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Some properties of Julia sets of transcendental entire functions with multiply-connected wandering domains

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Abstract

We study Julia components of transcendental entire functions with multiply-connected wandering domains. Under the assumption that the post singular set is contained in the Fatou set, it is shown that every repelling periodic point $p$ satisfies either

(1) $C(p) \supset \partial U$, where $C(p)$ is the Julia component containing $p$ and $U$ is an immediate attractive basin.

(2) $C(p) = \{p\}$ and this is a buried singleton component of $J(f)$.

§1 Introduction

Let $f$ be a transcendental entire function, $F(f)$ its Fatou set and $J(f)$ its Julia set. The following are some fundamental results on the connectivity of $J(f)$:

Proposition 1 If every Fatou component is bounded and simply connected, then $J(f) \subset \mathbb{C}$ is connected.

So it follows that if $J(f) \subset \mathbb{C}$ is disconnected, then either

(a) $f$ has an unbounded Fatou component or

(b) $f$ has a multiply-connected Fatou component.

For the case (a), the following holds. Note that an unbounded Fatou component $U$ is always simply connected (see [Ba1]) and so we can consider a Riemann map $\varphi : \mathbb{D} \to U$ of $U$.

Theorem 2 ([K, p.192, Main Theorem]) Suppose there exists an unbounded invariant Fatou component $U$ and let us consider the following conditions:
\( \infty \in \partial U \) is accessible in \( U \).

(B) There exist a finite point \( q \in \partial U \) with \( q \notin P(f) \), \( m_0 \in \mathbb{N} \) and a continuous curve \( C(t) \subset U \) (\( 0 \leq t < 1 \)) with \( C(1) = q \) which satisfies \( f^{m_0}(C) \supset C \), where

\[
P(f) = \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1}))
\]

is the post-singular set of \( f \).

(1) If \( U \) is either an attractive basin with (A) and (B), or a parabolic basin with (A) and (B), or a Siegel disk with (A), then the set

\[
\Theta_\infty := \{ e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty \} \subset \partial \mathbb{D}
\]

is dense in \( \partial \mathbb{D} \). In particular, \( J(f) \subset \mathbb{C} \) is disconnected.

(2) If \( U \) is a Baker domain with (B) and \( f|U \) is not univalent, then \( \Theta_\infty \) is dense in \( \partial \mathbb{D} \) or at least its closure \( \overline{\Theta_\infty} \) contains a certain perfect set in \( \partial \mathbb{D} \). In particular, \( J(f) \subset \mathbb{C} \) is disconnected.

Next result is a generalization of the above result:

**Theorem 3** ([BD1, p.439, Theorem 1.1, 1.2, Corollary 1.3] Theorem 2 holds without the assumption (B).

On the other hand, \( J(f) \subset \mathbb{C} \) can be connected nevertheless \( f \) has an unbounded Fatou component. For example,

\[
f(z) = 2 - \log 2 + 2z - e^z
\]

has a Baker domain but \( J(f) \) is connected ([K, p.194, Theorem 4]).

For the case (b), it is known that if \( f \) has a multiply-connected Fatou component \( U \), then \( U \) is a wandering domain and bounded (see, [Ba2, Theorem 3.1]) and therefore \( J(f) \subset \mathbb{C} \) is always disconnected. Furthermore \( J(f) \cup \{ \infty \} \subset \hat{\mathbb{C}} \) is also disconnected and actually this is the only case where \( J(f) \cup \{ \infty \} \subset \hat{\mathbb{C}} \) can be disconnected as follows:

**Proposition 4** ([K, p.191, Theorem 1]) \( J(f) \cup \{ \infty \} \subset \hat{\mathbb{C}} \) is disconnected if and only if \( f \) has a multiply-connected wandering domain.

In what follows, we will concentrate on the case (b), that is, the case where \( f \) has a multiply-connected wandering domain \( U \) and investigate some properties of connected components of the Julia set, which we call Julia components. We note the following fact (see, [Ba2, p.565, Theorem 3.1]):

**Proposition 5** If \( U \) is a multiply-connected wandering domain, then \( f^n|U \to \infty \).
Definition 6 (1) We call a connected component of $J(f)$ a Julia component.

(2) $z \in J(f)$ is called a buried point if $z$ satisfies $z \notin \partial U$ for any Fatou component $U$.

(3) We call the set

$$J_0(f) := \{z \in J(f) \mid z \text{ is a buried point}\}$$

the residual Julia set of $f$.

(4) A Julia component $C$ is called a buried component if $C \subset J_0(f)$.

For rational cases, the following are known:

Example 7 ([Mc]) Let $f(z) = z^2 + \frac{\lambda}{z^3}$, where $\lambda > 0$ is small. Then $J(f)$ is a Cantor set of nested quasi-circles. So there are buried components. In particular, $J_0(f) \neq \emptyset$.

Theorem 8 ([Mo, p.208, Theorem 3]) Let $f$ be a hyperbolic rational function. Then $J_0(f) \neq \emptyset$ if and only if

1. $F(f)$ has a completely invariant component, or
2. $F(f)$ consists of only two components.

Example 9 ([Mo, p.209]) Let $f(z) = \frac{-2z + 1}{(z - 1)^2}$, then the following hold:

1. The set $\{0, 1, \infty\}$ is a super-attracting cycle.
2. $f$ is hyperbolic.
3. Any Fatou component is a preimage of the super-attractive basin above.
4. $J(f)$ is connected.

So by Theorem 8, we have $J_0(f) \neq \emptyset$. But since $J(f)$ is connected, there is no buried component.

Example 10 ([U]) There exists a rational function $f$ whose Julia set is homeomorphic to a Sierpinski gasket. So $J_0(f) \neq \emptyset$, but again there is no buried component.

Here are some fundamental properties for buried points and residual Julia sets. Note that $f$ need not be rational and these hold also for transcendental entire functions and even for meromorphic functions.

Proposition 11 (1) If $F(f)$ has a completely invariant component, then $J_0(f) = \emptyset$.

(2) If there exists a buried component of $J(f)$, then $J(f)$ is disconnected.

(3) If $J_0(f) \neq \emptyset$, then $J_0(f)$ is completely invariant, dense in $J(f)$, and uncountable.

More information on residual Julia sets, see [DF].
§2 Results

Main result of this paper is as follows:

**Theorem A** Let \( f \) be a transcendental entire function. Assume that

\[
\text{P}(f) = \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1})) \subset F(f),
\]

(a) \( f \) has a multiply-connected wandering domain.

Then every repelling periodic point \( p \) satisfies either one of the following:

1. \( C(p) \supset \partial U \), where \( C(p) \) is the Julia component containing \( p \) and \( U \) is an immediate attractive basin.
2. \( \{p\} \) is a buried singleton component of \( J(f) \).

**Corollary B** Let \( f \) be a transcendental entire function. Assume the above conditions (a), (b) and also

(c) \( f^n(z) \to \infty \) for any \( z \in F(f) \).

Then every repelling periodic point \( p \) is a buried singleton component of \( J(f) \).

**Remark** \( f \) is called hyperbolic if

\[
\text{dist}_\mathbb{C}(P(f), J(f)) > 0,
\]

where \( \text{dist}_\mathbb{C} \) is the Euclidean distance on \( \mathbb{C} \). So the condition (a) in Theorem A is slightly weaker than hyperbolicity.

**(Outline of the Proof):** Let \( p \) be a repelling periodic point. For simplicity, we assume that \( p \) is a fixed point. Suppose that \( p \) does not satisfy (1). Let \( C(p) \subset J(f) \) be the Julia component containing \( p \). Then \( f(C(p)) = C(p) \) and we can show that \( C(p) \) is bounded. If there exists a Fatou component \( U \subset F(f) \) such that \( C(p) \cap \partial U \neq \emptyset \), then it follows that \( U \) is a wandering domain which satisfies \( f^n(U) \to \infty (n \to \infty) \). Then this contradicts the fact that \( C(p) \) is bounded. Hence \( C(p) \) is a buried component.

Next we can show that the complement of \( C(p) \) has no bounded component. Then since \( P(f) \subset F(f) \) and \( C(p) \) is bounded, we have

\[
\text{dist}_\mathbb{C}(C(p), P(f)) > 0.
\]

Then there exists a simply connected domain \( W \) such that \( C(p) \subset W \) and there exists a branch \( g_n \) of \( f^{-n} \) which satisfies \( g_n(p) = p \). It is well-known that \( \{g_n\}_{n=1}^{\infty} \) is a normal family and hence there exists a subsequence \( g_{n_k} \) converging to a constant function which must be the point \( p \). On the other hand, we have \( g_n(C(p)) = C(p) \), so we conclude that \( C(p) = \{p\} \). This completes the proof of Theorem A. Corollary B is an immediate consequence of Theorem A. \( \square \)
§3 Examples

Example 12 ([BD2, p.375, Theorem G]) There exists an $f(z)$ with the following form

$$f(z) = k \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right), \quad 0 < r_1 < r_2 < \ldots, \quad k > 0$$

such that for every repelling periodic point $p$ is a buried singleton component of $J(f)$.

Example 13 ([KS]) There exists a transcendental entire function $f$ with doubly-connected wandering domains, which satisfies the following: Every critical point $c$ satisfies $f^2(c) = 0$ and $0$ is a super-attracting fixed point. This implies that this $f$ satisfies the assumptions of Theorem A. Therefore every repelling periodic point $p$ satisfies either $C(p) \supset \partial U$ for the immediate attractive basin $U$ of the super-attractive fixed point $0$ or $\{p\}$ is a buried singleton component of $J(f)$.

Example 14 ([Be]) By using the similar method as in Example 13, Bergweiler constructed an example of transcendental entire function $f$ which has both a simply connected and a multiply connected wandering domain. Critical points of $f$ satisfy the following:

1. $c_0 = 0 < c_1 < c_2 < \ldots \to \infty$,
2. $f(0) = 0$, $f(c_i) = c_{i+1}$, $i = 1, 2, \ldots$
3. $c_i$ is contained in a simply connected wandering domain $U_i$ which satisfies

$$f(U_i) = U_{i+1}, \quad f^n|U_i \to \infty \quad (n \to \infty).$$

So this $f$ also satisfies the assumptions (a) and (b) of Theorem A.

Example C We can construct an $f$ which satisfies the assumptions (a), (b) and (c) by using the similar method as in Example 13. Hence every repelling periodic point $p$ is a buried singleton component of $J(f)$ from Corollary B. We omit the details.

References


