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Equivalence relations on measure spaces and von Neumann algebras
—the normality and the commensurability—

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1 Introduction

In the group theory, there exist two notions for each inclusion of groups $H \subseteq G$ —the normality and the commensurability. We say that H is normal in G if gHg^{-1} coincides with H for each $g \in G$, and H is commensurable in G if the index $[H : H \cap gHg^{-1}]$ is finite for each $g \in G$.

In this note, we apply these notions to ergodic discrete measured equivalence relations with corresponding von Neumann algebras.

The notion of the normality for discrete measured equivalence relations is introduced in [6]. But it was unknown to characterize this property in terms of operator algebras. So, as a characterization of the normality, we will give a definition of the normalizing groupoid for each inclusion of von Neumann algebras. Namely, we will show that, for each inclusion of ergodic discrete measured equivalence relations $\mathcal{S} \subseteq \mathcal{R}$, $\mathcal{S}$ is normal in $\mathcal{R}$ if the corresponding factor $W^*(\mathcal{R})$ is generated by the normalizing groupoid of the subfactor $W^*(\mathcal{S})$.

Moreover, we shall give a notion of the commensurability for each inclusion of ergodic discrete measured equivalence relations. We will show that the commensurability characterizes the discreteness of the corresponding inclusion of factors.

This note is a brief survey of [4] with some new examples.

2 Preparation

In this section, we summarize basic facts about measured groupoids and von Neumann algebras associated to them. Further details regarding these objects can be found in [3], [5], [6], [8].

We assume that all von Neumann algebras in this note have separable preduals, and

$$
\begin{aligned}
(X,\mu) &: \text{standard Borel space},\\
\mathcal{R} &: \text{discrete measured equivalence relation on } (X,\mu),\\
\nu &: \text{left counting measure on } \mathcal{R},\\
\omega &: \text{normalized 2-cocycle on } \mathcal{R},\\
\mathcal{R}(x) &:= \{y \in X \ : \ (x,y) \in \mathcal{R}\},\\
[\mathcal{R}]_* &:= \{\varphi : \text{bimeasurable nonsingular transformations}\\
&\qquad \text{from } \mathrm{Dom}(\varphi) \text{ to } \mathrm{Im}(\varphi) \text{ such that } \varphi(x) \text{ is in } \mathcal{R}(x) \text{ for a.e. } x \text{ in } X\},\\
\Gamma(\varphi) &:= \{(x,\varphi(x)) : x \in \mathrm{Dom}(\varphi)\} \quad (\varphi \in [\mathcal{R}]_*).
\end{aligned}
$$

Definition 1. (1) We define a von Neumann algebra $W^*(\mathcal{R},\omega)$ and a von Neumann subalgebra $W^*(X)$ which act on $L^2(\mathcal{R},\nu)$ by the following:

$$
\begin{aligned}
W^*(\mathcal{R},\omega) &:= \{L^\omega(f) : f \text{ is a left finite function on } \mathcal{R}\}'',\\
W^*(X) &:= \{L^\omega(d) : d \in L^\infty(X,\mu)\},
\end{aligned}
$$

where we regard $L^\infty(X,\mu)$ as functions on the diagonal of $\mathcal{R}$, and $L^\omega(f)$ is defined by

$$\{L^\omega(f)\xi\}(x,z) := \sum_{y\sim x} f(x,y)\xi(y,z)\omega(x,y,z).$$

(2) Let A be a von Neumann algebra and D be a subalgebra of A. We call D is a Cartan subalgebra of A if D satisfies the following:

(i) D is maximal abelian in A,

(ii) D is regular in A, i.e., the normalizer $N_A(D)$ generates A, where

$$N_A(D) := \{u \in A : u \text{ is unitary and } uDu^* = D\},$$

(iii) there exists a faithful normal conditional expectation E_D from A onto D.

It is known that, for each such a pair $(D \subseteq A)$, there exists a discrete measured equivalence relation $\mathcal{R}$ with a 2-cocycle ω such that $(D \subseteq A) \cong (W^*(X) \subseteq W^*(\mathcal{R},\omega))$ ([5, Theorem 1]).

2.1 Borel 1-cocycles with coactions

Let $\mathcal{R}$ be a discrete measured equivalence relation. A Borel map c from $\mathcal{R}$ to a locally compact group K is called a 1-cocycle if c satisfies the following equations up to null sets:

$$c(x,y)c(y,z) = c(x,z), \quad c(x,x) = 1_K.$$

It is known that there exists a bijective correspondence between the set of Borel 1-cocycles from $\mathcal{R}$ to a locally compact group K and the set of coactions of K on $W^*(\mathcal{R},\omega)$ which fix each element in $W^*(X)$ ([3, Theorem 5.8]):

$$c : \mathcal{R} \to K \longrightarrow \alpha_c(L^\omega(f))\xi(k,x,z) := \sum_{y\sim x} f(x,y)\xi(c^{-1}(x,y)k, y, z),$$

$$W^*(X) \subseteq W^*(\mathcal{R},\omega)^\alpha \longrightarrow \alpha = \alpha_c \ (\exists c : \mathcal{R} \to K).$$

We further suppose that $\mathcal{R}$ is ergodic. For each measured equivalence subrelation $\mathcal{S}$ of $\mathcal{R}$, there exist countable functions $\{\varphi_i\}_{i\in I}$ on X such that $\mathcal{R}(x)$ is equal to a disjoint union of $\{\mathcal{S}(\varphi_i(x))\}_{i\in I}$ up to a null set. We call $\{\varphi_i\}_{i\in I}$ choice functions for $(\mathcal{S} \subseteq \mathcal{R})$.

For each choice functions $\{\varphi_i\}_{i\in I}$, we define a 1-cocycle σ from $\mathcal{R}$ to the full permutation group on I by the following rule:

$$\sigma(x,y)(i) = j \quad \Longleftrightarrow \quad (\varphi_i(y), \varphi_j(x)) \in \mathcal{S}.$$

We call σ the index cocycle.

A subrelation $\mathcal{S}$ of $\mathcal{R}$ is called normal in $\mathcal{R}$ if the index cocycle cobounds up to a null set. It is known that $\mathcal{S}$ is normal in $\mathcal{R}$ if there exists a 1-cocycle c from $\mathcal{R}$ to a discrete group K such that $\mathrm{Ker}(c) := c^{-1}(1_K)$ coincides with $\mathcal{S}$ up to a null set([6, Theorem 2.2]).

2.2 Basic extension and the discreetness

Let $B \subseteq A$ be an inclusion of factors with a conditional expectation $E_B : A \to B$, the modular conjugation J and the Jones projection e_B. Put $A_1 := JB'J = A\vee\{e_B\}$ and $\widehat{E_B} := \mathrm{Ad}(J)\circ E_B^{-1}\circ \mathrm{Ad}(J)$, the operator valued weight from $A_1 \to A$. We note that, if B contains a Cartan subalgebra of A, then the conditional expectation E_B always exists uniquely ([1, Theorem 1]).

By [8], the relative commutant $A_1 \cap B'$ is decomposed as follows:

$$A_1 \cap B' = \mathcal{A} \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \mathcal{C},$$

where $\mathcal{A} \oplus \mathcal{B}_1$ is the semifinite part of $\widehat{E_B}|_{A_1 \cap B'}$ and $J\mathcal{B}_1 J = \mathcal{B}_2$. The inclusion $B \subseteq A$ is called discrete if $\widehat{E_B}$ is semifinite on $A_1 \cap B'$. It is known that the $B \subseteq A$ is discrete if and only if there exist minimal partitions $\{f_n\}_{n\geq 1}$ in $A_1 \cap B'$ such that $\sum_{n\geq 1} f_n = 1$ and $\widehat{E_B}(f_n) < \infty$.

3 A characterization of the normality

In the rest of this note, we fix an inclusion of the ergodic discrete measured equivalence relations $\mathcal{S} \subseteq \mathcal{R}$ on (X, μ) with a 2-cocycle ω and the corresponding factors with a Cartan subalgebra $(D \subseteq B \subseteq A) := (W^*(X) \subseteq W^*(\mathcal{S}, \omega) \subseteq W^*(\mathcal{R}, \omega))$.

We have the following

Theorem 2. *Let $\mathcal{S}$ be an ergodic Borel subrelation of $\mathcal{R}$, and $B := W^*(\mathcal{S}, \omega)$ be the associated subfactor of A. Then the following are equivalent:*

(1) The subrelation $\mathcal{S}$ is normal in $\mathcal{R}$.

(2) The normalizing groupoid $\mathcal{GN}_A(B)$ of B in A generates A, where

$$\mathcal{GN}_A(B) := \{v \in A : v \ : \text{ partial isometry},\ v^*v, vv^* \in B,\ vBv^* = vv^*Bvv^*\}.$$

Proof. (1)$\Rightarrow$(2): If $\mathcal{S}$ is normal in $\mathcal{R}$, then there exists a 1-cocycle c from $\mathcal{R}$ to a discrete group K such that $\mathrm{Ker}(c)$ is equal to $\mathcal{S}$ up to a null set. Set $\alpha := \alpha_c$. It is easy to check that $A^\alpha = B$ and the subspace of A generated by the spectral subspaces $\{A^\alpha(k)\}_{k\in K}$ is σ-strongly* dense in A. A direct computation shows that $A^\alpha(k) \cap \mathcal{GN}_A(D)$ is contained in $\mathcal{GN}_A(B)$ for each $k \in K$. For each element $v \in \mathcal{GN}_A(D)$, there exist projections $\{e_k\}_{k\in K}$ in D which satisfy $\sum_{k\in K} e_k = v^*v$ and $ve_k \in A^\alpha(k) \cap \mathcal{GN}_A(D)$ for each $k \in K$. So we obtain $\mathcal{GN}_A(B)'' \supseteq \mathcal{GN}_A(D)'' = A$.
(2)$\Rightarrow$(1): By using the same arguments as in the proof of [1, Lemma 3.1], for each $v \in \mathcal{GN}_A(B)$ and $w \in \mathcal{GN}_A(D)$, we have $\mathrm{Ad}(wE_D(w^*v))B \subseteq DvBv^*D \subseteq B$. It follows that $\mathcal{GN}_A(B)''$ coincides with $(\mathcal{GN}_A(B) \cap \mathcal{GN}_A(D))''$. So there exists countable elements $\{\rho_i\}_{i\in I}$ in $[\mathcal{R}]_*$ such that $\mathcal{R}$ is equal to a disjoint union of $\{\Gamma(\rho_i)\}_{i\in I}$ and $\mathcal{S}(\rho_i(x)) = \mathcal{S}(\rho_i(y))$ for each $(x, y) \in \mathcal{S}$ and $i \in I$ up to null sets. Since $\mathcal{S}$ is ergodic, there exists $\eta_i \in [\mathcal{S}]_*$ such that $\mathrm{Dom}(\eta_i) = X$ and $\mathrm{Im}(\eta_i) = \mathrm{Dom}(\rho_i)$ and $\eta_i|_{\mathrm{Dom}(\rho_i)} = id_{\mathrm{Dom}(\rho_i)}$. Set $\varphi_i := \rho_i \circ \eta_i$ for each $i \in I$. By construction, we have that $\mathcal{R}(x) = \bigcup_{i\in I} \mathcal{S}(\varphi_i(x))$ and $\mathcal{S}(\varphi_i(x)) = \mathcal{S}(\varphi_i(y))$ hold for each $(x, y) \in \mathcal{S}$ and $i \in I$ up to null sets.

Now we claim that there exists a subset J of I such that $\{\varphi_i\}_{i\in J}$ are choice functions for $\mathcal{S} \subseteq \mathcal{R}$. Indeed, for any pair $(i,j) \in I \times I$, define a Borel subset $X_{i,j}$ of X by

$$X_{i,j} := \{x \in X : (\varphi_i(x), \varphi_j(x)) \in \mathcal{S}\}.$$

Since φ_i "normalizes" $\mathcal{S}$, $X_{i,j}$ is $\mathcal{S}$-invariant, so that it is either null or conull. Now we define the subset J by the following:

$$J := \{0\} \cup \{j \in I \setminus \{0\} : X_{i,j} \text{ is null for all } i < j\}.$$

By definition, each $\{\mathcal{S}(\varphi_j(x))\}_{j\in J}$ are mutually disjoint for a.e. $x \in X$. We will show that, for a.e. $x \in X$, $\mathcal{R}(x)$ is equal to the disjoint union of $\{\mathcal{S}(\varphi_j(x))\}_{j\in J}$. Indeed, if there exists a Borel subset F of X such that $\mu(F) > 0$ and the disjoint union of $\{\mathcal{S}(\varphi_j(x))\}_{j\in J}$ is not equal to $\mathcal{R}(x)$ for each $x \in F$. Then there exists a Borel function $k : F \mapsto\in I \setminus J$ such that $\varphi_{k(x)}(x)$ is not in $\bigcup_{j\in J} \mathcal{S}(\varphi_j(x))$. Since I is countable, we can choose $i \in I \setminus J$ such that $k^{-1}(i)$ is of positive measure. It follows that $(\varphi_j(x), \varphi_i(x))$ is not in $\mathcal{S}$ for all $j \in J$ and $x \in k^{-1}(i)$. In particular, the Borel subset $X_{j,i}$ is not conull for all $j \in J$. So $X_{j,i}$ must be a null set. It follows that $X_{j,i}$ is a null set for all $j < i$. This contradicts the definition of J. So we conclude that $\mathcal{R}(x)$ is a disjoint union of $\{\mathcal{S}(\varphi_j(x))\}_{j\in J}$ for a.e. $x \in X$. Hence $\{\varphi_j\}_{j\in J}$ are choice functions for $\mathcal{S} \subseteq \mathcal{R}$.

Let σ be the index cocycle for $\mathcal{S} \subseteq \mathcal{R}$ determined by $\{\varphi_j\}_{j\in J}$. Since $0 \in J$ and $\varphi_0 = id_X$, $\mathcal{S}$ contains $\sigma^{-1}(e)$. On the other hands, for each $j \in J$ and for a.e. $(x,y) \in \mathcal{S}$, $(\varphi_j(x), \varphi_j(y))$ is also in $\mathcal{S}$. So we obtain $\sigma^{-1}(e) = \mathcal{S}$. Moreover, since the map $X \ni x \mapsto \sigma(\varphi_i(x), x)$ is $\mathcal{S}$-invariant, we may assume that σ is a 1-cocycle to a countable group. Hence $\mathcal{S}$ is normal in $\mathcal{R}$. □

By using this theorem, we have the following

Definition 3. For each inclusion of ergodic discrete measured equivalence relation–subrelation $\mathcal{S} \subseteq \mathcal{R}$, there exists a intermediate subrelation $N_{\mathcal{R}}(\mathcal{S})$, which is the largest, up to a null set, among the Borel subrelation of $\mathcal{R}$ containing $\mathcal{S}$ as a normal subrelation. We call $N_{\mathcal{R}}(\mathcal{S})$ the normalizer of $\mathcal{S}$ in $\mathcal{R}$. Indeed, $N_{\mathcal{R}}(\mathcal{S})$ corresponds to the intermediate subfactor $\mathcal{GN}_A(B)''$.

4 The commensurability and the discreteness

As before, we fix an ergodic Borel subrelation $\mathcal{S}$ of $\mathcal{R}$. Put $A := W^*(\mathcal{R}, \omega)$ and $B := W^*(\mathcal{S}, \omega)$. It follows that the relative commutant $A_1 \cap B'$ is abelian. Moreover, we have the following:

Lemma 4. *(1) The relative commutant $A_1 \cap B'$ is contained in $L^\infty(\mathcal{R})$.*

Let $\mathcal{E}$ be a Borel subset of $\mathcal{R}$,

(2) The projection $\chi_\mathcal{E} \in L^\infty(\mathcal{R})$ on $L^2(\mathcal{R})$ is in B' if and only if $\chi_\mathcal{E}$ satisfies the following:

$$\chi_\mathcal{E}(x,z) = \chi_\mathcal{E}(y,z) \quad \textit{for a.e. } (x,y) \in \mathcal{S} \textit{ and } \forall z \in \mathcal{R}(x).$$

(3) The projection $\chi_\mathcal{E} \in L^\infty(\mathcal{R})$ on $L^2(\mathcal{R})$ is in $A_1 \cap B'$ if and only if $\chi_\mathcal{E}$ satisfies the following:

$$\chi_\mathcal{E}(x,z) = \chi_\mathcal{E}(y,z),\ \chi_\mathcal{E}(z,x) = \chi_\mathcal{E}(z,y) \quad \textit{for a.e. } (x,y) \in \mathcal{S} \textit{ and } \forall z \in \mathcal{R}(x).$$

Proof. (1) We have

$$A_1 \cap B' = JB'J \cap B' \subseteq JD'J \cap D' = (JDJ \vee D)' = L^\infty(\mathcal{R})' = L^\infty(\mathcal{R}).$$

Thus (1) has been proven.

(2) For any $a = L^\omega(f) \in A$ and $\xi \in L^2(\mathcal{R})$, we have

$$\{\chi_\mathcal{E} a\xi\}(x,z) = \sum_{y\sim x} f(x,y)\chi_\mathcal{E}(x,z)\xi(y,z)\omega(x,y,z),$$
$$\{a\chi_\mathcal{E}\xi\}(x,z) = \sum_{y\sim x} f(x,y)\chi_\mathcal{E}(y,z)\xi(y,z)\omega(x,y,z).$$

So we conclude that $\chi_\mathcal{E}$ belongs to B' if and only if $\chi_\mathcal{E}(x,z) = \chi_\mathcal{E}(y,z)$ for a.e. $(x,y) \in \mathcal{S}$ and $\forall z \in \mathcal{R}(x)$, which implies (2).

(3) The projection $\chi_\mathcal{E}$ is in A_1 if and only if $J\chi_\mathcal{E}J$ is in B'. Moreover, note that $J\chi_\mathcal{E}J = \chi_{\mathcal{E}^{-1}}$, where $\mathcal{E}^{-1} = \{(x,y) \in \mathcal{R} : (y,x) \in \mathcal{E}\}$. Now the conclusion follows from (2). $\square$

We set $T := E_B \circ \widehat{E_B}$, which is a faithful normal semifinite operator valued weight from A_1 to B. Let $\xi_0 := \chi_\mathcal{D} \in L^2(\mathcal{R})$, where $\mathcal{D}$ is the diagonal subset $\{(x,x) : x \in X\}$. So ξ_0 is a cyclic and separating unit vector for A. We write θ for the vector state on A given by ξ_0. Note that $A_1 \cap B'$ is contained in $L^\infty(\mathcal{R})$. Since the modular automorphism $\sigma_t^{\widehat{\theta}}$ of the weight $\widehat{\theta} := \theta \circ \widehat{E_B}$ is implemented on $L^2(\mathcal{R})$ by $\delta^{it} \in L^\infty(\mathcal{R})$ (see [8]), the restriction of $\sigma^{\widehat{\theta}}$ to $A_1 \cap B'$ is the identity. So, in particular, $(A_1 \cap B')_T = A_1 \cap B'$.

For a nonzero $a \in A$, consider the closed subspace $[BaB\xi_0]$. It is clearly B-invariant. Since $Jx\xi_0 = \sigma^{\theta}_{-i/2}(x^*)\xi_0$ $(x \in A)$ and B is globally invariant under σ^{θ}, we have, for any b, b_1 and $b_2 \in B$:

$$JbJ(b_1ab_2\xi_0) = b_1ab_2JbJ\xi_0 = b_1ab_2\sigma^{\theta}_{-i/2}(b^*)\xi_0 \in [BaB\xi_0].$$

This shows that $[BaB\xi_0]$ is also JBJ-invariant. Hence the projection z_a onto $[BaB\xi_0]$ belongs to $B' \cap (JBJ)' = A_1 \cap B'$.

Definition 5. We define $\mathcal{CG}(B)$ to be the set of all partial isometries $v \in A$ satisfying the following two conditions:

(1) Both v^*v and vv^* belong to B.

(2) The projections z_v and z_{v^*} belong to $\mathfrak{m}^+_{\widehat{E_B}}$.

We call $\mathcal{CG}(B)$ the *commensurability groupoid* of B in A.

It is easy to check that $\mathcal{CG}(B)$ is closed under the $*$-operation. Moreover, we have the following:

Lemma 6. *Let the notations be as above.*

(1) Each element in $\mathcal{CG}(B)$ belongs to $(\mathcal{CG}(B) \cap \mathcal{GN}_A(D))''$, i.e., $(\mathcal{CG}(B) \cap \mathcal{GN}_A(D))''$ coincides with $\mathcal{CG}(B)''$.

(2) $\mathcal{CG}(B) \cap \mathcal{GN}_A(D)$ is closed under the product operation.

Sketch of the proof. (1) A direct computation shows that, for each $a \in A$ and $w \in \mathcal{GN}_A(D)$, the inequality $z_{E_D(vw^*)w} \le z_v$ holds. So we have that each element in $\mathcal{CG}(B)'' \cap \mathcal{GN}_A(D)$ belongs to $(\mathcal{CG}(B) \cap \mathcal{GN}_A(D))''$ and get the conclusion.

(2) It suffices to show that, for each $v_1, v_2 \in \mathcal{CG}(B) \cap \mathcal{GN}_A(B)$, v_1v_2 also belongs to $\mathcal{CG}(B)$. We may and do assume that $v_1v_2 \neq 0$. By using the maximal arguments, for $i = 1, 2$, there exist unitaries $\{w_{i,n}\}_{n\ge1}$ in $\mathcal{GN}_B(D)$ such that $z_{v_i} = \sum_{n\ge1} w_{i,n}v_ie_Bv_i^*w_{i,n}^*$. So we have the following inequality:

$$z_{v_1v_2} \le \sum_{n,m\ge1} w_{1,n}v_1w_{2,m}v_2e_Bv_2^*w_{2,m}^*v_1^*w_{1,n}^*.$$

Hence we get

$$\begin{aligned}\widehat{E_B}(z_{v_1v_2}) &\leq \sum_{n,m\geq 1} w_{1,n}v_1\widehat{E_B}(w_{2,m}v_2e_Bv_2^*w_{2,m}^*)v_1^*w_{1,n}^* \\ &= \sum_{n\geq 1} w_{1,n}v_1\widehat{E_B}(z_{v_2})v_1^*w_{1,n}^* \\ &= \widehat{E_B}(z_{v_2})\sum_{n\geq 1} w_{1,n}v_1v_1^*w_{1,n}^* \\ &= \widehat{E_B}(z_{v_1})\widehat{E_B}(z_{v_2}) < \infty\end{aligned}$$

Thus we get the conclusion.

□

So we get the following

Theorem 7. *Let $\mathcal{S}$ be an ergodic Borel subrelation of $\mathcal{R}$, and $B := W^*(\mathcal{S},\omega)$ be the associated subfactor of A. Then the following are equivalent:*

(1) The commensurability groupoid of B in A generates A.

(2) The inclusion $B \subseteq A$ is discrete.

Sketch of the proof. (1)⇒(2): There exist countable elements $\{w_n\}_{n\geq 1}$ of $\mathcal{CG}(B) \cap \mathcal{GN}_A(D)$ such that $\vee_{n\geq 1} z_{w_n}$ is equal to 1. It follows that the relative commutant $A_1 \cap B'$ is discrete.
(2)⇒(1): Since $A_1 \cap B'$ is discrete, there exist minimal projections $\{f_n\}_{n\geq 1}$ in $A_1 \cap B'$ such that $\widehat{E_B}(f_n)$ is finite for all $n \geq 1$. Since $A_1 \cap B'$ is contained in $L^\infty(\mathcal{R})$, there exists a Borel partition $\{F_n\}_{n\geq 1}$ of $\mathcal{R}$ such that f_n is equal to χ_{F_n} for each $n \geq 1$. Since each F_n is not a null set, there exists $\rho_n \in [\mathcal{R}]_*$ such that F_n contains $\Gamma(\rho_n)$. It follows that for each $v \in \mathcal{GN}_A(D)$, there exist countable projections $\{e_n\}_{n\geq 1}$ such that $\sum_{n\geq 1} e_n = v^*v$ and $z_{ve_n} = f_n$, i.e., $ve_n \in \mathcal{CG}(B)$. So we conclude that A is generated by $\mathcal{CG}(B)$. □

By using this theorem, we have the following

Definition 8. For each inclusion of ergodic discrete measured equivalence relation–subrelation $\mathcal{S} \subseteq \mathcal{R}$, there exists a intermediate subrelation $\mathrm{Comm}_{\mathcal{R}}(\mathcal{S})$, which is the largest, up to a null set, among the Borel subrelation of $\mathcal{R}$ such that the inclusion $B \subseteq A$ is discrete. We call $\mathrm{Comm}_{\mathcal{R}}(\mathcal{S})$ the commensurability subrelation of $\mathcal{S}$ in $\mathcal{R}$. Indeed, $\mathrm{Comm}_{\mathcal{R}}(\mathcal{S})$ corresponds to the intermediate subfactor $\mathcal{CG}(B)''$.

5 Examples

We conclude this note with some concrete examples. These examples justify the terminologies "normality" and "commensurability" for equivalence relations and corresponding von Neumann algebras.

Firstly, we consider that the inclusion $\mathcal{S} \subseteq \mathcal{R}$ comes from the crossed product of a discrete measured equivalence relation by an inclusion of countable groups.

Proposition 9. *Suppose that $(\mathcal{S} \subseteq \mathcal{R})$ is equal to crossed products $(\mathcal{P} \ltimes H \subseteq \mathcal{P} \ltimes G)$, where $\mathcal{P}$ is a discrete measured equivalence relation with an outer action of G and a subgroup H. Then the following equations hold up to null sets:*

$$N_{\mathcal{R}}(\mathcal{S}) = \mathcal{P} \ltimes N_G(H), \quad \mathrm{Comm}_{\mathcal{R}}(\mathcal{S}) = \mathcal{P} \ltimes \mathrm{Comm}_G(H).$$

Proof. By assumption, there exist canonical choice functions $\{\varphi_g\}_{g \in G}$ for $\mathcal{P} \subseteq \mathcal{R}$. A direct computation shows that $\Gamma(\varphi_g)$ belongs to $N_{\mathcal{R}}(\mathcal{S})$ if $g \in N_G(H)$. So $N_{\mathcal{R}}(\mathcal{S})$ contains $\mathcal{P} \ltimes N_G(H)$. Conversely, for each $\rho \in [N_{\mathcal{R}}(\mathcal{S})]_*$, there exists a partition $\{E_g\}_{g \in G}$ of $\mathrm{Dom}(\rho)$ such that $\rho|_{E_g}(x) \in \mathcal{S}(\varphi_g(x))$ for each $g \in G$ up to null sets. On the other hands, it is easy to check that $Du_g \cap \mathcal{GN}_A(B) = \{0\}$ if $g \notin N_G(H)$. So we conclude that E_g is null if $g \notin N_G(H)$ and $N_{\mathcal{R}}(\mathcal{S})$ is equal to $\mathcal{P} \ltimes N_G(H)$ up to a null set.

The second half assertion follows from the same arguments. $\square$

We note that the above proposition treats the case that $W^*(\mathcal{S})$ is regular in $W^*(N_{\mathcal{R}}(\mathcal{S}))$, i.e., $\mathcal{GN}_A(B)''$ coincides with $N_A(B)''$. The next example shows that there exists a normal subrelation such that the corresponding subfactor is not regular in A.

Example 1. Let $\mathcal{R}$ be an ergodic type III_λ equivalence relation with a II_1 subrelation $\mathcal{S}$ with regard to an admissible measure $(0 < \lambda < 1)$. Then, by [9], B is singular, i.e., $N_A(B)$ is equal to the set of unitaries in B. But $\mathcal{S}$ is normal in $\mathcal{R}$ because $\mathcal{S}$ is equal to the kernel of the Radon–Nikodym derivative. So we have $\mathcal{GN}_A(B)'' = A$.

Finally, we will treat the case which can not be expressed as the group measure space constructions.

Example 2. For $n \geq 3$ and $\lambda \in \mathbf{R} \setminus \mathbf{Q}$, consider actions $(H \subseteq G) := (SL_n(\mathbf{Q}) \subseteq (SL_n(\mathbf{Q}) \vee T_\lambda))$ on $X := \mathbf{R}^n/\mathbf{Z}^n$, where $T_\lambda(x_n) := (x_n + \lambda)$. It is known that $H \subseteq G$ act freely and ergodically on X. Moreover, a direct computation shows that $[H : H \cap T_{-\lambda} H T_\lambda]$ is infinite. It follows that the equation $\mathrm{Comm}_{\mathcal{R}_G|_Y}(\mathcal{R}_H|_Y) = \mathcal{R}_H|_Y$

holds for each non-null Borel subset Y of X, where $\mathcal{R}_G := \{(gx, x) : x \in X,\ g \in G\}$ and $\mathcal{R}_G|_Y := \mathcal{R}_G \cap Y \times Y$. We note that $\mathcal{R}_H|_Y$ does not come from the group measure space construction if $\mu(Y)/\mu(X) \notin \mathbf{Q}$ ([7]).

Remark . Our definition of the commensurability for equivalence relations depends on the theory of operator algebras. In the recent work of the author, he succeeds in characterizing this property in terms of measure theoretical arguments ([2]).

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