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# OHT — A software for the dynamics of the modular group action on the character variety

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## 1 Introduction

The purpose of this note is to describe the software (named "OHT") that I have presented in the workshop "Complex dynamics and related topics". See Figure 1. This software is based on a joint work with Ser Peow Tan on dynamics of the modular group action on the $SL(2,\mathbb{C})$ character variety of a one-holed torus. The detailed description of this work will be presented elsewhere.

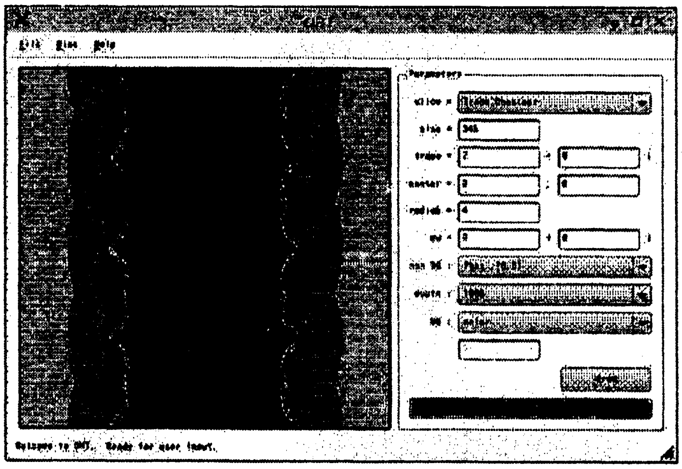

Figure 1: OHT

Let us describe the basic setting. See [9] for detail.

Let $T$ be a one-holed torus, and $\pi$ the fundamental group of $T$. $\pi$ is isomorphic to the free group of rank two and we fix a pair of standard generators for $\pi$ and denote them by $\alpha$ and $\beta$, i.e., $\pi = \langle\alpha, \beta\rangle$. Let $\mathcal{X} := \mathrm{Hom}(\pi, SL(2,\mathbb{C}))//SL(2,\mathbb{C})$ be the $SL(2,\mathbb{C})$ character variety of $\pi$, and $\kappa$-relative character variety $\mathcal{X}_\kappa$ is defined as follows

$$\mathcal{X}_\kappa := \{[\rho] \in \mathcal{X} \mid \mathrm{tr}\,\rho(\alpha\beta\alpha^{-1}\beta^{-1}) = \kappa\}.$$

Let $\Gamma$ be the mapping class group of $T$. Note that there is a natural action of $\Gamma$ on $\mathcal{X}$. Let $\mathcal{X}_{BQ}$ be the largest open subset of $\mathcal{X}$ on which $\Gamma$ acts properly discontinuously. The aim of the software OHT is to draw the pictures of $\mathcal{X}_{BQ}$ under the several settings that we would like to investigate.

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# 2 Character variety of one-holed torus

## 2.1 Brief history and motivation

$\mathcal{X}_{BQ}$ was first studied by Bowditch [2] as a subset of $\mathcal{X}_{-2}$, and Bowditch proposed the following conjecture.

**Conjecture 2.1 (Bowditch [2])** *$\mathcal{X}_{BQ} \cap \mathcal{X}_{-2}$ coincides with quasifuchsian space $\mathcal{QF}$ of once punctured torus.*

The question is still open. If the above conjecture is true, it gives a relatively simple algorithm to decide whether a given $\mathrm{SL}(2,\mathbb{C})$ representation of $\pi$ is in $\mathcal{QF}$ or not. See [11], [7], and [10], for the algorithms for the same problem based on Jorgensen's theory on punctured torus groups. See [4] and [1] for Jorgensen's theory. One of the aims of our joint work is to compare the pictures of $\mathcal{X}_{BQ} \cap \mathcal{X}_{-2}$ produced by the software of this paper with the pictures of $\mathcal{QF}$ which were presented in [7] and [11].

## 2.2 Character variety and Farey tessellation

In our software, we use the following well-known coordinate for $\mathcal{X}$.

**Fact 2.2 (Fricke)** *The character variety $\mathcal{X}$ can be identified with $\mathbb{C}^3$. The identification is given as follows; for $[\rho] \in \mathcal{X}$, $(x, y, z) := (\operatorname{tr}\rho(\alpha), \operatorname{tr}\rho(\beta), \operatorname{tr}\rho(\alpha\beta))$, where $\alpha$ and $\beta$ are the fixed pair of generators for $\pi$.*

By a trace identity for $\mathrm{SL}(2,\mathbb{C})$, we have $\operatorname{tr}\rho(\alpha\beta\alpha^{-1}\beta^{-1}) = x^2+y^2+z^2-xyz-2$. Thus

$$\mathcal{X}_\kappa = \{(x,y,z) \in \mathbb{C}^3 \mid x^2+y^2+z^2-xyz-2=\kappa\}.$$

For an element $\phi$ of the mapping class group $\Gamma$, the action of $\phi$ on the character variety $\mathcal{X}$ is given by

$$\phi([\rho]) := [\rho \circ \phi_*^{-1}],$$

where $\phi_*$ is the induced action of $\phi$ on the fundamental group $\pi$. This action preserves $\operatorname{tr}\rho(\alpha\beta\alpha^{-1}\beta^{-1})$. Hence, $\phi$ acts on $\mathcal{X}_\kappa$ for any $\kappa \in \mathbb{C}$.

Let $\mathcal{C}$ be the set of free homotopy classes of non-trivial, non-peripheral simple closed curves on $T$. Consider $\mathbb{Z}+\mathbb{Z}$ cover of $T$. We use the convention that the preimage of $\alpha$ is vertical lines and the preimage of $\beta$ is horizontal lines. Then each free homotopy class gives rise to a "slope" in $\mathbb{Z}+\mathbb{Z}$ cover and $\mathcal{C}$ can be identified with $\widehat{\mathbb{Q}} := \mathbb{Q} \cup \{1/0\}$. For example, the slopes of $[\alpha]$, $[\beta]$, $[\alpha\beta]$, and $[\alpha\beta^{-1}]$ are $1/0$, $0/1$, $1/1$, and $-1/1$, respectively.

Let $\mathcal{C}(T)$ be the pants graph of $T$. The set of vertices of $\mathcal{C}(T)$ is $\mathcal{C} = \widehat{\mathbb{Q}}$, and two vertices $c_1, c_2 \in \mathcal{C}$ is connected by an edge if and only if the geometric intersection number of $c_1$ and $c_2$ is equal to 1. $\mathcal{C}(T)$ can be viewed as Farey tessellation of the hyperbolic place $\mathbb{H}^2$. See Figure 2. The mapping class group $\Gamma$ acts naturally on $\mathcal{C}$ and $\mathcal{C}(T)$. The action of $\Gamma$ is realized by the action of $\mathrm{PGL}(2, \mathbb{Z})$ on $\widehat{\mathbb{Q}}$.

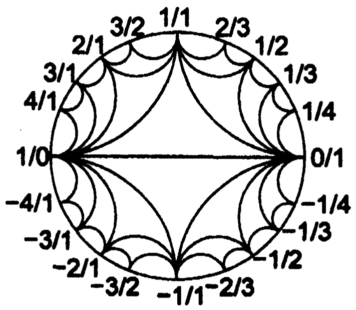



Figure 2: Farey tessellation

For $[\rho] \in \mathcal{X}$ and $X \in \mathcal{C} = \widehat{\mathbb{Q}}$, $\operatorname{tr}\rho(X)$ is well-defined. Starting with

$$(x, y, z) := (\operatorname{tr}\rho(\alpha), \operatorname{tr}\rho(\beta), \operatorname{tr}\rho(\alpha\beta)),$$

we can calculate $\operatorname{tr}\rho(X)$ for any $X \in \mathcal{C}$ by the trace identity.

## 2.3 Bowditch Q-condition and related theorems

In this subsection, we redefine $\mathcal{X}_{BQ}$ that fits computer calculation and review basic theorems used in our calculation.

**Definition 2.3 (Bowditch, Tan-Wong-Zhang)** *$\mathcal{X}_{BQ}$ is a subset of $\mathcal{X}$ consisting of characters $[\rho]$ satisfying the following two conditions.*

$$\operatorname{tr}\rho(X) \notin [-2, 2] \text{ for all } X \in \mathcal{C}. \tag{1}$$

$$|\operatorname{tr}\rho(X)| \leq 2 \text{ for only finitely many } X \in \mathcal{C}. \tag{2}$$

**Theorem 2.4 (Tan-Wong-Zhang)** *$\Gamma$ acts properly discontinuously on $\mathcal{X}_{BQ}$. $\mathcal{X}_{BQ}$ is the largest open subset of $\mathcal{X}$ for which this holds.*

**Theorem 2.5 (Bowditch, Tan-Zhang)** *There exists a finite criterion for recognizing that a given $(x, y, z) \in \mathcal{X}$ lies in $\mathcal{X}_{BQ}$.*

**Theorem 2.6 (Ng-Tan [8])** *For $[\rho] \in \mathcal{X}_{-2}$, $[\rho] \in \int(\mathcal{X} \setminus \mathcal{X}_{BQ})$ if there exists $X \in \mathcal{C}$ such that $|\operatorname{tr}\rho(X)| < 0.5$.*

## 2.4 Relative BQ-condition

Suppose that $[\rho] \in \mathcal{X}$ is stabilized by an element $\theta \in \Gamma$. In other words, there exists an element $A \in \mathrm{SL}(2, \mathbb{C})$ such that for $g \in \pi$,

$$\theta(\rho)(g) = A \cdot \rho(g) \cdot A^{-1}.$$

Then, $\operatorname{tr} \rho(X)$ is well defined on the classes $[X] \in \mathcal{C}/\langle\theta\rangle$.

**Definition 2.7 (Tan-Wong-Zhang)** *The following two conditions are called* relative BQ-condition *for $[\rho] \in \mathcal{X}$ which is stabilized by $\theta \in \Gamma^+$.*

$$\operatorname{tr} \rho(X) \notin [-2, 2] \text{ for all } X \in \mathcal{C}/\langle\theta\rangle. \tag{3}$$
$$|\operatorname{tr} \rho(X)| \leq 2 \text{ for only finitely many } X \in \mathcal{C}/\langle\theta\rangle. \tag{4}$$

One of the main purpose is to investigate relative BQ-conditions for hyperbolic, parabolic, and elliptic elements in $\Gamma$.

# 3 OHT

In this section, we describe the software OHT. See Figure 1.

The main window consists of two parts. The left part is used to show the picture of $\mathcal{X}_{BQ}$. The right part is used to specify the parameters of $\mathcal{X}$. Since complex dimension of $\mathcal{X}$ is three, we have to take some slice to get one dimensional slice. User can choose a type of slice at "slice" menu, and each type of slice will be described in 3.1 – 3.11

To detect that a given input $(x, y, z) \in \mathcal{X}$ does not belong to $\mathcal{X}_{BQ}$, we use theorem 2.6. Note that we can use theorem 2.6 only when $\mu = 0$. When $\mu \neq 0$, we can avoid using 0.5 condition in this theorem or change the value from 0.5 to 0.1. These choices can be made at "non BQ" menu.

The algorithm given in theorem 2.5 may not stop in a finite time if given input is not in $\mathcal{X}_{BQ}$. Depth first search for $\mathcal{C}(T)$ is used during the calculation. In practice, we must stop our calculation when the depth of the search tree becomes too big. User can specify the maximal depth allowed at "depth" menu.

Recall that $\mathcal{X} \cong \mathbb{C}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{C}\}$. Put

$$\mu := x^2 + y^2 + z^2 - xyz. \tag{5}$$

To describe some of the slices mentioned below, we need the following definition.

**Definition 3.1** *Let $\mathcal{D}$ be a subset of $\mathcal{C}$. We say that $[\rho] \in \mathcal{X}$ satisfies* BQ-condition *with respect to $\mathcal{D}$ if following two conditions hold.*

$$\operatorname{tr} \rho(X) \notin [-2, 2] \text{ for all } X \in \mathcal{D}. \tag{6}$$
$$|\operatorname{tr} \rho(X)| \leq 2 \text{ for only finitely many } X \in \mathcal{D}. \tag{7}$$

BQ-condition with respect to $\mathcal{C}$ is equal to the original definition of BQ-condition defined in 2.3.

## 3.1 Trace constant

In "trace constant" slice, we fix $x$ and $\mu$ and draw $y$-plane. The value of $x$ is specified at "trace" input form and $\mu$ is specified at "mu" input form. The region of $y$-plane to draw is specified at "center" and "radius" input form. Thus, for each pixel of the picture, we know the values of $x$ and $y$. The value of $z$ is determined by (5). This equation is a quadratic polynomial for $z$ and, in general, we have two solutions, say $z_1$ and $z_2$. But, this is not a problem, because different choice of the solution corresponds to changing the marking (generator) of $\pi$. $(x, y, z_1)$ is in $\mathcal{X}_{BQ}$ if and only if $(x, y, z_2)$ is in $\mathcal{X}_{BQ}$.

See Figure 3 in section 4.

## 3.2 Riley

In "Riley" slice, we set $x = 0$ and fix $\mu$ and draw a complex plane — say $w$-plane. The value of $\mu$ is specified at "mu" input form. Then, we set $y = \sqrt{w}$ and use (5) to get $z$. Again, the choice of $y$ and $z$ does not matter.

We modify the definition of Bowditch Q-condition for this slice. Otherwise, since $x = 0$, we always have $(0, y, z) \notin \mathcal{X}_{BQ}$. We draw the picture of BQ-set with respect to $\mathcal{D} = \{p/q \in C \mid 0/1 \leq p/q \leq 1/1\}$. Note that $\operatorname{tr}\rho(x) = \operatorname{tr}\rho(1/0) = 0$, and $\mathcal{D}$ corresponds to the vertices of $C$ between $y$ and $z$.

## 3.3 Fixed by $\{\{1, n\}, \{0, 1\}\}$

In this slice, we consider relative BQ-condition. Recall that the action of $\Gamma$ on $\mathcal{X}$ is realized by an element of $\mathrm{PGL}(2, \mathbb{Z})$ and the action of this matrix is given by the corresponding Möbius map on $\widehat{\mathbb{Q}}$.

In this slice, we consider relative BQ-condition for

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

The integer value of $n$ can be set at "n" input form. Then $x = 2\cos(\pi/n)$ and we draw $y$-plane. $\mu$ is specified at "mu" input form, and for $z$, we solve the same quadratic polynomial as before.

## 3.4 Earle

In this slice, we draw Earle slice. This is a one dimensional slice of $\mathcal{X}$. For details about the Earle slice see [3]. See also [6].

## 3.5 Fixed by $\{\{2, 1\}, \{1, 1\}\}$

In this slice, we consider relative BQ-condition for

$$A_{2111} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then the subset of $\mathcal{X}$ that is stabilized by $A_{2111}$ is $\{(x, x/(x-1), x) \mid x \in \mathbb{C}\}$, and we draw $x$-plane. In $\mathcal{C}$, a fundamental region of the action of $A_{2111}$ is

$$\mathcal{D} = \{p/q \in \mathcal{C} \mid -1/1 \leq p/q \leq 0/1 \text{ or } 1/1 \leq p/q \leq 1/0\}.$$

In practice, we draw the picture of BQ-set with respect to $\mathcal{D}$ for each $(x, x/(x-1), x)$.

This slice has a close connection to the hyperbolic Dehn surgery space of figure eight knot.

See Figure 4 in section 4.

## 3.6 Fixed by $\{\{2,1\},\{3,2\}\}$

In this slice, we consider relative BQ-condition for

$$A_{2132} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

Then the subset of $\mathcal{X}$ stabilized by $A_{2132}$ is $\{(x, x^2/(x^2-2), 2x/(x^2-2)) \mid x \in \mathbb{C}\}$, and we draw $x$-plane. A fundamental region of the action of $A_{2132}$ is

$$\mathcal{D} = \{p/q \in \mathcal{C} \mid 1/0 \leq p/q \leq -1/1 \text{ or } 1/1 \leq p/q \leq 1/0\}.$$

See Figure 4 in section 4.

## 3.7 Maskit with pleating ray

In this slice, we draw Maskit slice, i.e., trace constant slice with $x = 2$. We also draw pleating ray. See [5] for details about pleating ray.

## 3.8 Imaginary

In this slice, we draw imaginary slice. It is discussed in [9]. If we name the complex plane we draw as $w$-plane, we set $x = \Re(w) \cdot i$, $y = \Im(w) \cdot i$, where $i = \sqrt{-1}$. $\mu$ is specified using input form and $z$ is calculated as before.

## 3.9 xxx

This slice is one parameter family $\{(x, x, x) \mid x \in \mathbb{C}\}$. This slice corresponds to the relative BQ-condition which is fixed by order three elliptic element of $\Gamma$.

## 3.10 xyy

This slice is two parameter family $\{(x, y, y) \mid x, y \in \mathbb{C}\}$. Thus we must impose one more condition to get a picture of complex dimension one. There are several choices. (Fixing $\mu$ and drawing $x$-plane. Fixing $\mu$ and drawing $y$-plane. Fixing $x/y$ and drawing $x$-plane. Fixing $x$ and drawing $y$-plane.) This slice corresponds to the relative BQ-condition which is fixed by order two elliptic element of $\Gamma$.

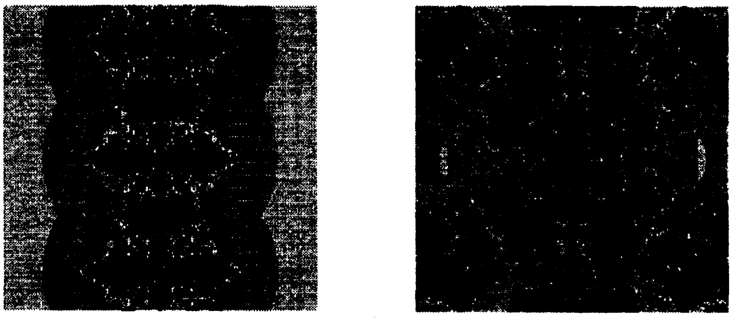
### 3.11 p/q

In $\{\{1,n\},\{0,1\}\}$ slice, we considered $x=2\cos(\pi/n)$ and we drew $y$-plane. In this slice we set $x=2\cos(\pi p/q)$ and we draw $y$-plane. $\mu$ is specified at "mu" input form, and for $z$, we solve the same quadratic polynomial as before.

## 4 Gallery

In this section, we present some of the pictures that was produced by OHT and shown in our talk in the workshop. Gray part of the pictures corresponds to $\mathcal{X}_{BQ}$ and black part corresponds to outside of $\mathcal{X}_{BQ}$.

In Figure 3, we present two examples of trace constant slices. In (a), we set $x=2$ and $\mu=-5$. If $\mu=0$, the picture generated by OHT looks exactly like the classical Maskit slice. Since we have set $\mu=-5$ in the picture below, our picture looks a lot more complicated. In (b), $x=100$, $mu=-4$ and we get an even more complicated picture.



(a) $x=2$, $\mu=-5$ (b) $x=100$, $\mu=-4$

Figure 3: Trace constant slices

In Figure 4, we present pictures of relative BQ-condition for $\{\{2,1\},\{1,1\}\}$ and $\{\{2,1\},\{3,2\}\}$.

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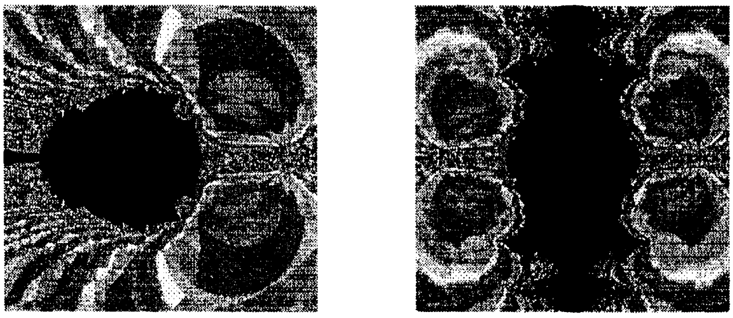

(a) $\{\{2,1\},\{1,1\}\}$ (b) $\{\{2,1\},\{3,2\}\}$

Figure 4: Relative BQ-conditions


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