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Random complex dynamics and rational semigroups

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Abstract

We investigate the random dynamics of rational maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of semigroups of rational maps on $\hat{\mathbb{C}}$. We will see that the both fields are related to each other very deeply. Moreover, we investigate singular functions in the complex plane.

1 Introduction

In this paper, we investigate the random dynamics of rational maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of semigroups of rational maps on $\hat{\mathbb{C}}$. We will see that the both fields are related to each other very deeply.

One of the motivations of the research of complex dynamical systems is to describe some mathematical models on ethology. For example, the behavior of the population of a certain species can be described as the dynamical system of a polynomial $f(z) = az(1-z)$ such that f preserves the unit interval and the postcritical set in the plane is bounded (cf. [7]). However, according to the change of the natural environment, some species have several strategies to survive in the nature. From this point of view, it is very important to consider the random dynamics of such polynomials.

In order to consider the random dynamics of polynomials on $\hat{\mathbb{C}}$, let $T_\infty(z)$ be the probability of tending to $\infty \in \hat{\mathbb{C}}$ starting with the initial value

$z \in \hat{\mathbb{C}}$. In this paper, we will see that under some condition, the function $T_\infty : \hat{\mathbb{C}} \to [0,1]$ is continuous on $\hat{\mathbb{C}}$ and has some singular properties (for instance, varies only inside a fractal set, so called the Julia set of a polynomial semigroup), and this function is a complex analogue of the devil's staircase (Cantor function) or Lebesgue's singular functions (see figure 2, 3, 4). Moreover, in this paper we will see that under some condition, for any fixed $z \in \hat{\mathbb{C}}$, $T_\infty(z)$ is real-analytic with respect to the probability parameter, and the partial derivative of $T_\infty(z)$ with respect to the probability parameter can be regarded as a complex analogue of the Takagi function (see figure 5). Before going into the detail, let us recall the definition of the devil's staircase (Cantor function), Lebesgue's singular function, and the Takagi function.

Definition 1.1 ([25]). Let $\varphi : \mathbb{R} \to [0,1]$ be the unique bounded function which satisfies the following functional equation:

$$\frac{1}{2}\varphi(3x) + \frac{1}{2}\varphi(3x-2) \equiv \varphi(x), \ \varphi|_{(-\infty,0]} \equiv 0, \ \varphi|_{[1,+\infty)} \equiv 1. \tag{1}$$

The function $\varphi|_{[0,1]} : [0,1] \to [0,1]$ is called the **devil's staircase (or Cantor function)**.

Remark 1. The above $\varphi : \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R}$ and varies only on the Cantor middle third set. Moreover, it is monotone (see figure 1).

Definition 1.2 ([25]). Let $0 < a < 1$ be a constant. We denote by $\psi_a : \mathbb{R} \to [0,1]$ the unique bounded function which satisfies the following functional equation:

$$a\psi_a(2x) + (1-a)\psi_a(2x-1) \equiv \psi_a(x), \ \psi_a|_{(-\infty,0]} \equiv 0, \ \psi_a|_{[1,+\infty)} \equiv 1 \tag{2}$$

The function $L_a := \psi_a|_{[0,1]} : [0,1] \to [0,1]$ is called **Lebesgue's singular function** with respect to the parameter a.

Remark 2. $\psi_a : \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R}$ and monotone. Moreover, for almost every $x \in [0,1]$ with respect to the one-dimensional Lebesgue measure, the derivative of ψ_a at x is equal to zero (see figure 1). Moreover, in [13], it was shown that for each fixed $x \in [0,1]$, the function $a \mapsto L_a(x)$ is real-analytic on $(0,1)$.

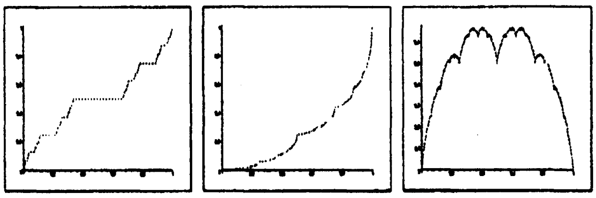
Definition 1.3 ([25]). Let $0 < a < 1$ be a constant. We denote by $\phi : \mathbb{R} \to \mathbb{R}$ be the unique bounded function which satisfies the following functional equation:

$$\frac{1}{2}\phi(2x)+\frac{1}{2}\phi(2x-1)+\psi_{1/2}(2x)-\psi_{1/2}(2x-1) \equiv \phi(x), \ \phi|_{(-\infty,0]\cup[1,+\infty)} \equiv 0. \tag{3}$$

The function $S := \frac{\phi}{2}|_{[0,1]} : [0,1] \to \mathbb{R}$ is called the **Takagi function**.

Remark 3. The Takagi function is continuous on $[0, 1]$ but non-differentiable at every point of $[0, 1]$ (see figure 1). Moreover, in [11], it was shown that the function $x \mapsto \frac{1}{2}\frac{\partial L_a(x)}{\partial a}|_{a=1/2}$ on $[0, 1]$ is equal to the Takagi function.

Figure 1: (From left) Devil's staircase, Lebesgue's singular function, Takagi function.



These singular functions defined on $[0, 1]$ can be redefined by using the random dynamical systems on $\mathbb{R}$ as follows. Let $f_1(x) := 3x, f_2(x) := 3(x-1)+1$ $(x \in \mathbb{R})$ and we consider the random dynamical system on $\mathbb{R}$ such that at every step we choose f_1 with probability $1/2$ and f_2 with probability $1/2$. We set $\hat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We denote by $T_{+\infty}(x)$ the probability of tending to $+\infty \in \hat{\mathbb{R}}$ starting with the initial value $x \in \mathbb{R}$. Then, we can see that the function $T_{+\infty}|_{[0,1]} : [0, 1] \to [0, 1]$ is equal to the devil's staircase.

Similarly, let $g_1(x) := 2x, g_2(x) := 2(x-1)+1$ $(x \in \mathbb{R})$ and let $0 < a < 1$ be a constant. We consider the random dynamical system on $\mathbb{R}$ such that at every step we choose the map g_1 with probability a and the map g_2 with probability $1-a$. Let $T_{+\infty,a}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$. Then, we can see that the function $T_{+\infty,a}|_{[0,1]} : [0, 1] \to [0, 1]$ is equal to Lebesgue's singular function L_a with respect to the parameter a. Therefore, as in Remark 3, the function $x \mapsto \frac{1}{2}\frac{\partial T_{+\infty,a}(x)}{\partial a}|_{a=1/2}$ defined on $[0, 1]$ is equal to the Takagi function. In particular, the Takagi function is equal to the half of the partial derivative of the function of probability of tending to $+\infty$ with respect to the probability parameter a.

In fact, it is easy to show that $T_{+\infty}, T_{+\infty,a}$ and $\frac{\partial T_{+\infty,a}(x)}{\partial a}|_{a=1/2}$ satisfies (1), (2), and (3), respectively. We remark that in most of the researches, the theory of random dynamical systems has not been used directly, in order to investigate these singular functions on the interval, although some researchers have used it implicitly.

One of the main purposes of this paper is to consider the complex analogue of the above story. In order to do that, we have to investigate the i.i.d. random dynamics of rational maps and the dynamics of semigroups of rational maps on $\hat{\mathbb{C}}$, simultaneously. We will develop both the theory of random dynamics of rational maps and that of the dynamics of semigroups of rational maps. The author thinks this is the best strategy. In fact, when we want to investigate the i.i.d. random dynamics of rational maps, we need to investigate the dynamics of semigroups of rational maps, and when we want to investigate the dynamics of semigroups of rational maps, we need to investigate the i.i.d. random dynamics of rational maps.

2 Preliminaries

In this section, we give some basic definitions and notations on the dynamics of semigroups of rational maps and the i.i.d. random dynamics of rational maps.

A **rational semigroup** is a semigroup generated by a family of non-constant rational maps on the Riemann sphere $\hat{\mathbb{C}}$ with the semigroup operation being functional composition([12, 10]). A **polynomial semigroup** is a semigroup generated by a family of non-constant polynomial maps.

Definition 2.1 ([12, 10]). Let G be a rational semigroup.

- The **Fatou set** of G is defined to be $F(G) :=$ $\{z \in \hat{\mathbb{C}} \mid \exists \text{ nbd } U \text{of } z \text{ s.t. } \{g|_U : U \to \hat{\mathbb{C}}\}_{g\in G} \text{ is equicontinuous on } U\}$.
- The **Julia set** of G is defined to be $J(G) := \hat{\mathbb{C}} \setminus F(G)$.
- If G is generated by $\{g_i\}_i$, then we write $G = \langle g_1, g_2, \ldots\rangle$.
- For a rational map g, we set $J(g) := J(\langle g\rangle)$.

Lemma 2.2. *Let G be a rational semigroup. Then for each $h \in G$, $h(F(G)) \subset F(G)$ and $h^{-1}(J(G)) \subset J(G)$. Note that the equality does not hold in general.*

Definition 2.3. We set $\text{Rat} : = \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-constant rational map}\}$ endowed with the topology induced by uniform convergence on $\hat{\mathbb{C}}$. Moreover, we set $\mathcal{Y} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g \text{ is a polynomial}, \deg(g) \geq 2\}$ endowed with the relative topology from Rat.

Definition 2.4. For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\text{Rat})^{\mathbb{N}}$, we set

$$F_\gamma := \{z \in \hat{\mathbb{C}} \mid \exists \text{ nbd } U \text{ of } z \text{ s.t. } \{\gamma_n \circ \cdots \circ \gamma_1\}_{n\in\mathbb{N}} \text{ is equicontinuous on } U\}$$

and $J_\gamma := \hat{\mathbb{C}} \setminus F_\gamma$. This J_γ is called the **Julia set** of the sequence γ.

Definition 2.5. For a topological space X, we denote by $\mathfrak{M}_1(X)$ the space of all Borel probability measures on X endowed with the weak topology. Note that if X is a compact metric space, then $\mathfrak{M}_1(X)$ is a compact metric space.

For any $\tau \in \mathfrak{M}_1(\text{Rat})$, we will consider the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a rational map according to τ.

Definition 2.6. Let $\tau \in \mathfrak{M}_1(\text{Rat})$.

1. We denote by supp τ the support of τ. Moreover, we set $X_\tau := (\text{ supp } \tau)^{\mathbb{N}}$ $(= \{\gamma = (\gamma_1, \gamma_2, \ldots) \mid \gamma_j \in \text{supp } \tau\})$ endowed with the product topology. Furthermore, we set $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau$. This is a Borel probability measure on X_τ. We denote by G_τ the rational semigroup generated by supp τ.

2. Let $C(\hat{\mathbb{C}})$ be the Banach space of all continuous functions on $\hat{\mathbb{C}}$ endowed with the supremum norm. Let M_τ be the operator on $C(\hat{\mathbb{C}})$ defined by $M_\tau(\varphi)(z) := \int_{\text{supp}\,\tau} \varphi(g(z))\, d\tau(g)$. Moreover, let $(M_\tau)_* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the dual of M_τ. Thus for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ and each open subset V of $\hat{\mathbb{C}}$, we have $(M_\tau)_*(\mu)(V) = \int_{\text{supp}\,\tau} \mu(g^{-1}(V))\, d\tau(g)$.

3. We denote by $F_{meas}(\tau)$ the set of $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ satisfying that there exists a neighborhood B of μ in $\mathfrak{M}_1(\hat{\mathbb{C}})$ such that the sequence $\{(M_\tau)_*^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n\in\mathbb{N}}$ is equicontinuous on B.

4. We set $J_{meas}(\tau) := \mathfrak{M}_1(\hat{\mathbb{C}}) \setminus F_{meas}(\tau)$.

5. We denote by $F^0_{meas}(\tau)$ the set of $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ satisfying that the sequence$\{(M_\tau)_*^n : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n\in\mathbb{N}}$ is equicontinuous at the one point μ. Note that $F_{meas}(\tau) \subset F^0_{meas}(\tau)$.

6. We set $J^0_{meas}(\tau) := \mathfrak{M}_1(\hat{\mathbb{C}}) \setminus F^0_{meas}(\tau)$.

Definition 2.7. Let $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the topological embedding defined by: $\Phi(z) := \delta_z$, where δ_z denotes the Dirac measure at z. Using this topological embedding $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$, we regard $\hat{\mathbb{C}}$ as a compact subset of $\mathfrak{M}_1(\hat{\mathbb{C}})$.

Definition 2.8. Let $\tau \in \mathfrak{M}_1(Rat)$. Regarding $\hat{\mathbb{C}}$ as a compact subset of $\mathfrak{M}_1(\hat{\mathbb{C}})$ as above, we use the following notation.

1. We denote by $F_{pt}(\tau)$ the set of $z \in \hat{\mathbb{C}}$ satisfying that there exists a neighborhood B of z in $\hat{\mathbb{C}}$ such that the sequence$\{(M_\tau)_*^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n\in\mathbb{N}}$ is equicontinuous on B.

2. We set $J_{pt}(\tau) := \hat{\mathbb{C}} \setminus F_{pt}(\tau)$.

3. Similarly, we denote by $F^0_{pt}(\tau)$ the set of $z \in \hat{\mathbb{C}}$ such that the sequence $\{(M_\tau)^n_*|_{\hat{\mathbb{C}}} : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n\in\mathbb{N}}$ is equicontinuous at the one point $z \in \hat{\mathbb{C}}$. Note that $F_{pt}(\tau) \subset F^0_{pt}(\tau)$.

4. We set $J^0_{pt}(\tau) := \hat{\mathbb{C}} \setminus F^0_{pt}(\tau)$.

Remark 4. We have $F_{pt}(\tau) \subset F^0_{pt}(\tau)$, $F_{meas}(\tau) \subset F^0_{meas}(\tau)$, $J^0_{pt}(\tau) \subset J_{pt}(\tau) \cap J^0_{meas}(\tau)$, and $J^0_{meas}(\tau) \subset J_{meas}(\tau)$.

Remark 5. If supp $\tau = \{h\}$ with $h \in \mathrm{Rat}$ and $\deg(h) \geq 2$, then $J_{meas}(\tau) \neq \emptyset$. In fact, using the embedding $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$, we have $\emptyset \neq \Phi(J(h)) \subset J_{meas}(\tau)$.

Definition 2.9. Let $\tau \in \mathfrak{M}_1(\mathcal{Y})$. For each $z \in \hat{\mathbb{C}}$, we set

$$T_{\infty,\tau}(z) := \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{Y}^{\mathbb{N}} \mid \gamma_n \circ \cdots \circ \gamma_1(z) \to \infty \text{ as } n \to \infty\}).$$

This is the **probability of tending to ∞ starting with the initial value** $z \in \hat{\mathbb{C}}$, with respect to the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a polynomial map according to the probability distribution τ.

The following is the key to investigate the random complex dynamics.

Definition 2.10. Let G be a rational semigroup. We set $J_{\ker}(G) := \bigcap_{g\in G} g^{-1}(J(G))$. This is called the **kernel Julia set** of G.

Remark 6.

1. $J_{\ker}(G)$ is a compact subset of $J(G)$.

2. For each $h \in G$, $h(J_{\ker}(G)) \subset J_{\ker}(G)$. If $F(G) \neq \emptyset$, then int $J_{\ker}(G) = \emptyset$.

3. If G is generated by a single map or if G is a group, then $J_{\ker}(G) = J(G)$. However, for a general rational semigroup G, it may happen that $J_{\ker}(G) \neq J(G)$.

Definition 2.11. We denote by Leb_2 the two-dimensional Lebesgue measure on $\hat{\mathbb{C}}$.

3 Results

In this section, we present the main results.

Theorem 3.1. *Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ be such that supp τ is compact. Suppose that $J_{\ker}(G_\tau) = \emptyset$. Then, $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$, and for almost every $\gamma \in (\mathrm{Rat})^{\mathbb{N}}$ with respect to $\tilde{\tau}$, $\mathrm{Leb}_2(J_\gamma) = 0$.*

By Theorem 3.1, we obtain the following result.

Theorem 3.2. *Let $\tau \in \mathfrak{M}_1(\mathcal{Y})$ be such that* $\mathrm{supp}\,\tau$ *is compact. Suppose that $J_{\ker}(G_\tau) = \emptyset$. Then, the function $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ is continuous on the whole $\hat{\mathbb{C}}$.*

Remark 7. Let $h \in \mathcal{Y}$ and let $\tau := \delta_h$. Then, $T_{\infty,\tau}(\hat{\mathbb{C}}) = \{0,1\}$ and $T_{\infty,\tau}$ is not continuous at every point in $J(h) \neq \emptyset$.

On the one hand, we have the following, due to Vitali's theorem.

Lemma 3.3. *Let $\tau \in \mathfrak{M}_1(\mathcal{Y})$ be such that* $\mathrm{supp}\,\tau$ *is compact. Then, for each connected component U of $F(G_\tau)$, there exists a constant $C_U \in [0,1]$ such that $T_{\infty,\tau}|_U \equiv C_U$.*

Remark 8. Higher dimensional version of Theorem 3.1 can be shown. Moreover, higher dimensional (and modified) version of Theorem 3.2 and Lemma 3.3 can be shown. However, we omit the detail.

Remark 9. Combining Theorem 3.2 and Lemma 3.3, it follows that under the assumption of Theorem 3.2, if $T_{\infty,\tau} \not\equiv 1$, then the function $T_{\infty,\tau}$ is continuous on $\hat{\mathbb{C}}$ and varies only inside the Julia set $J(G_\tau)$ of G_τ. In this case, the function $T_{\infty,\tau}$ is called the **devil's coliseum** (see figure 3, 4). This is a complex analogue of the devil's staircase or Lebesgue's singular functions. We will see the monotonicity of this function $T_{\infty,\tau}$ in Theorem 3.6.

In order to present the result on the monotonicity of the function $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$, the level set of $T_{\infty,\tau}|_{J(G_\tau)}$ and the structure of the Julia set $J(G_\tau)$, we need the following notations.

Definition 3.4. Let K_1, K_2 be two non-empty compact subsets of $\hat{\mathbb{C}}$.

1. "$K_1 <_s K_2$" indicates that K_1 is included in the union of all bounded components of $\mathbb{C} \setminus K_2$.

2. "$K_1 \leq_s K_2$" indicates that $K_1 <_s K_2$ or $K_1 = K_2$.

Remark 10. This "$\leq_s$" is a partial order in the space of all non-empty compact subsets of $\hat{\mathbb{C}}$. This "$\leq_s$" is called the **surrounding order.**

Definition 3.5. Let G be a polynomial semigroup. We set

$$\hat{K}(G) := \{z \in \mathbb{C} \mid \{g(z) \mid g \in G\} \text{ is bounded in } \mathbb{C}\}.$$

Moreover, if $\infty \in F(G)$, then we denote by $F_\infty(G)$ the connected component of $F(G)$ containing ∞. (Note that if G is a polynomial semigroup generated by a compact subset of $\mathcal{Y}$, then $\infty \in F(G)$.)

By Theorem 3.2 and Lemma 3.3, we obtain the following result.

Theorem 3.6. *(Monotonicity of $T_{\infty,\tau}$ and the structure of $J(G_\tau)$) Let $\tau \in \mathfrak{M}_1(\mathcal{Y})$ be such that supp τ is compact. Suppose that $T_{\infty,\tau} \not\equiv 1$ on $\hat{\mathbb{C}}$ and $J_{\ker}(G_\tau) = \emptyset$. Then, we have all of the following.*

1. $\mathrm{int}(\hat{K}(G_\tau)) \neq \emptyset$.
2. $T_{\infty,\tau}(J(G_\tau)) = [0,1]$.
3. *For each $t_1, t_2 \in [0,1]$ with $0 \leq t_1 < t_2 \leq 1$, we have* $T_{\infty,\tau}^{-1}(\{t_1\}) <_s T_{\infty,\tau}^{-1}(\{t_2\}) \cap J(G_\tau)$.
4. *For each $t \in (0,1)$, we have* $\hat{K}(G_\tau) <_s T_{\infty,\tau}^{-1}(\{t\}) \cap J(G_\tau) <_s \overline{F_\infty(G_\tau)}$.

Remark 11. If G is generated by a single map $h \in \mathcal{Y}$, then $\partial\hat{K}(G) = \partial F_\infty(G) = J(G)$ and so $\hat{K}(G)$ and $\overline{F_\infty(G)}$ cannot be separated. However, under the assumption of Theorem 3.6, the theorem implies that $\hat{K}(G)$ and $\overline{F_\infty(G)}$ are separated by the uncountably many level sets $\{T_{\infty,\tau}|_{J(G_\tau)}^{-1}(\{t\})\}_{t\in(0,1)}$, and that these level sets are totally ordered with respect to the surrounding order, respecting the usual order in $(0,1)$.

Definition 3.7. Let G be a rational semigroup.

- We set $P(G) := \overline{\bigcup_{g\in G}\{\text{all critical values of } g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}$ $(\subset \hat{\mathbb{C}})$. This is called the **postcritical set** of G.
- We say that G is **hyperbolic** if $P(G) \subset F(G)$.
- For a polynomial semigroup G, we set $P^*(G) := P(G) \setminus \{\infty\}$.

Definition 3.8. Let G be a rational semigroup.

- Let N be a positive integer. We denote by $SH_N(G)$ the set of points $z \in \hat{\mathbb{C}}$ satisfying that there exists a positive number δ such that for each $g \in G$, $\deg(g : V \to B(z,\delta)) \leq N$, for each connected component V of $g^{-1}(B(z,\delta))$. Moreover, we set $UH(G) := \hat{\mathbb{C}} \setminus \cup_{N \in \mathbb{N}} SH_N(G)$.

- We say that G is **semihyperbolic** if $UH(G) \subset F(G)$.

Remark 12. We have $UH(G) \subset P(G)$. If G is hyperbolic, then G is semi-hyperbolic.

The following theorem generalizes some results in [2, 4].

Theorem 3.9. *Let $\tau \in \mathfrak{M}_1(\mathcal{Y})$ be such that* $\operatorname{supp}\tau$ *is compact. Suppose that $P^*(G_\tau)$ is not bounded in $\mathbb{C}$. Then, for almost every $\gamma \in \mathcal{Y}^{\mathbb{N}}$ with respect to $\tilde{\tau}$, the Julia set J_γ of γ has uncountably many connected components.*

Question 1. What happens if $T_{\infty,\tau} \equiv 1$?

We present a necessary and sufficient condition for $T_{\infty,\tau}$ to be the constant function 1.

Lemma 3.10. *Let $\tau \in \mathfrak{M}_1(\mathcal{Y})$ be such that* $\operatorname{supp}\tau$ *is compact. Then, the following are equivalent.*

1. $T_{\infty,\tau} \equiv 1$.
2. $T_{\infty,\tau}|_{J(G_\tau)} \equiv 1$.
3. $\hat{K}(G_\tau) = \emptyset$.

Definition 3.11. Let $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{Y}^{\mathbb{N}}$. We set

$$K_\gamma := \{z \in \mathbb{C} \mid \{\gamma_n \circ \cdots \circ \gamma_1(z)\}_{n \in \mathbb{N}} \text{ is bounded in } \mathbb{C}\}.$$

Theorem 3.12. *Let $\tau \in \mathfrak{M}_1(\mathcal{Y})$ be such that* $\operatorname{supp}\tau$ *is compact. Suppose that $T_{\infty,\tau}|_{J(G_\tau)} \equiv 1$ (for example, suppose $\hat{K}(G_\tau) = \emptyset$). Then, we have all of the following 1,2, and 3.*

1. $J_{\ker}(G_\tau) = \emptyset$.
2. $T_{\infty,\tau} \equiv 1$ *on* $\hat{\mathbb{C}}$.
3. *For almost every $\gamma \in \mathcal{Y}^{\mathbb{N}}$ with respect to $\tilde{\tau}$,*
    (a) $\mathrm{Leb}_2(K_\gamma) = 0$,
    (b) $K_\gamma = J_\gamma$, *and*

(c) $K_\gamma = J_\gamma$ *has uncountably many connected components.*

Even if $J_{\ker}(G_\tau) \neq \emptyset$, we have the following.

Theorem 3.13. *Let* $\tau \in \mathfrak{M}_1(\mathcal{Y})$ *be such that supp* τ *is compact. Suppose that* $J_{\ker}(G_\tau)$ *is included in the unbounded component of* $\mathbb{C} \setminus (UH(G_\tau) \cap J(G_\tau))$. *Then, for almost every* $\gamma \in X_\tau$ *with respect to* $\tilde{\tau}$, $\mathrm{Leb}_2(J_\gamma) = 0$.

Corollary 3.14. *Let* $\tau \in \mathfrak{M}_1(\mathcal{Y})$ *be such that supp* τ *is compact. Suppose that* $J_{\ker}(G_\tau)$ *is included in the unbounded component of* $\mathbb{C} \setminus (P(G_\tau) \cap J(G_\tau))$. *Then, for almost every* $\gamma \in X_\tau$ *with respect to* $\tilde{\tau}$, $\mathrm{Leb}_2(J_\gamma) = 0$.

Question 2. When $J_{\ker}(G) = \emptyset$?

Lemma 3.15. *Let* Γ *be a subset of* Rat *such that the interior of* Γ *with respect to the topology of* Rat *is not empty. Let* G *be a rational semigroup generated by* Γ. *Suppose that* $F(G) \neq \emptyset$. *Then,* $J_{\ker}(G) = \emptyset$.

Lemma 3.16. *Let* Γ *be a subset of* $\mathcal{Y}$ *such that the interior of* Γ *with respect to the topology of* $\mathcal{Y}$ *is not empty. Let* G *be a polynomial semigroup generated by* Γ. *Then,* $J_{\ker}(G) = \emptyset$.

Definition 3.17. For a metric space X, we denote by $\mathrm{Cpt}(X)$ the space of all non-empty compact subsets of X, endowed with the Hausdorff topology.

Lemma 3.18. *Let* $\tau \in \mathfrak{M}_1(\mathcal{Y})$ *be such that* $\mathrm{supp}\,\tau$ *is compact. Let* B_1 *be any neighborhood of* τ *in* $\mathfrak{M}_1(\mathcal{Y})$ *and* B_2 *any neighborhood of* $\mathrm{supp}\,\tau$ *in* $\mathrm{Cpt}(\hat{\mathbb{C}})$. *Then, there exists an element* $\rho \in \mathfrak{M}_1(\mathcal{Y})$ *such that* $\rho \in B_1$, $\mathrm{supp}\,\rho \in B_2$, $\sharp \mathrm{supp}\,\rho < \infty$, *and* $J_{\ker}(G_\rho) = \emptyset$.

Theorem 3.19. *Let* G *be a polynomial semigroup generated by a subset of* $\mathcal{Y}$. *Suppose that* $P^*(G)$ *is bounded in* $\mathbb{C}$ *and* $J(G)$ *is disconnected. Then* $J_{\ker}(G) = \emptyset$ *and* $T_{\infty,\tau} \not\equiv 1$ *for each* $\tau \in \mathfrak{M}_1(\mathcal{Y})$ *with* $G_\tau = G$.

Proposition 3.20. *Let* $(h_1, \dots, h_m) \in (\mathrm{Rat})^m$ *and suppose* $\deg(h_j) \geq 2$ *for each* $j = 1, \dots, m$. *Let* $G = \langle h_1, \dots, h_m \rangle$. *Suppose that* $J_{\ker}(G) = \emptyset$ *and* G *is hyperbolic. Then there exists an open neighborhood* U *of* $(h_1, \dots, h_m)$ *in* $(\mathrm{Rat})^m$ *such that for each* $(g_1, \dots, g_m) \in U$, *we have* $J_{\ker}(\langle g_1, \dots, g_m \rangle) = \emptyset$.

Question 3. What happens if $J_{\ker}(G_\tau) \neq \emptyset$?

Theorem 3.21. *Let* $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ *be such that* supp τ *is compact. Suppose that* G_τ *is semihyperbolic,* $F(G_\tau) \neq \emptyset$, *and for each* $g \in$ supp τ, $\deg(g) \geq 2$. *Then, we have all of the following.*

1. $\mathrm{Leb}_2(J^0_{pt}(\tau)) = 0$.

2. $J_{\ker}(G_\tau) \subset J^0_{pt}(\tau)$.

3. *If, in addition to the assumption,* $J_{\ker}(G_\tau) \neq \emptyset$, *then* $J_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$.

Corollary 3.22. *Let* $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ *be such that* $\mathrm{supp}\,\tau$ *is compact. Suppose that* G_τ *is hyperbolic and for each* $g \in \mathrm{supp}\,\tau$, $\deg(g) \geq 2$. *Then, we have all of the following.*

1. $\mathrm{Leb}_2(J^0_{pt}(\tau)) = 0$.

2. $J_{\ker}(G_\tau) \subset J^0_{pt}(\tau)$.

3. *If, in addition to the assumption,* $J_{\ker}(G_\tau) \neq \emptyset$, *then* $J_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$.

Theorem 3.23. *Let* $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ *be such that* $\sharp\,\mathrm{supp}\,\tau < \infty$. *Suppose that* G_τ *is semihyperbolic,* $F(G_\tau) \neq \emptyset$, *and for each* $g \in \mathrm{supp}\,\tau$, $\deg(g) \geq 2$. *Then, we have all of the following.*

1. $\cup_{g \in G_\tau} g^{-1}(J_{\ker}(G_\tau)) \subset J^0_{pt}(\tau)$.

2. *Either* $J_{meas}(\tau) = \emptyset$ *or* $J_{pt}(\tau) = J(G_\tau)$.

Corollary 3.24. *Let* $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ *be such that* $\sharp\,\mathrm{supp}\,\tau < \infty$. *Suppose that* G_τ *is hyperbolic and for each* $g \in \mathrm{supp}\,\tau$, $\deg(g) \geq 2$. *Then, we have all of the following.*

1. $\cup_{g \in G_\tau} g^{-1}(J_{\ker}(G_\tau)) \subset J^0_{pt}(\tau)$.

2. *Either* $J_{meas}(\tau) = \emptyset$ *or* $J_{pt}(\tau) = J(G_\tau)$.

We now present some results on the case $\sharp\,\mathrm{supp}\,\tau = 2$.

Definition 3.25. We use the following notation.

- $\mathcal{B} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid P^*(\langle h_1, h_2 \rangle)$ is bounded in $\mathbb{C}\}$.
- $\mathcal{C} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid J(\langle h_1, h_2 \rangle)$ is connected$\}$.
- $\mathcal{D} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid J(\langle h_1, h_2 \rangle)$ is disconnected$\}$.
- $\mathcal{H} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid \langle h_1, h_2 \rangle$ is hyperbolic$\}$.
- $\mathcal{I} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid J(h_1) \cap J(h_2) \neq \emptyset\}$.
- $\mathcal{Q} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid J(h_1) = J(h_2)$, and $J(h_1)$ and $J(h_2)$ are quasicircles$\}$.

- For each $(h_1, h_2) \in \mathcal{Y}^2$, $0 < p < 1$, and $z \in \hat{\mathbb{C}}$, we set

$$T(h_1, h_2, p, z) := T_{\infty, p\delta_{h_1} + (1-p)\delta_{h_2}}(z).$$

Lemma 3.26. *The sets $\mathcal{H}, \mathcal{B} \cap \mathcal{H}, \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$ are non-empty open subsets of $\mathcal{Y}^2$.*

Theorem 3.27. *We have all of the following.*

1. *There exists a neighborhood U of $(\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$ in $\mathcal{Y}^2$ such that for each $(h_1, h_2) \in U$, $J_{\ker}(\langle h_1, h_2 \rangle) = \emptyset$ and $T(h_1, h_2, p, \cdot) \not\equiv 1$ (hence we can apply Theorem 3.1, 3.2, and 3.6) .*

2. *There exists a neighborhood U of $(\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$ in $\mathcal{Y}^2$ such that for each $(h_1, h_2) \in U$, $\dim_H(J(\langle h_1, h_2 \rangle)) < 2$. Here, $\dim_H$ denotes the Hausdorff dimension with respect to the spherical metric.*

3. *We have $\overline{\mathrm{int}(\mathcal{C})} \cap \mathcal{B} \cap \mathcal{H} = \mathcal{C} \cap \mathcal{B} \cap \mathcal{H}$.*

4. *Suppose that $h_1 \in \mathcal{Y}$, $P^*(\langle h_1 \rangle)$ is bounded in $\mathbb{C}$, and $\langle h_1 \rangle$ is hyperbolic. Let $d \in \mathbb{N}, d \geq 2$ and suppose $(\deg(h_1), d) \neq (2, 2)$. Then, there exists an element $h_2 \in \mathcal{Y}$ such that $(h_1, h_2) \in ((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$ and $\deg(h_2) = d$.*

5. *For each $(h_1, h_2) \in (\mathcal{D} \cap \mathcal{B}) \cup ((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H})$ and each $0 < p < 1$, $J(\langle h_1, h_2 \rangle) = \{z_0 \in \hat{\mathbb{C}} \mid \forall \text{ nbd } V \text{ of } z_0,\ T(h_1, h_2, p, \cdot)|_V \text{ is not constant}\}$.*

6. *Let $(h_1, h_2) \in (\mathcal{D} \cap \mathcal{B}) \cup (((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I})$. Then, for each $z \in \hat{\mathbb{C}}$, the function $p \mapsto T(h_1, h_2, p, z)$ is real analytic on $(0, 1)$. Moreover, for each $n \in \mathbb{N} \cup \{0\}$, the function $(p, z) \mapsto (\partial^n T/\partial p^n)(h_1, h_2, p, z)$ is continuous on $(0, 1) \times \hat{\mathbb{C}}$.*

Definition 3.28 ([17]). Let Γ be a non-empty compact subset of Rat. We define a map $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ as follows: For a point $(\gamma, y) \in \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ where $\gamma = (\gamma_1, \gamma_2, \ldots)$, we set $f(\gamma, y) := (\sigma(\gamma), \gamma_1(y))$, where $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is the shift map, that is, $\sigma(\gamma_1, \gamma_2, \ldots) = (\gamma_2, \gamma_3, \ldots)$. The map $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ is called the **skew product associated with the generator system** Γ. Moreover, we use the following notation.

1. Let $\pi : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}}$ and $\pi_{\hat{\mathbb{C}}} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the canonical projections.

2. For each $\gamma \in \Gamma^{\mathbb{N}}$, we set $J^\gamma := \{\gamma\} \times J_\gamma$ $(\subset \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$.

3. We set $\tilde{J}(f) := \overline{\cup_{\gamma \in \Gamma^{\mathbb{N}}} J^\gamma}$, where the closure is taken in the product space $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$. (Note that $f^{-1}(\tilde{J}(f)) = \tilde{J}(f) = f(\tilde{J}(f))$.)

4. For each $\gamma \in \Gamma^{\mathbb{N}}$, we set $\hat{J}^{\gamma,\Gamma} := \pi^{-1}\{\gamma\} \cap \tilde{J}(f)$ and $\hat{J}_{\gamma,\Gamma} := \pi_{\hat{\mathbb{C}}}(\hat{J}^{\gamma,\Gamma})$. Note that $J_\gamma \subset \hat{J}_{\gamma,\Gamma}$.

5. For each $z = (\gamma, y) \in \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$, we set $f'(z) := (\gamma_1)'(y)$.

Remark 13. Under the above notation, let G be the rational semigroup generated by Γ.

1. $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) \subset J(G)$. Moreover, the following diagram commutes.

$$\begin{array}{ccc} \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} & \xrightarrow{f} & \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \\ \pi \downarrow & & \downarrow \pi \\ \Gamma^{\mathbb{N}} & \xrightarrow[\sigma]{} & \Gamma^{\mathbb{N}} \end{array}$$

2. If $\sharp J(G) \geq 3$, then $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) = J(G)$.

Definition 3.29. Let $(h_1, h_2) \in \mathcal{Y}^2$ and $0 < p < 1$ and we set $\Gamma := \{h_1, h_2\}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be an f-invariant Borel probability measure. We define a function $\tilde{p} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \mathbb{R}$ by

$$\tilde{p}(\gamma, y) := \begin{cases} p & \text{if} \quad \gamma_1 = h_1, \\ 1 - p & \text{if} \quad \gamma_1 = h_2, \end{cases}$$

(where $\gamma = (\gamma_1, \gamma_2, \ldots)$), and we set

$$u(h_1, h_2, p, \mu) := \frac{-(\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log \tilde{p} \, d\mu)}{\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log |f'| \, d\mu}$$

(when the integral converges).

Definition 3.30. Let V be a non-empty open subset of $\mathbb{C}$. Let $\varphi : V \to \mathbb{R}$ be a function and let $y \in V$ be a point. We set

$$\text{Höl}(\varphi, y) := \sup\{\beta \in \mathbb{R} \mid \limsup_{z \to y} \frac{|\varphi(z) - \varphi(y)|}{|z - y|^\beta} = 0\}$$

and this is called the **pointwise Hölder exponent of φ at y.**

Remark 14. If $\text{Höl}(\varphi, y) < 1$, then φ is non-differentiable at y. If $\text{Höl}(\varphi, y) > 1$, then φ is differentiable at y and the derivative at y is equal to 0.

Theorem 3.31. (**Non-differentiability of $T(h_1, h_2, p, \cdot)$ at the points in $J(G_\tau)$**) *Let $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B}$ and $0 < p < 1$ and we set $\Gamma := \{h_1, h_2\}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := p\delta_{h_1} + (1-p)\delta_{h_2} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{Y})$ and we set $G = \langle h_1, h_2 \rangle$. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be the maximal relative entropy measure of $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ with respect to $(\sigma, \tilde{\tau})$ (Note that the existence and the uniqueness of the maximal relative entropy measure has been shown in [14]). Moreover, let $\lambda := (\pi_{\hat{\mathbb{C}}})_*(\mu) \in \mathfrak{M}_1(\hat{\mathbb{C}})$. Then, we have all of the following.*

1. int $J(G) = \emptyset$.
2. supp $\lambda = J(G)$.
3. *For each $z \in J(G)$, $\lambda(\{z\}) = 0$.*
4. *For almost every $z_0 \in J(G)$ with respect to λ,*

$$\begin{aligned}\text{Höl}(T(h_1, h_2, p, \cdot), z_0) &= u(h_1, h_2, p, \mu) \\ &= \frac{-(p \log p + (1-p)\log(1-p))}{p \log(\deg(h_1)) + (1-p)\log(\deg(h_2))} < 1.\end{aligned}$$

In particular, there exists an uncountable dense subset A of $J(G)$ such that for each $z \in A$, $T(h_1, h_2, p, \cdot)$ is non-differentiable at z.

Remark 15. $T(h_1, h_2, p, \cdot)$ is a **complex analogue of devil's staircase or Lebesgue singular functions** (see figure 3, 4). Moreover, $z \mapsto \frac{\partial T}{\partial p}(h_1, h_2, p, z)$ is a **complex analogue of Takagi function** (see figure 5). For the definition and the properties of the devil's staircase, Lebesgue singular functions, the Takagi function, and further singular functions on $\mathbb{R}$, see [25, 1] etc. (however, in these references, the relation between the singular functions on $\mathbb{R}$ and the random dynamical systems was not written).

Remark 16. In the proof of Theorem 3.31, we use the Birkhoff ergodic theorem and the Koebe distortion theorem, in order to show $\text{Höl}(T(h_1, h_2, p, \cdot), z_0) = u(h_1, h_2, p, \mu)$. Moreover, we apply potential theory in order to calculate $u(h_1, h_2, p, \mu)$ by p and $\deg(h_i)$.

4 Tools

In this section, we give some basic tools to prove the main results.

Lemma 4.1 ([15]). *Let G be a rational semigroup generated by a compact subset Γ of* Rat. *Then, $J(G) = \cup_{h\in\Gamma} h^{-1}(J(G))$. In particular, if $G = \langle h_1, \ldots, h_m\rangle$, then $J(G) = \cup_{j=1}^{m} h_j^{-1}(J(G))$. This property is called the* **backward self-similarity**.

Lemma 4.2. *Let $\tau \in \mathfrak{M}_1(\text{Rat})$. Then, we have the following.*

1. *$(M_\tau)_*^{-1}(F_{meas}(\tau)) \subset F_{meas}(\tau)$, and $(M_\tau)_*^{-1}(F^0_{meas}(\tau)) \subset F^0_{meas}(\tau)$.*
2. *Let $y \in \hat{\mathbb{C}}$ be a point. Then, $y \in F_{pt}(\tau)$ if and only if for any $\phi \in C(\hat{\mathbb{C}})$, there exists a neighborhood U of y in $\hat{\mathbb{C}}$ such that the sequence $\{z \mapsto M_\tau^n(\phi)(z)\}_{n\in\mathbb{N}}$ of functions on U is equicontinuous on U. Similarly, $y \in F^0_{pt}(\tau)$ if and only if for any $\phi \in C(\hat{\mathbb{C}})$, the sequence $\{z \mapsto M_\tau^n(\phi)(z)\}_{\in\mathbb{N}}$ of functions on $\hat{\mathbb{C}}$ is equicontinuous at the one point y.*
3. *$F_{meas}(\tau) \cap \hat{\mathbb{C}} \subset F_{pt}(\tau)$ and $F^0_{meas}(\tau) \cap \hat{\mathbb{C}} = F^0_{pt}(\tau)$.*
4. *$F(G_\tau) \subset F_{pt}(\tau)$.*
5. *Let $y \in \hat{\mathbb{C}}$ be a point. Suppose that* supp *τ is compact, and that $\tilde{\tau}\left(\{\gamma = (\gamma_1, \gamma_2, \gamma_3, \ldots) \in X_\tau \mid y \in \cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G_\tau))\}\right) = 0$. Then, we have that $y \in F^0_{pt}(\tau) = F^0_{meas}(\tau) \cap \hat{\mathbb{C}}$.*
6. *$F^0_{pt}(\tau) = \hat{\mathbb{C}}$ if and only if $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$.*

Lemma 4.3 ([22]). *Let Γ be a compact subset of* Rat *and let G be the rational semigroup generated by Γ. Suppose that $\sharp(J(G)) \geq 3$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Then, $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) = J(G)$ and for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$, we have $\hat{J}_{\gamma,\Gamma} = \cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$.*

Lemma 4.4. *Let $\tau \in \mathfrak{M}_1(\text{Rat})$ be such that* supp *τ is compact. Let V be a non-empty open subset of $\hat{\mathbb{C}}$ such that for each $g \in G_\tau$, $g(V) \subset V$. For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in X_\tau$, we set $L_\gamma := \cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(\hat{\mathbb{C}} \setminus V)$. Moreover, we set $L_{\ker} := \cap_{g\in G_\tau} g^{-1}(\hat{\mathbb{C}} \setminus V)$. Let $y \in \hat{\mathbb{C}}$ be a point. Then, we have that*

$$\tilde{\tau}(\{\gamma \in X_\tau \mid y \in L_\gamma,\ \liminf_{n\to\infty} d(\gamma_n \circ \cdots \circ \gamma_1(y), L_{\ker}) > 0\}) = 0.$$

(When $L_{\ker} = \emptyset$, we set $d(z, L_{\ker}) := \infty$ for each $z \in \hat{\mathbb{C}}$.)

Remark 17. As we see in this paper, both the theory of rational semigroups and that of the random dynamics of rational maps are related to each other very deeply. For the research of polynomial semigroups, see [22] and [20]. In [20], many new phenomena which can hold in the dynamics of polynomial semigroups (or random dynamics of polynomial maps) but cannot hold in the usual dynamics of a single polynomial map were found and systematically investigated. For example, in [22], it was shown that for each $n \in \mathbb{N} \cup \{\aleph_0\}$, there exists a finitely generated polynomial semigroup $G \subset \mathcal{Y}$ such that the cardinality of the set of all connected components of $J(G)$ is equal to n. This cardinality is related to a new cohomology theory on the backward self-similar systems, which the author initiated (see [19, 23]).

Figure 2: The Julia set of $G = \langle h_1, h_2 \rangle$, where $g_1(z) := z^2 - 1, g_2(z) := z^2/4, h_1 := g_1^2, h_2 := g_2^2$. G satisfies the assumption of Theorem 3.19. Hence $J_{\text{ker}}(G) = \emptyset$.

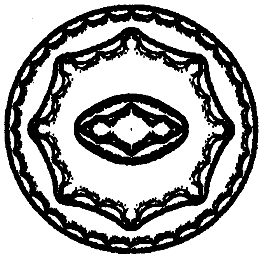


Figure 3: The graph of $T_{\infty,\tau}$, where $\tau = \sum_{i=1}^{2} \frac{1}{2}\delta_{h_i}$ with the same h_i as above. $T_{\infty,\tau}$ is continuous on $\hat{\mathbb{C}}$. The set of varying points of $T_{\infty,\tau}$ is equal to $J(G)$ in figure 2. A "devil's coliseum" (A complex analogue of the devil's staircase).

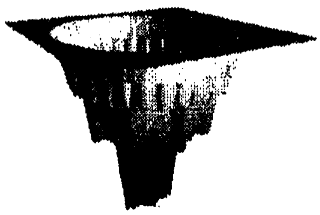


Figure 4: The upside down figure of figure 3.

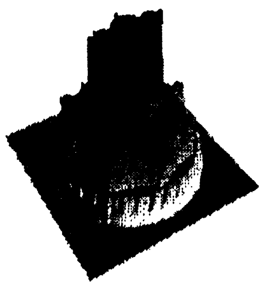
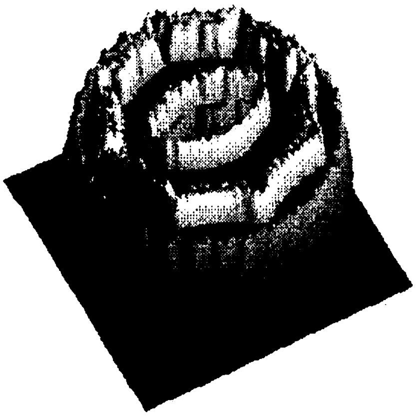


Figure 5: The graph of $z \mapsto \frac{\partial T}{\partial p}(h_1, h_2, 1/2, z)$. A complex analogue of the Takagi function.



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