcal{L}_0^*: \mathcal{O}_0(F) \to \mathcal{O}_0(F)$ can be described in an explicit manner as a pull-back operator of differential forms.

<u>Proposition</u> 2.1. If R is a polynomial map, then

$$(\mathcal{L}_0^* h)(z) = h(R(z))R'(z)$$

for $h \in \mathcal{O}_0(F)$.

<u>proof</u> In the following computations, variable ζ is used for functions in $\mathcal{O}(J)$ and considered as $\zeta \in U$. Variable z is used for functions in $\mathcal{O}_0(F)$ and considered as $z \in F$. Variable τ is used in the integration along γ_J or γ_F. If $\tau \in \gamma_J$ $(\doteqdot \partial U)$, then $R(\tau) \in F \setminus U$. Now, by a direct computation, we have the following.

$$(\mathcal{L}_0^* h)(z) = (\tilde{\mathcal{L}}_0^* h^*)[-\chi_z] = h^*[\mathcal{L}_0[-\chi_z]]$$

$$= h^*\left[\frac{1}{2\pi i}\int_{\gamma_J} \frac{R'(\tau)}{R(\tau)-\zeta}(-\chi_z(\tau))\,d\tau\right]$$
$$= \frac{1}{2\pi i}\int_{-\gamma_F} h(\zeta)\left(\frac{1}{2\pi i}\int_{\gamma_J} \frac{R'(\tau)}{R(\tau)-\zeta}\,\frac{d\tau}{z-\tau}\right)d\zeta$$
$$= \frac{1}{2\pi i}\int_{\gamma_J} \frac{R'(\tau)}{z-\tau}\left(\frac{1}{2\pi i}\int_{-\gamma_F} \frac{h(\zeta)}{R(\tau)-\zeta}\,d\zeta\right)d\tau$$
$$= \frac{1}{2\pi i}\int_{\gamma_J} \frac{R'(\tau)}{z-\tau}\, h(R(\tau))\,d\tau$$
$$= h(R(z))\,R'(z)$$

Here, $h(\zeta)$ is holomorphic in F and $R(\tau)\in F$ with $-\gamma_F \doteqdot \partial F$, we used the Cauchy formula

$$\frac{1}{2\pi i}\int_{-\gamma_F} \frac{h(\zeta)}{R(\tau)-\zeta}\,d\zeta = h(R(\tau)).$$

The last equality holds for $z\in F\setminus U$ and $\tau\in\gamma_J \doteqdot \partial U$. As R has no poles in F, the only residue comes from $\tau=z$. The residue at the infinity vanishes since $R'(\tau) \sim D\tau^{D-1}$ and $h(R(\tau)) \sim \frac{1}{R(\tau)}$ as $\tau\to\infty$.

This proposition indicates that the dual operator $\mathcal{L}_0^*$ acts like the pull back operator R^* acting on differential forms, i.e.,

$$R^*(h(z)dz) = h(R(z))\,R'(z)\,dz.$$

In the case of hyperbolic rational map, we must take care of the poles and the behavior of the integrand near the infinity. If the infinity is an attractive or super-attractive fixed point, then the infinity does not give rise to non-zero residue, since

$$\frac{h(R(\tau))\cdot R'(\tau)}{z-\tau} \sim O\left(\frac{1}{\tau^2}\right), \quad (\tau\to\infty).$$

If $R(\infty)$ is finite, then $R'(\infty)=0$ and $h(R(\infty))$ is bounded. So the residue at the infinity vanishes, too. Finally, let us consider the residue at a pole, p, of R. Suppose

$$R(\tau) \sim \frac{a}{(\tau-p)^k} \quad \text{near } \tau=p. \text{ Then } R'(\tau) \sim \frac{-ka}{(\tau-p)^{k+1}}.$$

Hence, $\dfrac{h(R(\tau))\cdot R'(\tau)}{z-\tau} \sim \dfrac{(\tau-p)^k}{z-p}\cdot\dfrac{-k\cdot h_\infty}{(\tau-p)^{k+1}} \sim \dfrac{-k h_\infty}{(z-p)(\tau-p)}$

near $\tau = p$. Here h_∞ is a constant satisfying

$$h(z) = \frac{h_\infty}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty.$$

This term gives rise to the residue at p

$$Res_p = \frac{-k h_\infty}{z-p},$$

which defines a rational function in $O(J)$. This fraction is exactly the pole part of $\mathcal{L}_0^* h = h\circ R\cdot R'$. Observe that

$$h(R(z))\, R'(z) = \frac{1}{2\pi i}\int_{\gamma_J \cup \gamma_F} \frac{h(R(\tau))\, R'(\tau)}{\tau - z}\, d\tau$$

holds, and we should set

$$(\mathcal{L}_0^* h)(z) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{h(R(\tau))\, R'(\tau)}{\tau - z}\, d\tau$$

in the case of hyperbolic rational map R. This formula shows that

$$\mathcal{L}_0^* h \in O_0(F).$$

We have proved the following.

<u>Proposition</u> 2.2. If R is a hyperbolic rational function and the infinity is an attractive periodic point, then

$$(\mathcal{L}_0^* h)(z) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{h(R(\tau))\, R'(\tau)}{\tau - z}\, d\tau$$

for $h \in O_0(F)$ and γ_F is a multi-curve passing near J with $z \in F$.

Note that the "transpose" of $\mathcal{L}_0$ defines an exactly same operator $({}^t\mathcal{L}_0 h)(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_F} \frac{h(\tau)\, R'(z)}{R(z) - \tau}\, d\tau$.

§3. Brolin-Lyubich measure

We denote the unit pole at x by $\chi_x(z) = \dfrac{1}{z-x}$, where x is a given point and χ_x is a rational function in z.

The dual operator $\mathcal{L}_0^*$ of $\mathcal{L}_0$, considered as a pull-back map operating on the space of measures,

$$\delta_x \longmapsto \frac{1}{D} \sum_{y \in R^{-1}(x)} \delta_y$$

for the Dirac's masses. The limit of probability measures

$$\lim_{n\to\infty} \frac{1}{D^n} \sum_{y \in R^{-n}(x)} \delta_y = \mu$$

exists for any choice of x which is not exceptional. The limit is called the Lyubich measure.

In our context, unit pole χ_x, considered as a "hyper-function" defined by differential form $\chi_x(z)\,dz$ corresponds to Dirac's mass δ_x at x.

Let $x \in J$ and for $z \in F$,

$$(\mathcal{L}_0^* \chi_x)(z) = \frac{1}{2\pi i} \int_{\gamma_J} \frac{R'(\tau)}{\tau - z} \frac{d\tau}{R(\tau) - x}$$
$$= \sum_{y \in R^{-1}(x)} \frac{-1}{y - z} = \sum_{y \in R^{-1}(x)} \chi_y(z).$$

Or, simply

$$\mathcal{L}_1^* : \chi_x \longmapsto \sum_{y \in R^{-1}(x)} \chi_y$$

The existence of Lyubich measure tells that $\mathcal{L}_0^*$ has D as an eigenvalue and

$$\left(\frac{1}{D} \mathcal{L}_0^*\right)^n \chi_x$$

converges to the Cauchy transform of Lyubich measure. This limit gives the eigenfunctional.

In the following sections, we study the "eigenvalue problem" of our transfer operator.

§4. Trace of transfer operator $\mathcal{L}_0$

Let us fix an open neighborhood U of J and let $\gamma = \partial U$ be the path of integration looking J on the left hand side. Operator $\mathcal{L}_0$ can be expressed in an integral operator form: $(\mathcal{L}_0 f)(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, R'(\zeta)}{R(\zeta) - z}\, d\zeta$.

If we regard $\mathcal{L}_0$ as a completely continuous linear operator on the space of continuous functions on γ, we can apply the Fredholm Theory to $\mathcal{L}_0$. The trace of $\mathcal{L}_0$ is computed as follows.

$$\mathrm{Tr}\,\mathcal{L}_0 = \frac{1}{2\pi i}\int_\gamma \frac{R'(z)}{R(z)-z}\,dz$$

$$= \sum_{x\in \mathrm{Fix}(R)\cap J} \frac{R'(x)}{R'(x)-1} = \sum_{x\in \mathrm{Fix}(R)\cap J} \frac{1}{1-\frac{1}{R'(x)}}.$$

This gives $\mathrm{Tr}\,\mathcal{L}_0$ in terms of the multipliers at the fixed points of R. However, since the integrand is a rational function of z, we can apply the residue formula to the complement $\bar{\mathbb{C}}\setminus \bar{U}$.

Proposition 4.1. If R is a polynomial of degree D, with $D\geq 2$,

$$\mathrm{Tr}\,\mathcal{L}_0 = D + \sum_{p\in \mathrm{Fix}(R)\cap F} \frac{\sigma_p}{1-\sigma_p},$$

where σ_p denotes the multiplier of R at p.

proof. $R(z)\sim a z^D$ as $z\to\infty$, hence

$$\frac{1}{2\pi i}\int_{\gamma_\infty} \frac{R'(z)}{R(z)-z}\,dz = D,$$

where the integration path γ_∞ turns around the Julia set of R passing near the infinity. At each attracting fixed point p in $\mathbb{C}$,

$$\frac{1}{2\pi i}\int_{\gamma_p} \frac{R'(z)}{R(z)-z}\,dz = \frac{\sigma_p}{1-\sigma_p}$$

where integration path turns around the fixed point p in the clockwise direction, and σ_p represents the multiplier at the fixed point p.

Proposition 4.2 If R is a hyperbolic rational map of degree D, with $D\geq 2$, then

$$\mathrm{Tr}\,\mathcal{L}_0 = D + \sum_{p\in \mathrm{Fix}(R)\cap F} \frac{\sigma_p}{1-\sigma_p},$$

where σ_p denotes the multiplier of R at P.

proof. The formula is same as in proposition 4.2.

We only need to check the residues related to the infinity. If the infinity is not a fixed point, i.e., $R(\infty) \neq \infty$, then $R'(\infty) = 0$ and the residue at ∞ vanishes:

$$\frac{1}{2\pi i}\int_{\gamma_\infty} \frac{R'(\zeta)}{R(\zeta)-\zeta}\, d\zeta = 0.$$

Residues at poles in this case are computed as follows. Let p be a pole of ~~of~~ order m, with

$$R(z) = a(z-p)^{-m} + o((z-p)^{-m})$$

and

$$R'(z) = -ma(z-p)^{-m-1} + o((z-p)^{-m-1})$$

as $z \to p$. Then, for integration path γ_p turning around the pole p in the clockwise direction, we have

$$\frac{1}{2\pi i}\int_{\gamma_p} \frac{R'(\zeta)}{R(\zeta)-\zeta}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{\gamma_p} \frac{-ma(\zeta-p)^{-m-1} + o((\zeta-p)^{-m-1})}{a(\zeta-p)^{-m} + o((\zeta-p)^{-m}) - \zeta}\, d\zeta$$

$$= m.$$

Next, let us consider the case when the infinity is an attracting (and not super-attracting) fixed point with multiplier σ_∞, i.e., $R(\infty) = \infty$, and $\lim_{z\to\infty} R'(z) = \frac{1}{\sigma_\infty}$ with $|\sigma_\infty| < 1$. Then

$$R(z) = \frac{1}{\sigma_\infty} z + o(z) \quad \text{and} \quad R'(z) = \frac{1}{\sigma_\infty} + o(1)$$

holds as $z \to \infty$. Hence, for integration path γ_∞ turning around the infinity in the counterclockwise direction (in $\mathbb{C}$),

$$\frac{1}{2\pi i}\int_{\gamma_\infty} \frac{R'(\zeta)}{R(\zeta)-\zeta}\, d\zeta = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{\frac{1}{\sigma_\infty} + o(1)}{(\frac{1}{\sigma_\infty} - 1)\zeta + o(\zeta)}\, d\zeta$$

$$= \frac{\frac{1}{\sigma_\infty}}{\frac{1}{\sigma_\infty} - 1} = \frac{1}{1-\sigma_\infty} = 1 + \frac{\sigma_\infty}{1-\sigma_\infty}.$$

Third, let us consider the case when the infinity is a super-attracting fixed point of order k, with $k \geq 2$,

$$R(z) = az^k + o(z^k)$$

and

$$R'(z) = kaz^{k-1} + o(z^{k-1})$$

as $z \to \infty$. Then, for integration path γ_∞ as in the previous case, we have

$$\frac{1}{2\pi i}\int_{\gamma_\infty} \frac{R'(\zeta)}{R(\zeta)-\zeta}\,d\zeta = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{k a \zeta^{k-1} + o(\zeta^{k-1})}{a\zeta^k + o(\zeta^k)}\,d\zeta = k.$$

In this case, if we set the multiplier $\sigma_\infty = 0$, this residue at the infinity can be considered as

$$k = k + \frac{\sigma_\infty}{1-\sigma_\infty}.$$

Summing up all the above calculations, we see that the residues contributing to $\mathrm{Tr}\,\mathcal{L}_0$ consists of the sum of the degrees of poles and the multiplicity of fixed point at the infinity, which make the degree D of the rational map R, and the terms $\frac{\sigma_p}{1-\sigma_p}$ for each fixed point in F. This proves the proposition.

Note that, we obtained

$$\mathrm{Tr}\,\mathcal{L}_0 = \sum_{x \in \mathrm{Fix}(R)\cap J} \frac{R'(x)}{R'(x)-1} = D + \sum_{p \in \mathrm{Fix}(R)\cap F} \frac{\sigma_p}{1-\sigma_p}.$$

§5. Iteration of $\mathcal{L}_0$ and $\mathcal{L}_0^*$.

Iteration of $\mathcal{L}_0$ and its dual operator $\mathcal{L}_0^*$ are easily obtained as follows.

$$(\mathcal{L}_0^n f)(z) = \sum_{y \in R^{-n}(z)} f(y).$$

$\mathcal{L}_0^n$ and $(\mathcal{L}_0^*)^n$ can be represented as an integral operator.

Proposition 5.1. For $n = 1, 2, \dots$,

$$(\mathcal{L}_0^n f)(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,(R^{\circ n})'(\zeta)}{R^{\circ n}(\zeta) - z}\,d\zeta$$

$$\left((\mathcal{L}_0^*)^n h\right)(z) = \frac{1}{2\pi i}\int_\gamma \frac{h(R^{\circ n}(\tau))\,(R^{\circ n})'(\tau)}{z-\tau}\,d\tau$$

proof. Simply apply similar arguement to $\mathcal{L}_0^n$ and $(\mathcal{L}_0^*)^n$ in place of $\mathcal{L}_0$ and $\mathcal{L}_0^*$.

Proposition 5.2 For hyperbolic rational map R of degree D,

$$\mathrm{Tr}\,\mathcal{L}_0^n = D^n + \sum_{p \in \mathrm{Fix}(R^{\circ n})\cap F} \frac{\sigma_p}{1-\sigma_p}$$

Proof. Proof is similar to that of proposition 4.2.

Note that $$\operatorname{Tr} \mathcal{L}_0^n = \sum_{x \in \operatorname{Fix}(R^{\circ n}) \cap J} \frac{(R^{\circ n})'(x)}{(R^{\circ n})'(x) - 1}.$$

§6. Fredholm determinant by the trace formula.

As is well known, for completely continuous operator on a Banach space, the Fredholm determinant can be expressed by the so-called trace formula for sufficiently small λ, i.e.,

$$\det(I - \lambda L) = \exp\left(-\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \operatorname{trace}(L^n)\right).$$

Theorem 6.1. Formal Fredholm determinant of $\mathcal{L}_0$ converges in a neighborhood of $\lambda = 0$ and extends holomorphically to an entire function given by

$$\det(I - \lambda \mathcal{L}_0) = (1 - D\lambda) \prod_{j=1}^{r} \prod_{k=1}^{\infty} (1 - \sigma_j^k \lambda^{P_j}),$$

where r is the number of attracting cycles, P_j is the period of the j-th cycle and σ_j is the multiplier of the cycle.

Proof. For an attracting cycle $C = \{p_1, \cdots, p_P\}$ of period P and multiplier $\sigma = (R^{\circ P})'(p_i)$ in the Fatou set, define function $Z_C(\lambda)$ by

$$Z_C(\lambda) = \exp\left(-\sum_{m=1}^{\infty} \frac{\lambda^{mP}}{mP} \sum_{p \in C} \frac{\sigma^m}{1 - \sigma^m}\right).$$

Then, we have

$$Z_C(\lambda) = \exp\left(-\sum_{m=1}^{\infty} \frac{(\lambda^P)^m}{m} \sum_{k=1}^{\infty} (\sigma^k)^m\right)$$

$$= \exp\left(-\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\sigma^k \lambda^P)^m}{m}\right)$$

$$= \prod_{k=1}^{\infty} (1 - \sigma^k \lambda^P).$$

This computation is justified since $|\sigma| < 1$. As $|\sigma| < 1$, $Z_C(\lambda)$ defines an entire function of λ.

The Fredholm determinant of $\mathcal{L}_0$ is computed as follows. Let $C_1, \cdots, C_r$ denote the attracting cycles in the Fatou set with periods $p_1, \dots, p_r$ and multipliers $\sigma_1, \dots, \sigma_r$ respectively. Here, for each j,

$$\sigma_j = \prod_{i=1}^{p_j} R'(p_i).$$

Then, by the trace formula, we have the Fredholm determinant

$$\exp\left(-\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \mathrm{Tr}(\mathcal{L}_0^n)\right)$$

$$= \exp\left(-\sum_{n=1}^{\infty} \frac{\lambda^n}{n}\left(D^n + \sum_{p\in \mathrm{Fix}(R^{\circ n})\cap F} \frac{\sigma_p}{1-\sigma_p}\right)\right)$$

$$= (1-D\lambda)\prod_{j=1}^{r} Z_{C_j}(\lambda)$$

$$= (1-D\lambda)\prod_{j=1}^{r}\prod_{k=1}^{\infty}(1-\sigma_j^k \lambda^{p_j}).$$

§7. Eigenfunctions and co-eigenfunctions.

By the general theory of completely continuous operator on a Banach space, the zero's of the Fredholm determinants are related to eigenvalues and eigenfunctions.

<u>Theorem</u> 7.1. All zero's of the Fredholm determinant of $\mathcal{L}_0$ are inverses of eigenvalues of $\mathcal{L}_0$. For each eigenvalue, eigenfunction is a rational function in $\mathcal{O}(\hat{\mathbb{C}}\setminus(\bigcup_{j=1}^{r} C_j))$.

Here, $\mathcal{O}(\hat{\mathbb{C}}\setminus(\bigcup_{j=1}^{r} C_j))$ denote the space of holomorphic functions in the complement of the set of attracting periodic points. For each attracting cycle C_j, we denote the space of functions

$$\mathcal{H}_{C_j} = \{ f \in \mathcal{O}(\hat{\mathbb{C}}\setminus C_j) \mid f(\infty) = 0\}$$

and $\mathcal{H}_{C_\infty} = \mathcal{O}(\hat{\mathbb{C}}\setminus C_\infty)$, then $\mathcal{O}(\hat{\mathbb{C}}\setminus(\bigcup_{j=1}^{r} C_j))$ can be decomposed into a "direct sum"

$$\mathcal{O}(\hat{\mathbb{C}}\setminus(\bigcup_{j=1}^{r} C_j)) = \bigoplus_{j=1}^{r} \mathcal{H}_{C_j}.$$

<u>Proposition</u> 7.2. Eigenfunctions of $\mathcal{L}_0$ belong to $\mathcal{O}(\hat{\mathbb{C}}\setminus(\bigcup C_j))$.

<u>Proposition</u> 7.3. For each j, $\mathcal{H}_{C_j}$ is invariant under $\mathcal{L}_0$.

Proofs of propositions 7.2 and 7.3 are straightforward, and can be obtained by a direct computations.

Proof of Theorem 7.1. In the following, we compute the eigenfunctions for each zero of the Fredholm determinant. From the first factor $1-D\lambda=0$, we have an eigenvalue $D=1/\lambda$, whose eigenfunction is a constant function. Its co-eigenfunction is the Lyubich function defined by

$$\lim_{n\to\infty}\left(\frac{1}{D}\mathcal{L}_0^*\right)^n \chi_x .$$

Next, for eigenvalue λ satisfying $1-\sigma_\delta^k\lambda^{P_\delta}=0$, let us compute its eigenfunction. Here, note that even if the zero of the Fredholm determinant has multiplicities, each zero are related to attracting cycles and can be identified without ambiguities.

First, let us consider an attracting fixed point case. Suppose $p\in\mathbb{C}\cap F$, $p\neq\infty$ is a fixed point of R with multiplier $\sigma=R'(p)$. The period $P=1$. For $k=1$, $\sigma=\frac{1}{\lambda}$ is an eigenvalue and the eigenfunction is given by χ_p+c, with unit pole $\chi_p(z)=\frac{1}{z-p}$ and a constant c computed as follows. For $z\in J$, apply the residue formula to

$$(\mathcal{L}_0\chi_p)(z)=\frac{1}{2\pi i}\int_\gamma \frac{R'(\zeta)}{(R(\zeta)-z)(\zeta-p)}\,d\zeta$$

$$=-\frac{R'(p)}{R(p)-z}+\sum_{q\in R^{-1}(\infty)}\frac{m_q}{q-p}$$

$$=\sigma\chi_p(z)+\sum_{q\in R^{-1}(\infty)} m_q\,\chi_p(q)$$

Here, m_q denotes the order of pole at q. As $\mathcal{L}_0 1=D$, if we set the constant c to

$$c=\frac{1}{\sigma-D}\sum_{q\in R^{-1}(\infty)} m_q\chi_p(q),$$

then

$$\mathcal{L}_0(\chi_p+c)=\sigma\chi_p+(\sigma-D)c+D\cdot c=\sigma(\chi_p+c).$$

This shows that χ_p+c is an eigenfunction of $\mathcal{L}_0$ for eigenvalue σ.

In order to treat the case for $k\geq 2$, we consider the space of "polynomials" in unit pole χ_p.

Let $\mathcal{P}_p = \mathbb{C}[\mathcal{X}_p]$ denote the spae of polynomials in $\mathcal{X}_p$.

Proposition 7.4. $\mathcal{L}_0(\mathcal{P}_p) \subset \mathcal{P}_p$ and the linear map $\mathcal{L}_0$ restricted on $\mathcal{P}_p$ represented by an infinite size matrix with respected the basis $1, \mathcal{X}_p, \mathcal{X}_p^2, \cdots$ is of triangular form.

$$\begin{pmatrix} D & \sum_q m_q \mathcal{X}_p(q) & & & \\ 0 & \sigma & & * & \\ 0 & 0 & \sigma^2 & & \\ \vdots & \vdots & 0 & \sigma^3 & \\ & & \vdots & & \ddots \end{pmatrix}.$$

proof For $k \geq 2$, apply the residue formula to

$$\mathcal{L}_0(\mathcal{X}_p^k)(z) = \frac{1}{2\pi i}\int_\gamma \frac{R'(\zeta)}{(R(\zeta)-z)(\zeta-p)^k}\, d\zeta.$$

The integration path makes a turn around the Julia set of R. As we consider $z \in J$, the inverseimage of z is included in J. So, only poles in the Fatou set are our attracting fixed point p and the poles, say q_j of order m_{q_j}, of R. The residue related to pole q_j is given by $m_q(\mathcal{X}_p(q_j))^k$. The residue at p is comuted as follows. First recall the Cauchy's integration formula for the derivatives

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}\, d\zeta,$$

where $f(z)$ is a holomorphic function and Γ makes a one turn around z. Let γ_p denote the integration path turning around p once in the clockwise direction, the residue

$$\frac{1}{2\pi i}\int_{\gamma_p} \frac{R'(\zeta)}{(R(\zeta)-z)(\zeta-p)^k}\, d\zeta = \frac{-1}{(k-1)!}\left(\left(\frac{d}{d\zeta}\right)^{(k-1)}\left(\frac{R'(\zeta)}{R(\zeta)-z}\right)\right)\Bigg|_{\zeta=p}$$

is a "polynomial" of unit pole $\mathcal{X}_p$ of degree k without constant term. More precisely, this residue is of the form

$$\frac{\sigma^k}{(z-p)^k} + \left(\text{polynomial of } \frac{1}{p-z} \text{ without constant term}\right).$$

Just to show how it is comuted, we try the Laurent expansion.

$$\frac{R'(\zeta)}{(R(\zeta)-z)(\zeta-p)^k} = \frac{1}{(\zeta-p)^k}\, \frac{\sigma + \sum_{j=2}^{\infty} \frac{R^{(j)}(p)}{(j-1)!}(\zeta-p)^{j-1}}{p-z+\sigma(\zeta-p)+\sum_{j=2}^{\infty}\frac{R^{(j)}(p)}{j!}(\zeta-p)^j}$$

$$= \frac{1}{(\zeta-p)^k} \frac{\sigma + \sum_{j=2}^{\infty} \frac{R^{(j)}(p)}{(j-1)!}(\zeta-p)^{j-1}}{(p-z)\left(1+\frac{\sigma}{p-z}(\zeta-p)+\frac{1}{p-z}\sum_{j=2}^{\infty}\frac{R^{(j)}(p)}{j!}(\zeta-p)^j\right)}$$

$$= \frac{1}{(\zeta-p)^k}\frac{1}{p-z}\left(\sigma+\sum_{j=2}^{\infty}\frac{R^{(j)}(p)}{(j-1)!}(\zeta-p)^{j-1}\right)\left(1-\left(\frac{\sigma}{p-z}(\zeta-p)+\frac{1}{p-z}\sum_{j=2}^{\infty}\frac{R^{(j)}(p)}{j!}(\zeta-p)^j\right)\right.$$

$$\left.+\cdots+(-1)^{k-1}\left(\frac{\sigma}{p-z}(\zeta-p)+\frac{1}{p-z}\sum_{j=2}^{\infty}\frac{R^{(j)}(p)}{j!}(\zeta-p)^j\right)^{k-1}+\cdots\right).$$

The coefficient of $\frac{1}{\zeta-p}$ in the Laurent expansin is given by

$$(-1)^{k-1}\frac{\sigma^k}{(p-z)^k} + \left(\text{polynomial of } \text{“}\frac{1}{p-z}\text{”} \text{ of degee } k-1 \text{ without constant term}\right).$$

These coefficients gives the precise values of the upper triangular matrix representation of $\mathcal{L}_0$.

<u>Proposition</u> 7.5 Eigenfunction of $\mathcal{L}_0$ for eigenvalue σ^k related to the fixed point p is a "polynomial" of degree k of χ_p.

<u>proof</u>. The uppertriangular matrix representation shows that, if we restrict $\mathcal{L}_0$ to the subspace spanned by $1, \chi_p, \cdots,$ and χ_p^k is of upper triangular linear map. The eigenvetor for σ^k is a linear compination of the basis vectors, which is a "polynomial" of degree k of the unit pole χ_p.

<u>Proposition</u> 7.6 Co-eigenfunction of $\mathcal{L}_0^*$ for eigenvalue σ^k is given by $((\mathcal{X}^k)'(z))$, where $\mathcal{X}$ is the Schröder's linearlizing function $\mathcal{X}$: (Basin of p) $\longrightarrow \mathbb{C}$, satisfying $\mathcal{X}\circ R = \sigma\cdot\mathcal{X}$.

<u>proof</u> As $\mathcal{X}(R(z)) = \sigma\cdot\mathcal{X}(z)$, Schröder's linearlizing function $\mathcal{X}$ is an eigenfunction of the pull-back operator. For $k=1,2,\cdots$, by differentiating $(\mathcal{X}(R(z)))^k = \sigma^k(\mathcal{X}(z))^k$, we have

$$(\mathcal{X}^k)'(R(z))\cdot R'(z) = \sigma^k\cdot(\mathcal{X}^k)'(z),$$

which shows that the derivatives are eigenfunctions of the pull-back operator of differential forms. Thus we have

$$\mathcal{L}_0^*[(\mathcal{X}^k)'] = \sigma^k((\mathcal{X}^k)'.$$

Let us now continue our proof of theorem 7.1. Next case to consider is the case where the infinity is an attracting fixed point with multiplier $\sigma_\infty \neq 0$. In this case we suppose

$$R(z) \sim \frac{1}{\sigma_\infty} z + a + o(1)$$

as $z \to \infty$. Function $\chi_\infty(z) = z$ is a unit pole at the infinity.

$$\mathcal{L}_0[\chi_\infty](z) = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{(\frac{1}{\sigma_\infty} + o(\frac{1}{\zeta}))\chi_\infty(\zeta)}{\frac{1}{\sigma_\infty}\zeta + a + o(1) - z}\, d\zeta + \sum_{g \in R^{-1}(\infty)\cap\mathbb{C}} m_g \chi_\infty(g).$$

The residue at $\zeta = \infty$ can be computed by using the Laurent expansion as follows.

$$\frac{\frac{1}{\sigma_\infty} + o(\frac{1}{\zeta}) \cdot \chi_\infty(\zeta)}{\frac{1}{\sigma_\infty}\zeta + a + o(1) - z} = \frac{1 + o(\frac{1}{\zeta})}{1 + \sigma_\infty(a-z)\frac{1}{\zeta} + o(\frac{1}{\zeta})}$$

$$= \left(1 + o(\tfrac{1}{\zeta})\right)\left(1 - \sigma_\infty(a-z)\frac{1}{\zeta} + o(\tfrac{1}{\zeta})\right)$$

$$= 1 - \sigma_\infty(a-z)\frac{1}{\zeta} + o(\tfrac{1}{\zeta}).$$

Hence, we have

$$\mathcal{L}_0[\chi_\infty](z) = -\sigma_\infty(a-z) + \sum_{g \in R^{-1}(\infty)\cap\mathbb{C}} m_g\, g$$

$$= \sigma_\infty z - a\sigma_\infty + \sum_g m_g\, g.$$

By adjusting the constant term, we find that

$$\chi_\infty(z) + C = z + \frac{1}{\sigma_\infty - D}\left(\sum_g m_g\, g - a\sigma_\infty\right)$$

is an eigenfunction of $\mathcal{L}_0$ for eigenvalue σ_∞.

For $k = 2, 3, \cdots$, we can try to compute in a similar way.

$$\mathcal{L}_0[\chi_\infty^k](z) = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{(\frac{1}{\sigma_\infty} - o(\frac{1}{\zeta}))(\chi_\infty(\zeta))^k}{\frac{1}{\sigma_\infty}\zeta + a + o(1) - z}\, d\zeta + \sum_{g \in R^{-1}(\infty)\cap\mathbb{C}} m_g(\chi_\infty(g))^k.$$

The Laurent expansion of the integrand near ∞ is given by

$$\frac{\frac{1}{\sigma_\infty} + o(\frac{1}{\zeta}) \cdot \zeta^k}{\frac{1}{\sigma_\infty}\zeta + a + o(1) - z} = \frac{(1 + o(\frac{1}{\zeta}))\zeta^{k-1}}{1 + \sigma_\infty(a-z)\frac{1}{\zeta} + o(\frac{1}{\zeta})}$$

$$= (1 + o(\tfrac{1}{\zeta}))\zeta^{k-1}\left(1 + \left(\frac{\sigma_\infty}{\zeta}z - \frac{\sigma_\infty a}{\zeta} + o(\tfrac{1}{\zeta})\right) + \left(\frac{\sigma_\infty}{\zeta}z - \frac{\sigma_\infty}{\zeta}a + o(\tfrac{1}{\zeta})\right)^2 + \right.$$

$$\left. \cdots + \left(\frac{\sigma_\infty}{\zeta}z - \frac{\sigma_\infty}{\zeta}a + o(\tfrac{1}{\zeta})\right)^k + \cdots \right) = \cdots + (\sigma_\infty^k z^k + \cdots)\frac{1}{\zeta} + \cdots$$

This implies that $\mathcal{L}_0[\chi_\infty^k]$ is a polynomial of z of degree k,

$$\mathcal{L}_0[\chi_\infty^k](z) = \sigma_\infty^k z^k + \cdots .$$

By considering a triangular matrix representation of $\mathcal{L}_0$ on the space of polynomials in z, we see that the eigenfunction related to σ_∞^k is a polynomial of degree k. So, we obtained the following proposition.

<u>proposition</u> 7.7 Eigenfunction of $\mathcal{L}_0$ for eigenvalue σ_∞^k related to the fixed point ∞ is a polynomial of degree k.

In the above, we computed eigenfunctions and co-eigenfunctions. For attracting periodic points, eigenfunctions are easily obtained. For a cycle C of period P, just consider $\mathcal{L}_0^P$, then as periodic points of period P are fixed points of R^P, the above procedure for fixed point can be applied. Consider a factor $(1-\sigma_j^k \lambda^{P_j})$ of the Fredholm determinant. The inverse of a zero of this factor.

$$\frac{1}{\lambda} = (\sigma_j^k)^{\frac{1}{P_j}}$$

If φ is an eigenfunction of a fixed point of R^P, then

$$\Phi_\ell = \varphi + e^{\frac{\ell}{P_j} i}\mathcal{L}_0\varphi + e^{\frac{2\ell}{P_j} i}\mathcal{L}_0\varphi + \cdots + e^{\frac{(P_j-1)\ell}{P_j} i}\mathcal{L}_0\varphi$$

gives an eigenfunction for $\frac{1}{\lambda} = (\sigma_j^k)^{\frac{1}{P_j}} \cdot e^{\frac{\ell}{P_j} i}$, $\ell = 0, \cdots, P_j - 1$.
This completes the proof of Theorem 7.1.

§8. Transfer operator $\mathcal{L}_1 : \mathcal{O}(J) \to \mathcal{O}(J)$.

In the preceeding sections, we considered linear operator $\mathcal{L}_0$. The suffix 0 of $\mathcal{L}_0$ stands for the order of the weight of the derivative, and it will be extended in this section as follows. Transfer operator $\mathcal{L}_1 : \mathcal{O}(J) \to \mathcal{O}(J)$ is defined by

$$(\mathcal{L}_1 f)(z) = \sum_{y \in R^{-1}(z)} \frac{f(y)}{R'(y)}.$$

This operator can be rewritten in an integral operator form.

$$(\mathcal{L}_1 f)(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{R(\zeta)-z}\,d\zeta \qquad \text{for } f \in \mathcal{O}(J).$$

<u>Proposition</u> 8.1 If R is hyperbolic, then the dual operator $\mathcal{L}_1^*: \mathcal{O}_0(F) \to \mathcal{O}_0(F)$ is given by

$$(\mathcal{L}_1^* h)(z) = h(R(z)) - h(R(\infty))$$

for $h \in \mathcal{O}_0(F)$ and $z \in F$.

<u>Remark</u>. The transposed operator ${}^t\mathcal{L}_1$ defines exactly the same operator, i.e.,

$$({}^t\mathcal{L}_1 h)(z) = \frac{1}{2\pi i}\int_\gamma \frac{h(\zeta)}{R(z)-\zeta}\,d\zeta = h(R(z)) - h(R(\infty)).$$

<u>Proof</u>. For $h \in \mathcal{O}_0(F)$ and $z \in F$, a direct computation shows

$$(\mathcal{L}_1^* h)(z) = h^*(\mathcal{L}_1[-\chi_z])$$

$$= h^*\left[\frac{1}{2\pi i}\int_{-\gamma_F} \frac{1}{R(\tau)-\zeta}(-\chi_z(\tau))\,d\tau\right]_\zeta$$

$$= \frac{1}{2\pi i}\int_{\gamma_J} h(\zeta)\left(\frac{1}{2\pi i}\int_{-\gamma_F} \frac{1}{R(\tau)-\zeta}\,\frac{d\tau}{z-\tau}\right)d\zeta$$

$$= \frac{1}{2\pi i}\int_{-\gamma_F} \frac{1}{z-\tau}\left(\frac{1}{2\pi i}\int_{\gamma_J}\frac{h(\zeta)}{R(\tau)-\zeta}\,d\zeta\right)d\tau$$

$$= \frac{1}{2\pi i}\int_{-\gamma_F}\frac{h(R(\tau))}{z-\tau}\,d\tau = h(R(\tau)) - h(R(\infty)).$$

Where suffix ζ in the second line indicates the argument of h^* is considered as a function of ζ. The last equality takes accont the residue at the infinity. If R is a polynomial, then h is holomorphic in F with $h(\infty)=0$ and

$$h(z) = \frac{h_\infty}{z} + o\left(\frac{1}{z}\right),$$

then $\dfrac{h(R(\tau))}{z-\tau} = o\left(\dfrac{1}{z}\right)$, and the residue vanishes. If R is hyperbolic rational map and the infinity is attracting or superattracting fixed point, then the residue at the infinity vanishes as well. If R is hyperbolic and $R(\infty) \neq \infty$, then the residue at the infinity appears. Poles of R does not give rise to residues since $h(z)$ is holomorphic at ∞.

In the above proof, we obtained a nicer expression for $\mathcal{L}_1^*$ in an integral operator form.

Proposition 8.2 If R is a rational hyperbolic map,

$$(\mathcal{L}_1^* h)(z) = \frac{1}{2\pi i}\int_\gamma \frac{h(R(\tau))}{z-\tau}\, d\tau .$$

§9. Trace of $\mathcal{L}_1$.

Our transfer operator $\mathcal{L}_1$ is expressed in an integral operator form. We can compute the trace by applying the Cauchy's residue formula.

Proposition 9.1 If R is a hyperbolic rational map and the infinity is an attracting or superattracting periodic point, then

$$\mathrm{Tr}\,\mathcal{L}_1 = -1 + \sum_{p\in \mathrm{Fix}(R)\cap F} \frac{1}{1-\sigma_p},$$

where $\mathrm{Fix}(R)$ is the set of the fixed points of R and σ_p is the multiplier of fixed point p.

proof If R is a hyperbolic polynomial map, we have

$$\mathrm{Tr}\,\mathcal{L}_1 = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{R(\zeta)-\zeta} = \sum_{p\in \mathrm{Fix}(R)\cap J} \frac{1}{R'(p)-1}$$

$$= \sum_{p\in \mathrm{Fix}(R)\cap F\cap \mathbb{C}} \frac{1}{1-R'(p)}.$$

Note that the term $\frac{1}{1-R'(p)}$ is the so-called holomorphic index. If R is a polynomial of degree $D\geq 2$, the integration around the infinity has no residue and this part can be interpreted as

$$0 = -1 + \frac{1}{1-\sigma_\infty}$$

with $\sigma_\infty = 0$.

If R is a hyperbolic rational map and the infinity is an attracting or super-attracting fixed point with multiplier σ_∞, the residue at ∞ is given by

$$\frac{1}{2\pi i}\int_{\gamma_\infty} \frac{dz}{(\frac{1}{\sigma_\infty}-1)z + o(z)} = \frac{1}{\frac{1}{\sigma_\infty}-1} = \frac{1}{1-\sigma_\infty} - 1.$$

If $R(\infty)$ is finite, then the residue at the infinity is

$$\frac{1}{2\pi i}\int_{\gamma_\infty} \frac{dz}{R(\infty)+o(1)-z} = -1.$$

Note that poles of R do not contribute the trace. Putting all the above cases, we get the proof of proposition 9.1.

Similar arguement for $\mathcal{L}_1^n$ yields the following proposition.

<u>Proposition</u> 9.2 For $n=1,2,$ we have

$$\operatorname{Tr} \mathcal{L}_1^n = -1 + \sum_{p\in \operatorname{Fix}(R^{\circ n})\cap F} \frac{1}{1-\sigma_{p,n}},$$

where $\sigma_{p,n}$ represents the multiplier of $R^{\circ n}$ at p.

§10. Fredholm determinant of $\mathcal{L}_1$

For attracting periodic cycle C of period P with multiplier σ, we defined entire function $Z_C^0(z) = \prod_{k=1}^{\infty}(1-\sigma^k z^P)$ in §6. In this section we define entire function $Y_C(z)$ by

$$Y_C(z) = \exp\left(-\sum_{m=1}^{\infty} \frac{z^{mP}}{mP}\cdot P\cdot \frac{1}{1-\sigma^m}\right)$$
$$= \prod_{k=0}^{\infty}(1-\sigma^k z^P) = (1-z^P)\, Z_C(z).$$

Note that $\quad Y_C(z)/Z_C(z) = 1-z^P.$

<u>Theorem</u> 10.1. The formal Fredholm determinant of $\mathcal{L}_1$

$$\det(I-\lambda\mathcal{L}_1) = \exp\left(-\sum_{n=1}^{\infty}\frac{\lambda^n}{n}\operatorname{Tr}(\mathcal{L}_1^n)\right)$$

converges in $|\lambda|<1$ and extends to an entire function in λ.

Proof. Let $C_1, \cdots, C_r$ denote the attracting cycles of R with periods $p_1, \cdots, p_r$ respectively and with multipliers $\sigma_1, \cdots, \sigma_r$. The Fredholm determinant can be decomposed into factors related to each attracting cycle, since the trace of $\mathcal{L}_1^n$ is given as a sum of holomorphic invariants of periodic points of period p which divides n. For cycle C with (prime) period p and multiplier σ, the holomorphic invariant of the periodic points in this cycle appear in the trace of $\mathcal{L}_1^{mp}$ for $m=1,2,\ldots$. Hence, the factor related to such a cycle is given by

$$\exp\left(-\sum_{m=1}^{\infty} \frac{\lambda^{mp}}{mp}\cdot p\cdot \frac{1}{1-\sigma^m}\right) = \Upsilon_c(\lambda),$$

here note that there are p points in cycle C. Each cycle $C_1, \cdots, C_r$ gives rise to such a factor and the -1 term in $\mathcal{L}_1^n$ gives another factor. Putting these together, we get

$$\det(I-\lambda\mathcal{L}_1) = \frac{1}{1-\lambda}\prod_{j=1}^{r} \Upsilon_{c_j}(\lambda)$$
$$= \frac{1}{1-\lambda}\prod_{j=1}^{r}\prod_{k=0}^{\infty}(1-\sigma_j^k\lambda^{p_j}).$$

If R has more than one attracting cycle, the denominator $(1-\lambda)$ cancels out. As $|\sigma_j|<1$ for all attracting cycle, this expression as an infinite product gives an entire function.

§11. Eigenfunctions of $\mathcal{L}_1$

The inverses of zeros of the Fredholm determinant are eigenvalues of $\mathcal{L}_1$. The eigenfunctions are found in $\mathcal{O}(\hat{\mathbb{C}}\setminus(\mathrm{Per}(R)\cap F))$.

Lemma 11.1 $\quad \mathcal{L}_1\chi_p = \chi_{R(p)} - \chi_{R(\infty)}$

Proof.
$$(\mathcal{L}_1\chi_p)(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{(R(\zeta)-z)(\zeta-p)}$$
$$= \frac{1}{z-R(p)} + \frac{1}{R(\infty)-z} = \chi_{R(p)} - \chi_{R(\infty)}.$$

In order to consider the eigenfunctions of $\mathcal{L}_1$, let us begin with the simplest case with $k=0$.

If $p \in \mathbb{C} \cap F$ is a finite attracting fixed point, i.e., $P=1$, $p \neq \infty$, $R(p)=p$, $R'(p)=\sigma$. By the above lemma,

$$\mathcal{L}_1 \chi_p = \chi_{R(p)} - \chi_{R(\infty)} = \chi_p - \chi_{R(\infty)}.$$

In this case, if the infinity is an attracting or super-attracting fixed point, then $R(\infty)=\infty$ and $\chi_{R(\infty)} \equiv 0$, hence χ_p is an eigenfunction for eigenvalue $\lambda = 1$, i.e.,

$$\mathcal{L}_1 \chi_p = \chi_p.$$

If the infinity is an attracting or superattracting periodic point and not a fixed point, let P_∞ denote the period of the infinity and let $q_0 = \infty$,

$$q_1 = R(\infty),\ q_2 = R(q_1),\ \cdots,\ q_{P_\infty - 1} = R(q_{P_\infty-2}).$$

We have, for $j=1, \cdots, P_\infty - 2$,

$$(\mathcal{L}_1 \chi_{q_j})(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{(R(\zeta)-z)(\zeta - q_j)}$$

$$= \chi_{R(q_j)}(z) - \chi_{R(\infty)}(z) = \chi_{q_{j+1}}(z) - \chi_{q_1}(z).$$

And $$(\mathcal{L}_1 \chi_{q_{P_\infty-1}})(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{(R(\zeta)-z)(\zeta - q_{P_\infty-1})} = -\chi_{q_1}(z).$$

Hence $\chi_p - \frac{1}{P_\infty}(\chi_{q_1} + \cdots + \chi_{q_{P_\infty-1}})$ gives the eigenfunction of $\mathcal{L}_1$ for $\lambda = 1$, related to the fixed point p.

In fact. $$\mathcal{L}_1 (\chi_p - \frac{1}{P_\infty}(\chi_{q_1} + \cdots + \chi_{q_{P_\infty-1}}))$$

$$= \chi_{R(p)} - \chi_{R(\infty)} - \frac{1}{P_\infty}(\chi_{q_2} + \cdots + \chi_{q_{P_\infty-1}} - (P_\infty - 1)\chi_{q_1})$$

$$= \chi_p - \chi_{q_1} + \frac{P_\infty - 1}{P_\infty}\chi_{q_1} - \frac{1}{P_\infty}(\chi_{q_2} + \cdots + \chi_{q_{P_\infty-1}})$$

$$= \chi_p - \frac{1}{P_\infty}(\chi_{q_1} + \cdots + \chi_{q_{P_\infty-1}}).$$

Next, we consider the case $P \geq 2$ and the attracting periodic cycle $C = \{p_1, \cdots, p_P\}$ does not contain the infinity. Let σ denote the multiplier of the cycle and consider the case of $k=0$. In this case, the factor $(1-\lambda^P)$ of the Fredholm determinant is concerned.

It is easily verified that

$$\frac{1}{p}(\chi_{p_1}+\cdots+\chi_{p_p})-\frac{1}{p_\infty}(\chi_{q_1}+\cdots+\chi_{q_{p_\infty-1}})$$

is an eigenfunction for eigenvalue $\lambda=1$ related to this attracting cycle. In fact,

$$\mathcal{L}_1\left(\frac{1}{p}(\chi_{p_1}+\cdots+\chi_{p_p})-\frac{1}{p_\infty}(\chi_{q_1}+\cdots+\chi_{q_{p_\infty-1}})\right)$$

$$=\frac{1}{p}(\mathcal{L}_1\chi_{p_1}+\cdots+\mathcal{L}_1\chi_{p_p})-\frac{1}{p_\infty}(\mathcal{L}_1\chi_{q_1}+\cdots+\mathcal{L}_1\chi_{q_{p_\infty-1}})$$

$$=\frac{1}{p}((\chi_{p_2}-\chi_{q_1})+\cdots+(\chi_{p_p}-\chi_{q_1})+(\chi_{p_1}-\chi_{q_1}))$$

$$-\frac{1}{p_\infty}((\chi_{q_2}-\chi_{q_1})+\quad+(\chi_{q_{p_\infty-1}}-\chi_{q_1})+(-\chi_{q_1}))$$

$$=\frac{1}{p}(\chi_{p_1}+\cdots+\chi_{p_p})-\chi_{q_1}-\frac{1}{p_\infty}((\chi_{q_1}+\cdots+\chi_{q_{p_\infty-1}})-((p_\infty-1)+1)\chi_{q_1})$$

$$=\frac{1}{p}(\chi_{p_1}+\cdots+\chi_{p_p})-\frac{1}{p_\infty}(\chi_{q_1}+\cdots+\chi_{q_{p_\infty-1}}).$$

For eigenvalue λ satisfying $\lambda^p=1$ and $\lambda=1$, the eigenfunction for eigenvalue λ is given by

$$\sum_{j=1}^{p}\bar{\lambda}^j\chi_{p_j},$$

in fact, $\mathcal{L}_1(\sum_{j=1}^{p}\bar{\lambda}^j\chi_{p_j})=\sum_{j=1}^{p-1}\bar{\lambda}^j\chi_{p_{j+1}}+\bar{\lambda}^p\chi_{p_p}-(\sum_{j=1}^{p}\bar{\lambda}^j)\chi_{q_1}$

$$=\lambda(\sum_{j=1}^{p}\bar{\lambda}^j\chi_{p_j}).$$

Now, let us consider the attracting cycle containg the infinity of period $p_\infty\geq 2$, with $k=0$. We denote the multiplier of this cycle by σ_∞. As in the preceding consideration, we set $q_1=R(\infty)$, $q_2=R(q_1)$, $\cdots$, $q_{p_\infty-1}=R(q_{p_\infty-2})$, and $\infty=q_0=R(q_{p_\infty-1})$. The operator $\mathcal{L}_1$ maps the linear space spanned by $\chi_{q_1},\chi_{q_2},\cdots,\chi_{q_{p_\infty-1}}$ into itself and the matrix representation is given by the following.

$$(\mathcal{L}_1\chi_{q_1},\cdots,\mathcal{L}_1\chi_{q_{p_\infty-1}})=(\chi_{q_1},\cdots,\chi_{q_{p_\infty-1}})\begin{pmatrix}-1&-1&\cdots&-1\\1&0&\cdots&0\\&1&\ddots&\vdots\\&&\ddots&\vdots\\0&&1&0\end{pmatrix}$$

This space is invariant under $\mathcal{L}_1$ and the eigenvalues related to this invariant space is given by equation

$$\begin{vmatrix} \lambda+1 & 1 & \cdots & & 1 \\ -1 & \lambda & 0 & \cdots & 0 \\ 0 & -1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & \lambda \end{vmatrix} = \frac{\lambda^{p_\infty}-1}{\lambda-1} = 0.$$

The eigenvalues are those λ with $\lambda^{p_\infty}=1$ and $\lambda \neq 1$. And $\sum_{j=1}^{p_\infty-1} \lambda^{-j}\chi_{q_j}$ is an eigenfunction for λ.

In fact,

$$\mathcal{L}_1\left(\sum_{j=1}^{p_\infty-1} \lambda^{-j}\chi_{q_j}\right) = \sum_{j=1}^{p_\infty-2} \lambda^{-j}\chi_{q_{j+1}} - \left(\sum_{j=1}^{p_\infty-1} \lambda^{-j}\right)\chi_{q_1}$$
$$= \lambda\left(\sum_{j=1}^{p_\infty-1} \lambda^{-j}\chi_{q_j}\right).$$

We go back to the attracting finite fixed point case.

In order to proceed to $k \geq 1$ cases, let us compute $\mathcal{L}_1\chi_p^{k+1}$!

<u>proposition</u> 11.2 $\mathcal{L}_1\chi_p^{k+1}$ is a polynomial of χ_p of degree $k+1$ without terms of degree 1 or 0. The coefficient of the leading term χ_p^{k+1} is equal to σ^k, i.e.,

$$\mathcal{L}_1\chi_p^{k+1} = \sigma^k\chi_p^{k+1} + a_k\chi_p^k + \cdots + a_2\chi_p^2.$$

<u>proof</u>. Apply the Cauchy's formula for derivatives.

$$(\mathcal{L}_1\chi_p^{k+1})(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{(R(\zeta)-z)(\zeta-p)^{k+1}} = \frac{-1}{k!}\left(\frac{d^k}{d\zeta^k}\left(\frac{1}{R(\zeta)-z}\right)\right)\Bigg|_{\zeta=p}$$

Inductively, we got

$$((R(\zeta)-z)^{-1})' = -(R(\zeta)-z)^{-2}R'(\zeta)$$

$$((R(\zeta)-z)^{-1})'' = (-1)^2 2!(R(\zeta)-z)^{-3}(R'(\zeta))^2 - (R(\zeta)-z)^{-2}R''(\zeta)$$

$$\cdots$$

$$((R(\zeta)-z)^{-1})^{(k)} = (-1)^k k!\,(R(\zeta)-z)^{-k-1}(R'(\zeta))^k + \cdots$$
$$\cdots + (R(\zeta)-z)^{-2}R^{(k)}(\zeta).$$

Hence,

$$\mathcal{L}_1\chi_p^{k+1}(z) = \frac{-1}{k!}\cdot(-1)^k k!\left(\frac{1}{p-z}\right)^{k+1}\sigma^k + \cdots + \left(\frac{1}{p-z}\right)^2 \cdot R^{(k)}(p)$$
$$= \left(\frac{1}{z-p}\right)^{k+1}\sigma^k + \cdots + \left(\frac{1}{z-p}\right)^2 R^{(k)}(p) = \sigma^k\chi_p^{k+1}(z) + \cdots + R^{(k)}(p)\,\chi_p^2(z).$$

<u>Theorem</u> 11.3 If p is an attracting fixed point of R with $p \neq \infty$, then $\mathcal{L}_1$ maps the space spanned by $\chi_p^2, \cdots, \chi_p^{k+1}$ is mapped into itself for $k = 1, 2, \cdots$. For each $k = 1, 2, \cdots$, eigenfunction for σ^k is found in this space as a polynomial of χ_p of degree $k+1$.

<u>Proof</u> By proposition 11.2, $\mathcal{L}_1$ restricted to the space spanned by $\chi_p^2, \cdots, \chi_p^{k+1}$ is represented by a triangular matrix with diagonal components $\sigma, \sigma^2, \cdots, \sigma^k$. The eigenfunction belonging to σ^k is given by a linear combination of $\chi_p^2, \cdots, \chi_p^{k+1}$.

In order to proceed to the case $k \geq 1$ and $\ell \geq 2$, we consider a bounded attracting cycle $C = \{p_1, \cdots, p_\ell\}$ with multiplier σ.

<u>Proposition</u> 11.4. $\mathcal{L}_1 \chi_{p_j}^{k+1}$ is a polynomial of $\chi_{p_{j+1}}$ (we set $p_{\ell+1} = p_1$) of degree $k+1$ without first order term nor constant term. The coefficient of degree k term is $(R'(p_j))^k$.

<u>Proof</u>. The proof is similar to proposition 11.2. We apply the Cauchy's integral formula.

$$(\mathcal{L}_1 \chi_{p_j}^{k+1})(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{(R(\zeta) - z)(\zeta - p_j)^{k+1}}$$

$$= \frac{-1}{k!} \left(\frac{d^k}{d\zeta^k} \left(\frac{1}{R(\zeta) - z} \right) \Big|_{\zeta = p_j} \right)$$

$$= (R'(p_j))^k \chi_{p_{j+1}}^{k+1}(z) + \cdots + R^{(2)}(p_j) \chi_{p_{j+1}}^2 (z).$$

<u>Theorem</u> 11.5. Let $C = \{p_1, \cdots, p_\ell\}$ be a bounded attracting cycle with multiplier σ, $0 < |\sigma| < 1$, and of period ℓ. For $k \geq 1$ and for eigenvalue λ satisfying $\lambda^\ell = \sigma^k$, the eigenfunction of $\mathcal{L}_1$ is given by a sum of polynomials of χ_{p_j}'s of degrees $k+1$ without linear and constant terms.

More precisely, the eigenfunction for this eigenvalue is obtained as follows.

Periodic points $p_1, \cdots, p_\ell$ of R are fixed points of $R^{\circ\ell}$. Let

$$\varphi_j = \chi_{p_j}^{k+1} + \cdots$$

denote the monic polynomial of χ_{p_j} which is an eigenfunction of $\mathcal{L}_1^\ell$ obtained as in theorem 11.3 of attracting fixed point case. Then, by looking at the coefficient of the leading terms, we see that

$$\mathcal{L}_1 \varphi_j = (R'(p_j))^k \varphi_{j+1}.$$

Hence the sum

$$\varphi_1 + \lambda^{-1} R'(p_1)^k \varphi_2 + \cdots + \lambda^{-\ell+1} (R'(p_1) \cdots R'(p_{\ell-1}))^k \varphi_\ell$$

gives an eigenfunction of $\mathcal{L}_1$ fore eigenvalue λ (with $\lambda^\ell = \sigma^k$). This complete theorem 11.5.

To complete the comutation of eigenfunctions of $\mathcal{L}_1$, we examine the cases of the cycle containing the infinity. The case of $\ell_\infty = 1$ and $k = 0$ is already treated. In the case where the infinity is an attracting fixed point of R with multiplier σ_∞, the constant function is an eigenfunction. In fact,

$$(\mathcal{L}_1 1)(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{R(\zeta) - z} = \sigma_\infty.$$

If the infinity is an attracting periodic point of period ℓ_∞ and multiplier σ_∞, then the infinity is a fixed point of $R^{\circ \ell_\infty}$ and

$$\mathcal{L}_1^{\ell_\infty} 1 = \sigma_\infty.$$

By the way, as a direct computation shows, we have

$$\mathcal{L}_1 1 = -a_1 \chi^2_{R(\infty)},$$

where a_1 is the Laurent coefficient of $R(\zeta)$ expanded as a function of ζ near $\zeta = \infty$, i.e.,

$$R(\zeta) = R(\infty) + a_1 \zeta^{-1} + a_2 \zeta^{-2} + \cdots.$$

By a change of variables $\zeta = \frac{1}{w}$, $d\zeta = -\frac{1}{w^2} dw$, the integration path γ is transformed into a loop around the origin of w-plane turning in the clockwise direction.

So, we have

$$(\mathcal{L}_1 1)(z) = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{d\zeta}{R(\infty) - z} = \frac{1}{2\pi i}\int_{\gamma_0} \frac{-1}{w^2} \frac{dw}{R(\infty) + a_1 w + \cdots - z}$$

$$= \frac{1}{2\pi i}\int_{\gamma_0} \frac{1}{w^2} \frac{-1}{R(\infty) - z}\left(1 - \frac{a_1}{R(\infty)-z} w + \cdots \right) dw = -\frac{a_1}{(R(\infty) - z)^2}.$$

For $g = R^{\circ(p_\infty - 1)}(\infty)$, $R(g) = \infty$, we have

$$(\mathcal{L}_1 \chi_g^2)(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{(R(\zeta) - z)(\zeta - g)^2}$$

$$= \frac{1}{2\pi i}\int_{\gamma_g} \frac{d\zeta}{\left(\left(\frac{b_{-1}}{\zeta - g} + b_0 + b_1(\zeta - g) + \cdots\right) - z\right)(\zeta - g)^2}$$

$$= \frac{1}{2\pi i}\int_{\gamma_g} \frac{1}{b_{-1}(\zeta - g)}\left(1 - \frac{b_0 - z}{b_{-1}}(\zeta - g) + \cdots \right) d\zeta = \frac{-1}{b_{-1}},$$

where

$$R(\zeta) = \frac{b_{-1}}{\zeta - g} + b_0 + b_1(\zeta - g) + \cdots$$

is the Laurent expansion of R near $\zeta = g$. The eigenfunction for this attractive cycle is constructed as in the previous case. We have, as the eigenfunction,

$$1 + \lambda^{-1}(-a_1)\chi^2_{R(\infty)} + \lambda^{-2}(-a_1)\cdot R'(R(\infty))\,\chi^2_{R^2(\infty)} +$$

$$\cdots + \lambda^{-p_\infty + 1} R'(R(\infty)) \cdots R'(R^{\circ(p_\infty - 2)}(\infty))\,\chi^2_{R^{\circ(p_\infty - 1)}(\infty)}.$$

Finally, we compute the eigenfunctions related to the multiplier of the unbounded attracting cycle for $k \geq 2$. In the case where the infinity is an attractive (and not superattracting) fixed point of R with multiplier σ_∞ ($0 < |\sigma_\infty| < 1$), let

$$R(\zeta) = \frac{\zeta}{\sigma_\infty} + a_0 + a_{-1}\zeta^{-1} + a_{-2}\zeta^{-2} + \cdots$$

be the Laurent expansion of $R(\zeta)$ near the infinity. Then, compute the Laurent expansion of $\frac{1}{R(\zeta) - z}$ with respect to ζ near the infinity.

$$\frac{1}{R(\zeta) - z} = \frac{1}{\frac{\zeta}{\sigma_\infty} + a_0 - z + a_{-1}\zeta^{-1} + a_{-2}\zeta^{-2} + \cdots}$$

$$= \frac{\sigma_\infty}{\zeta} \frac{1}{1 - ((z - a_0)\sigma_\infty \zeta^{-1} - a_{-1}\sigma_\infty \zeta^{-2} - a_{-2}\sigma_\infty \zeta^{-3} + \cdots)}$$

$$= \frac{\sigma_\infty}{\zeta}\{1 + ((z-a_0)\sigma_\infty \zeta^{-1} - a_1\sigma_\infty\zeta^{-2} - a_{-2}\sigma_\infty\zeta^{-3} - \cdots)$$
$$+ ((z-a_0)\sigma_\infty\zeta^{-1} - \cdots)^2 + (\cdots)^3$$
$$+ \cdots + (\qquad)^k + \cdots \}$$
$$= \frac{\sigma_\infty}{\zeta}\left(1 + \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}(z-a_0)^m\sigma_\infty^m \times \sum_{s=0}^{n-m}\sigma_\infty^s \sum_{\substack{i_1,\dots,i_s \\ i_1+\cdots+i_s = n-m-s}} \prod_{t=1}^{s}(-a_{-i_t})\right)\zeta^{-n}\right)$$

With this formula, we have the following proposition.

__Proposition__ 11.6. For $k \geq 2$,

$$\left(\mathcal{L}_1[(z-a_0)^{k-1}]\right)(z) = \frac{1}{2\pi i}\int_\gamma \frac{(\zeta-a_0)^{k-1}}{R(\zeta)-z}\,d\zeta$$

is a polynomial in $(z-a_0)$ of degree $k-1$, whose coefficient of the leading term is σ_∞^k.

~~proof~~ $(\mathcal{L}_1[z^{k-1}])(z) = \frac{1}{2\pi i}\int_\gamma \frac{\zeta^{k-1}}{R(\zeta)-z}\,d\zeta$

is the coefficient of the Laurent expansion of $\frac{1}{R(\zeta)-z}$ with respect to ζ of degree $-k$, which is a polynomial in $(z-a_0)$ of the form (the coefficient of ζ^{-k} $(n=k-1)$)

$$\sigma_\infty^k(z-a_0)^{k-1} + \cdots.$$

Hence, for polynomial $(z-a_0)^{k-1}$, we have

$$\mathcal{L}_1[(z-a_0)^{k-1}] = \sigma_\infty^k(z-a_0)^{k-1} + \cdots.$$

The space of polynomials in $(z-a_0)$ is invariant under $\mathcal{L}_1$ and $\mathcal{L}_1$ is of the triangular matrix form with respect to the basis $\{(z-a_0)^{k-1}\}_{k\geq 1}$. And the diagonal components of this triangular matrix are σ_∞^k, $(k \geq 1)$.

__Theorem__ 11.7. If $P_\infty = 1$, for each $k \geq 1$, the eigenfunction of $\mathcal{L}_1$ for eigenvalue σ_∞^k is given by a polynomial in $(z-a_0)$ of degree $k-1$.

__Proof__ This theorem is just a restatement of proposition 11.6

Finally, the last piece of this section is to compute

the eigenfunctions in the case of $P_\infty \geq 2$ and $k > 1$.
All the eigenfunctions for λ with $\lambda^{P_\infty} = \sigma_\infty^k$ are constructed in the space of polynomials and poles at the orbit of ∞. The pricedure to obtain these eigenfunctions are similar to the case of $k=1$.

§12. Co-eigenfunctions.

The Fredholm determinat of $\mathcal{L}_1$ was

$$\det(I - \lambda \mathcal{L}_1) = \frac{1}{1-\lambda} \prod_{j=1}^{r} \prod_{k=0}^{\infty} (1 - \sigma_j^k \lambda^{P_j}),$$

where σ_j is the multiplier of attracting periodic cycle $C_j = \{P_0, \cdots, P_{P_j - 1}\}$ of period P_j. The dual operator of $\mathcal{L}_1$ has a canonical representation as an integral operator on a space of holomorphic functions via the Cauchy's transformation. As we verified in section 8, this dual operator is given as the transpose of the kernel of the integral operator. Hence the Fredholm determinant of $\mathcal{L}_1$ and its dual $\mathcal{L}_1^*$ are exactly same entire functions.

<u>Theorem</u> 12.1. Every zero of the Fredholm determinant is the inverse of eigenvalue of $\mathcal{L}_1^*$.

<u>proof</u> We compute the co-eigenfunctions in the following. (The case of $k=0$ will be treated afterwards.) For each attracting cycle C_j an $k \geq 1$, let $P = P_j$ denote the period of the cycle and $\sigma = \sigma_j$ its multiplier. Let Ψ_0 denote the Schröder's linearizing function of $R^{\circ P}$ at p_0, i.e.,

- Ψ_0 is defined and holomorphic in the attractive basin of p_0 with respect to $R^{\circ P}$ satisfying the Schröder's function equation

$$\Psi_0(R^{\circ P}(z)) = \sigma \Psi_0(z), \quad \Psi_0(p_0) = 0, \ \Psi_0'(p_0) = 1.$$

For the sake of manipulation of such functions we set

- $\Psi_0(z) = 0$ for z in the complement of the attractive basin of p_0.

Clearly, Schröder's function is an eigenfunction of pull-back operator operating on a space of functions. For $k \geq 1$, we see that ψ_0^k is an eigen function for eigenvalue σ^k,

$$(\psi_0(R^{\circ \ell}(z)))^k = \sigma^k(\psi_0(z))^k.$$

Let λ be a zero of Fredholm determinant satisfying $\lambda^\ell = \sigma^k$. Define functions ψ_m for $m = 1, 2, \ldots, \ell$ by

$$\psi_m(z) = \lambda^{m-\ell}\psi_0(R^{\circ(\ell-m)}(z)),$$

and let $\Psi_k(z) = \sum_{m=1}^{\ell}(\psi_m(z))^k$. Then we have

$$\begin{aligned}
(\mathcal{L}_1^* \Psi_k)(z) &= \Psi_k(R(z)) = \sum_{m=1}^{\ell}(\psi_m(R(z)))^k \\
&= \sum_{m=1}^{\ell}\lambda^{m-\ell}(\psi_0(R^{\circ(\ell-m+1)}(z)))^k \\
&= \sum_{m=0}^{\ell-1}\lambda^{m+1-\ell}(\psi_0(R^{\circ(\ell-m)}(z)))^k \\
&= \lambda\Psi_k(z) + \lambda^{1-\ell}(\psi_0(R^{\circ\ell}(z)))^k - \lambda(\psi_0(z))^k = \lambda\Psi_k(z).
\end{aligned}$$

This shows that Ψ_k is the eigenfunction of $\mathcal{L}_1$ (more precisely, Ψ_k should be decorated by λ and j).

For $k = 0$, the characteristic function of the attractive basin of the periodic cycle gives the co-eigenfunction, except for the cycle containing the infinity (because we restricted our space of functions $\mathcal{O}_0(F)$ by $f(\infty) = 0$).

We have $(r-1)$ linearly independent co-eigenfunctions for eigenvalue $\lambda = 1$.

If $\ell > 1$, then for $k = 0$, eigenfunctions for λ, with $\lambda^\ell = 1$, are constructed by a linear combination

$$\Psi_\ell(z) = \lambda^\ell X_0(R^{\circ(\ell-\ell)}(z)),$$

where $X_0(z)$ is the characteristic function of the attractive basin of p_0.

For $\ell_\infty > 1$ and $k = 0$, eigenfunction for eigenvalue λ with $\lambda^{\ell_\infty} = 1$ is constructed as follows. The cycle containing the infinity consists of periodic points, say $q_0 = \infty, q_1, q_2, \ldots, q_{\ell_\infty - 1}$. We denote by $X_j(z)$, $j = 0, \ldots, \ell_\infty - 1$, the characteristic function the attractive basin of g_j under $R^{\circ P_\infty}$. Define constants $h_0 = 0$, $h_1 = 1$, $h_{j+1} = \lambda h_j + h_1$, $j = 1, \cdots, P_\infty - 2$, and $h_{P_\infty - 1} = -\bar{\lambda}$. Then, the piecewise constant function

$$\Psi_\lambda(z) = \sum_{j=0}^{P_\infty - 1} h_j \chi_j(z)$$

gives the desired eigenfunction. This completes the construction of co-eigenfunctions.

§13. Artin-Mazur's dynamical ζ-function.

The famous ζ-function of Artin-Mazur is defined by

$$\zeta(\lambda) = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \#(\mathrm{Fix}(R^n) \cap J)\right).$$

We examined the Fredholm determinants of our transfer operators $\mathcal{L}_1$ and $\mathcal{L}_2$.

Theorem 13.1. The Artin-Mazur's dynamical ζ-function is given by

$$\zeta(\lambda) = \frac{\det(I - \lambda\mathcal{L}_0)}{\det(I - \lambda\mathcal{L}_1)}.$$

Proof. As we computed by the trace formula,

$$\det(I - \lambda\mathcal{L}_0) = (1 - D\lambda) \prod_{j=1}^{r} \prod_{k=1}^{\infty} (1 - \sigma_j^k \lambda^{P_j})$$

and

$$\det(I - \lambda\mathcal{L}_1) = \frac{1}{1-\lambda} \prod_{j=1}^{r} \prod_{k=0}^{\infty} (1 - \sigma_j^k \lambda^{P_j}).$$

And for traces, we have proved

$$\mathrm{Tr}(\mathcal{L}_0^n) = D^n + \sum_{p \in \mathrm{Fix}(R^{\circ n}) \cap F} \frac{\sigma_p}{1 - \sigma_p}$$

$$= \sum_{x \in \mathrm{Fix}(R^{\circ n}) \cap J} \frac{(R^{\circ n})'(x)}{(R^{\circ n})'(x) - 1},$$

$$\mathrm{Tr}(\mathcal{L}_1^n) = -1 + \sum_{p \in \mathrm{Fix}(R^{\circ n}) \cap F} \frac{1}{1 - \sigma_p}$$

$$= \sum_{x \in \mathrm{Fix}(R^{\circ n}) \cap J} \frac{1}{1 - (R^{\circ n})'(x)}$$

By comparing these traces, we have

$$\mathrm{Tr}(\mathcal{L}_0^n) - \mathrm{Tr}(\mathcal{L}_1^n) = \#\,(\mathrm{Fix}(R^{\circ n}) \cap J).$$

Hence we conclude that

$$\zeta(\lambda) = \exp\left(- \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \left(\mathrm{Tr}(\mathcal{L}_0^n) - \mathrm{Tr}(\mathcal{L}_1^n) \right) \right)$$

$$= \frac{\det (I - \lambda \mathcal{L}_0)}{\det (I - \lambda \mathcal{L}_1)}$$

$$= \frac{(1-\lambda)(1-D\lambda)}{\prod_{j=1}^{r} (1-\lambda^{p_j})}.$$

We remark that the final result itself can be obtained easily by a simple observation

$$\#(\mathrm{Fix}(R^{\circ n}) \cap J) = D^n + 1 - \#\left(\begin{array}{c}\text{attractive period pt.}\\ \text{of period } n\end{array}\right).$$

However, this result gives a relationship between the dynamical property on the Julia set and the information with respect to the attractors in the Fatou set.