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Hyperbolicity of critically finite maps on complex projective plane

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This is the abstract of my talk in the conference held at RIMS, September 3-6 2007. The results obtained in [M1] and [M2] will be explained.

Our main purpose is to give a necessary and sufficient condition for a critically finite map on complex projective plane to be Axiom A. This is helpful to understand the dynamics of a map $f$, which is obtained by a small perturbation of an Axiom A critically finite map $f_0$.

1 Repellers

We denote by $\mathbb{P}^k$ complex projective space of complex dimension $k(\geq 1)$ and by $\omega$ Fubini-Study form such that $\int_{\mathbb{P}^k} \omega^k = 1$. For a holomorphic self-map $f$ of $\mathbb{P}^k$, we define the degree of $f$ by the formula

$$\deg(f) := \int_{\mathbb{P}^k} f^* \omega \wedge \omega^{k-1}.$$ 

Because the dynamics of degree 1 maps can be understood by linear algebra, in this paper, we will focus on the case when $\deg(f) \geq 2$. Let $C$ denote the critical set of $f$. We consider the closure of the post-critical set and the critical limit set for $f$ which are respectively defined by

$$D := \bigcup_{n \geq 1} f^n(C), \ E := \bigcap_{n \geq 1} \bigcup_{i \geq n} f^i(C).$$

In this section, we will study the dynamics on invariant compact sets outside $D$. We will describe a 'semi-repelling' structure of such invariant compact sets.

**Definition 1.1.** Let $f$ be a holomorphic self-map of $\mathbb{P}^k$ of degree $\geq 2$. Let $T_p$ denote the holomorphic tangent space at $p \in \mathbb{P}^k$ and let $|\cdot|$ denote Fubini-Study metric.

We say that $p \in \mathbb{P}^k$ is repelling for $f$ if and only if

$$\min_{v \in T_p, |v|=1} |Df^j(v)| \to +\infty$$

as $j \to +\infty$, where $Df$ denote the derivative of $f$. 
We say that a compact set $K$ in $\mathbb{P}^{k}$ is a repeller for $f$ if and only if $f(K) = K$ and there are constants $c > 0$, $\lambda > 1$ such that

$$|Df^n(v)| \geq c\lambda^n |v|$$

for all $v \in \bigcup_{p \in K} T_p$ and all $n \geq 1$.

Let $D$ denote the unit disk in $\mathbb{C}$. We say that a holomorphic embedding $\phi : D \rightarrow \mathbb{P}^k$ is a Fatou disk if and only if $\{f^n \circ \phi\}_{n \geq 1}$ is a normal family in $D$. We say that a Fatou disk $\phi : D \rightarrow \mathbb{P}^k$ is noncontractive if and only if every limit map of $\{f^n \circ \phi\}_{n \geq 1}$ is nonconstant.

By the following theorem, we describe a 'semi-repelling' structure of an invariant compact set outside $D$, in terms of repelling points and Fatou disks.

**Theorem 1.2.** Let $f$ be a holomorphic self-map of $\mathbb{P}^k$ of degree $\geq 2$. Let $K$ be a compact set in $\mathbb{P}^k$ such that $f(K) \subset K$ and $K \cap D = \emptyset$. Then, there are subsets $K^u$, $K^c \subset K$ which satisfy the following properties:

(i) $K^u \cup K^c = K$, $K^u \cap K^c = \emptyset$;

(ii) $f(K^u) \subset K^u$, $f(K^c) \subset K^c$;

(iii) each point in $K^u$ is repelling;

(iv) for each $p \in E^c$, there is a noncontractive Fatou disk through $p$.

Moreover, if $f(K) = K$ and $K^c = \emptyset$, then $K$ is a repeller.

## 2 Maps with sparse critical orbits

Let $f$ be a holomorphic self-map of $\mathbb{P}^k$ of degree $\geq 2$. As in case when $k = 1$, we will consider the Fatou set and the Julia set for $f$.

**Definition 2.1.** We define the Fatou set $F$ for $f$ to be the domain of normality for the sequence of the iterates $\{f^n\}_{n \geq 1}$ and define the Julia set $J$ as $J := \mathbb{P}^k \setminus F$.

The limit $T := \lim_{n \rightarrow +\infty} \frac{1}{d^n}(f^*)^n \omega$ exists and we call $T$ the Green $(1,1)$ current for $f$. The $p$-fold wedge product $T^p := T \wedge \cdots \wedge T$ is called the Green $(p,p)$ current for $f$ and the support

$$J_p := \text{supp}(T^p)$$

is called the $p$-th Julia set.
By Fornæss-Sibony and Ueda, it is shown that $J_1 = J$. By Briend-Duval, it is shown that

$$J_k \subset \{\text{repelling periodic points}\}.$$

Interestingly, if $k \geq 2$, it is possible that $J_k$ is a proper subset of the one on the right hand side. So, when we study Axiom A maps in higher dimensions, we cannot avoid considering this phenomenon.

**Definition 2.2.** Let $f$ be a holomorphic self-map of $\mathbb{P}^k$ of degree $\geq 2$. We say that $f$ is critically finite if and only if $D$ is algebraic. We say that $f$ is critically sparse if and only if $D$ is pluripolar. (Obviously, critically finite maps are critically sparse.)

When $f$ is critically sparse, we can show that $J_k$ is the 'precise' locus of the distribution of repelling periodic points for $f$. Actually, we have a stronger theorem as follows.

**Theorem 2.3.** Suppose that $f$ is critically sparse. Then, all repellers for $f$ are contained in $J_k$. In particular,

$$J_k = \{\text{repelling periodic points}\}.$$

This theorem seems useful in many cases, not only for critically finite maps. For instance, let us see the following application.

**Example 2.4.** Let $P$ be a polynomial self-map of $\mathbb{C}^k$ of degree $\geq 2$ which extends holomorphically on $\mathbb{P}^k$. We put

$$K(P) := \{w \in \mathbb{C}^k \mid \{P^n(w)\}_{n \geq 0} \text{ is bounded}\}.$$

Suppose that $K(P) \cap C = \emptyset$, where $C$ is the critical set of (the extended) $P$. Since $K(P)$ is a repeller and $P$ is critically sparse in $\mathbb{P}^k$, we can apply Theorem 2.3. Hence, we obtain $K(P) = J_k$.

### 3 Critically finite maps and hyperbolicity

In this section, we will deal with holomorphic self-maps of $\mathbb{P}^2$. Our philosophy in this section is that a good behavior of critical orbits implies a good structure of global dynamics.

**Definition 3.1.** Let $f$ be a holomorphic self-map of $\mathbb{P}^2$ of degree $\geq 2$. (Then, $f$ is not invertible.)

Let $S$ be a surjectively forward invariant compact set in $\mathbb{P}^2$. We say that $S$ is hyperbolic if and only if the tangent bundle over the space $\tilde{S}$ of histories of points in $S$ has a hyperbolic splitting structure.

We say that $f$ is Axiom A if and only if the nonwandering set $\Omega$ for $f$ is hyperbolic and equals to the closure of the set of periodic points of $f$. 
When $f$ is Axiom A, we consider the decomposition of the nonwandering set
\[ \Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \]
where $\Omega_i$ is the part of unstable dimension $i$.

The following theorem states that a good behavior of critical orbits implies a good structure of Fatou set.

**Theorem 3.2.** Let $f$ be a holomorphic self-map of $\mathbb{P}^2$ of degree $\geq 2$. Suppose that $J \cap E$ is a hyperbolic set. Then, the Fatou set $F$ consists of the attractive basins for finitely many attracting cycles. Moreover, if the unstable dimension of $J \cap E$ is 1, then
\[ E = \{\text{attracting periodic points}\} \cup \bigcup_{\hat{p} \in J \cap E} W^u(\hat{p}) \]
where $W^u(\hat{p})$ is the unstable manifold for $\hat{p}$.

**Remark 3.3.** Theorem 3.2 is still true if we replace $J \cap E$ with the nonwandering part of $J \cap E$. Note that the hyperbolicity of the nonwandering part of $J \cap E$ is a necessary condition for $f$ to be Axiom A.

**Remark 3.4.** The first part of Theorem 3.2 can be generalized in any dimension $\geq 2$.

By integrating results above, we obtain our main theorems:

**Theorem 3.5.** Let $f$ be a critically finite map on $\mathbb{P}^2$. Then, $f$ is Axiom A if and only if $J \cap E$ is a hyperbolic set of unstable dimension 1.

**Theorem 3.6.** Let $f$ be a critically finite map on $\mathbb{P}^2$ which is Axiom A. Then, the following (1) - (7) hold:

1. all irreducible components of $E$ are rational;
2. $J_2$ is connected;
3. $\Omega_2 = J_2$;
4. $\Omega_1 = J \cap E$;
5. $\Omega_0 = \{\text{attracting periodic points}\} \neq \emptyset$;
6. $E = \{\text{attracting periodic points}\} \cup \bigcup_{p \in J \cap E} W^u(p)$;
7. $J = J_2 \cup \bigcup_{p \in J \cap E} W^*(p)$.

**Remark 3.7.** The degree of an irreducible component $X$ of $E$ can be any integer $\geq 1$. Thus, $X$ is not necessarily smooth.
References


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