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Blow-up at space infinity for nonlinear heat equations

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1 Introduction and main theorems

In this paper we gather the papers [5], [6] and [12] for our talk at Kyoto University. In particular we make the proofs of theorems in [5] easier by using the methods in [12] and other.

We consider solutions of the initial value problem for the equation

\[
\begin{cases}
  u_t = \Delta u + f(u), & x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}^n.
\end{cases}
\]

(1)

The nonlinear term \( f \in C^1(\mathbb{R}_+) \) satisfies that

\[
\int_0^\infty \frac{d\xi}{f(\xi)} < \infty \text{ with some } C \geq 0,
\]

(2)

and

\[
\begin{cases}
  \text{there exists a function } \Phi \in C^2(\mathbb{R}_+) \text{ such that} \\
  \Phi(v) > 0, \Phi'(v) > 0 \text{ and } \Phi''(v) \geq 0 \text{ for } v > 0, \\
  \int_1^\infty \frac{d\xi}{\Phi(\xi)} < \infty, \\
  \text{and } f'(v)\Phi(v) - f(v)\Phi'(v) \geq c\Phi(v)\Phi'(v) \text{ for } v > b \\
  \text{with some } b \geq 0 \text{ and } c \geq 0.
\end{cases}
\]

(3)

Remark. The conditions (2) and (3) were used in [12]. They are weaker than the conditions used in [5] and [6]:

\[
f(\delta b) \leq \delta^p f(b)
\]

for all \( b \geq b_0 \) and for all \( \delta \in (\delta_0, 1) \) with some \( b_0 > 0 \), some \( \delta_0 \in (0, 1) \) and some \( p > 1 \).
The initial data $u_0$ is assumed to be a measureable function in $\mathbb{R}^n$ satisfying

$$0 \leq u_0(x) \leq M \text{ a.e. in } \mathbb{R}^n$$

(4)

for some positive $M$. We are interested in initial data such that $u_0 \to M$ as $|x| \to \infty$ for $x$ in some sector of $\mathbb{R}^n$. We assume that there exists a sequence $\{x\}_{m=1}^{\infty} \subset \mathbb{R}^n$ such that

$$\lim_{m \to \infty} u_0(x + x_m) = M \quad \text{a.e. in } \mathbb{R}^n.$$ 

(5)

Remark. The condition (5) was given in [12]. This condition is equivalent to the condition in [5] with [6]:

$$\text{essinf}_{x \in B_m}(u_0(x) - M_m(x - x_m)) \geq 0 \quad \text{for } m = 1, 2, \ldots ,$$

where $B_m = B_{r_m}(x_m)$ with a sequence $\{r_m\}_{m=1}^{\infty}$, a sequence of functions $\{M_m(x)\}_{m=1}^{\infty}$ satisfying

$$\lim_{m \to \infty} r_m = \infty, \quad M_m(x) \leq M_{m+1}(x) \quad \text{for } m \geq 1$$

$$\lim_{m \to \infty} \inf_{x \in [1, r_m]} \frac{1}{|B_x|} \int_{|B_x(0)|} M_m(x) \, dx = M,$$

and some sequence of vectors $\{x_m\}_{m=1}^{\infty}$. Here $B_r(x)$ denotes the opened ball of radius $r$ centered at $x$.

Problem (1) has a unique bounded solution at least locally in time. However, the solution may blow up in finite time. For a given initial value $u_0$ and nonlinear term $f$ let $T^* = T^*(u_0, f)$ be the maximal existence time of the solution. If $T^* = \infty$, the solution exists globally in time. If $T^* < \infty$, we say that the solution blows up in finite time. It is well known that

$$\limsup_{t \to T^*} \|u(\cdot, t)\|_\infty = \infty,$$

(6)

where $\| \cdot \|_\infty$ denotes the $L^\infty$-norm in space variables.

In this paper we are interested in behavior of a blowing up solution near space infinity as well as location of blow-up directions defined below. A point $x_{BU} \in \mathbb{R}^n$ is called a blow-up point if there exists a sequence $\{(x_m, t_m)\}_{m=1}^{\infty}$ such that

$$t_m \uparrow T^*, \quad x_m \to x_{BU} \quad \text{and} \quad u(x_m, t_m) \to \infty \quad \text{as } m \to \infty.$$
If there exists a sequence $\{(x_m, t_m)\}_{m=1}^{\infty}$ such that
\[ t_m \uparrow T^*, \quad |x_m| \rightarrow \infty \quad \text{and} \quad u(x_m, t_m) \rightarrow \infty \quad \text{as} \quad m \rightarrow \infty, \]
then we say that the solution blows up to at space infinity.

A direction $\psi \in S^{n-1}$ is called a blow-up direction if there exists a sequence $\{(x_m, t_m)\}_{m=1}^{\infty}$ with $x_m \in \mathbb{R}^n$ and $t_m \in (0, T^*)$ such that $u(x_m, t_m) \rightarrow \infty$ as $m \rightarrow \infty$ and
\[ \frac{x_m}{|x_m|} \rightarrow \psi \quad \text{as} \quad m \rightarrow \infty. \quad (7) \]

We consider the solution $v(t)$ of an ordinary differential equation
\[
\begin{cases}
  v_t = f(v), & t > 0, \\
  v(0) = M.
\end{cases} \quad (8)
\]
Let $T_v = T^*(M, f)$ be the maximal existence time of solutions of (8), i. e.,
\[ T_v = \int_{M}^{\infty} \frac{ds}{f(s)}. \]

We are now in position to state our main results.

**Theorem 1.** Assume that $f \in C^1(\mathbb{R}_+)$ is nondecreasing function and locally Lipschitz in $\bar{\mathbb{R}}_+$. Let $u_0$ be a continuous function satisfying (4) and (5). Then there exists a subsequence of $\{x_m\}_{m=1}^{\infty}$, independent of $t$ such that
\[ \lim_{m \rightarrow \infty} u(x + x_m, t) = v(t) \quad \text{in} \quad \mathbb{R}^n. \quad (9) \]

The convergence is uniform in every compact subset of $\mathbb{R}^n \times [0, T_v)$. Moreover, the solution blows up at $T_v$.

For this theorem we should introduce the results of Gladkov [7]. In his paper there is the result [7, Theorem 1] relative to our first theorem. He considered the initial-boundary value problem:
\[
\begin{cases}
  u_t = u_{xx} + f(x, t, u), & x > 0, 0 < t < T_0, \\
  u(x, 0) = u_0(x), & x > 0, \\
  u(0, t) = \mu(t) & 0 < t < T_0,
\end{cases}
\]
and the ordinary differential equation
\[
\begin{cases}
  v_t = \hat{f}(t, u), & 0 < t < T_0, \\
  v(0) = M,
\end{cases}
\]
where \( T_0 \in (0, \infty], \) \( 0 \leq f(x,t,u) \leq \tilde{f}(t,u), \) \( \lim_{x \to \infty} f(x,t,u) = \tilde{f}(t,u), \)

\( 0 \leq u_0 \leq M \) and \( \lim_{x \to \infty} u_0(x) = M. \) For the equations he had \( u(x,t) \to v(t) \) as \( x \to \infty \) uniformly for \([0,T]\) with \( T < T_0. \) For the proof of this result, he used the fundamental solution of the heat equation.

In [5] the expression (9) was the weak sense:

\[
\lim_{n \to \infty} u(x_m, t) = v(t),
\]

(10)

After [5], (9) was used in [12]. However, for proving Theorems 2 and 3, we can select even the expression (10).

Our second main result is on the location of blow-up points.

**Theorem 2.** Assume the same hypotheses of Theorem 1 and that \( f \) satisfies (2) and (3). Let \( u_0 \not\equiv M \) a.e. in \( \mathbb{R}^n. \) Then the solution of (1) has no blow-up points with \( \infty \) in \( \mathbb{R}^n. \) (It blows up only at space infinity.)

There is a huge literature on location of blow-up points since the work of Weissler [15] and Friedman-McLeod [1]. (We do not intend to list references exhaustively in this paper.) However, most results consider either bounded domains or solutions decaying at space infinity; such a solution does not blow up at space infinity [2].

As far as the authors know, before the result of [4] the only paper discussing blow-up at space infinity is the work of Lacey [8]. He considered the Dirichlet problem in a half line. He studied various nonlinear terms and proved that a solution blows up only at space infinity. His method is based on construction of suitable subsolutions and supersolutions. However, the construction heavily depends on the Dirichlet condition at \( x = 0 \) and does not apply to the Cauchy problem even for the case \( n = 1. \)

As previously described, the Giga-Umeda [4] proved the statement of Theorems 1 and 2 assuming that \( \lim_{|x| \to \infty} u_0(x) = M \) for positive solutions of \( u_t = \Delta u + u^p. \) Later, Simojö[13] had the same results as in [4] by relaxing the assumptions of initial data \( u_0 \geq 0 \) which is similar to that in the present paper. His approach is a construction of a suitable supersolution which implies that \( a \in \mathbb{R}^n \) is not a blow-up point. Although he restricted himself for \( f(s) = s^p, \) his idea works our \( f \) under slightly strong assumption on \( u_0. \) Here we give a different approach.

By Simojö's results[13] it is natural to consider a problem of "blow-up direction" defined in (7). We next study this "blow-up direction" for the value \( \infty. \)

**Theorem 3.** Assume the same hypotheses of Theorem 1. Let a direction \( \psi \in S^{n-1}. \) If and only if there exists sequences \( \{y_m\}_{m=1}^\infty \) and satisfying
lim_{m \to \infty} y_m/|y_m| = \psi \text{ such that }
\lim_{m \to \infty} u_0(x + y_m) = M \text{ a.e. in } \mathbb{R}^n,
(11)
then \psi \text{ is a blow-up direction.}

After [5] there are some results in this field. Shimojö had the result of the upperbound and the lowerbound:

\[ v(t - \eta(x,t)) \leq u(x,t) \leq v(t - c\eta(x,t)) \]

with some function \( \eta \) and \( c \in (0,1) \). Moreover, he proved the complete blow-up of the solution. Seki-Suzuki-Umeda [12] and Seki [11] improved the results of [5] for the quasilinear parabolic equation:

\[ u_t = \Delta \varphi(u) + f(u). \]

In particular they had more results for more general case. In [3] some of the proofs of theorems in [5] were corrected.

This paper is organized as follows. In section 2 we prove Theorem 1 by using the fundamental solution of the heat equation. The proof of Theorem 2 is given in section 3 by using the argument used in [12]. In section 4 we show Theorem 3 using Theorem 1 and Lemma 3.2.

### 2 Behavior at space infinity

In this section we prove Theorem 1. We give proof of Theorem 1 which is inspired in private communication with Y. Seki and M. Shimojö.

**Proof of Theorem 1.** Put \( w = v - x \). Then, we have for \( t \in (0,T_0] \) with \( T_0 \in (0,T(M)) \),

\[ w_t = \Delta w + f(v(t)) - f(u(\cdot,t)) \leq \Delta w + C(v - u), \]

where

\[ C = \sup_{t \in [0,T_0]} \left\| \int_0^1 f'(\theta v(t) + (1 - \theta)u(\cdot,t))d\theta \right\|_{\infty}. \]

Then, by comparison we obtain

\[ w(x,t) \leq e^{CT_0} e^{\Delta t}(M - u_0(x)) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t}(M - u_0(y))dy. \]
From (5) we have
\[\lim_{m \to \infty} u(x + x_m, t) = v(t) \quad \text{in } \mathbb{R}^n. \] (12)

It remains to prove that \( u \) blows up at \( t = T_v \). For this purpose it suffices to prove that \( \lim_{m \to \infty} u(x_m, t_m) = \infty \) for some sequence \( t_m \to T_v \). We argue by contradiction. Suppose that \( \lim_{m \to \infty} u(x_m, t_m) \leq C \) for some \( C \in [M, \infty) \). Then we could take \( t_0 \in (0, T_v) \) satisfying \( v(t_0) \geq C \) and \( v_t(t) > 0 \) for \( t \geq t_0 \). By (12) we have
\[\lim_{m \to \infty} u \left( x_m, \frac{t_0 + T_v}{2} \right) = v \left( \frac{t_0 + T_v}{2} \right) > C,\]
which yields a contradiction. We thus proved that \( \lim_{m \to \infty} u(x_m, t_m) = \infty \), so that \( u(x, t) \) blows up at \( T_v \).

\[\square\]

3 No blow-up point in \( \mathbb{R}^n \)

In this section we prove Theorem 2. We use three lemmas for proving the theorem.

**Lemma 3.1.** Assume the same hypothesis of Theorem 1. Let \( u \) and \( v \) be solutions of (1) and (8) with \( u_0, M \) and \( f \) satisfying (2), (3) and (4). Then there exist \( \delta = \delta(a, t_0, u_0, f) \in (0, 1) \) such that for \((x, t) \in B_2(a) \times [t_0, T_v)\),
\[u(x, t) \leq \delta v(t)\]

with \( t_0 \in [0, T_v) \).

**Proof.** By (2) there exist \( M_f = M_f(f) > M \) and \( \delta_f = \delta_f(f) \in (0, 1) \) satisfying for \( r \geq M_f \) and \( \delta \in (\delta_f, 1) \),
\[f(\delta r) \leq \delta f(r). \] (13)

Let \( T_0 = T_0(u_0, f) \in (0, T_v) \) such that \( v(T_0) = M_f \). Since \( u_0 \leq M \) and \( u_0 \not\equiv M \) a.e. in \( \mathbb{R}^n \), we have \( u(x, T_0) < v(T_0) \). Note that \( u(x, t) < v(t) \) for \( t \in (0, T_0] \). Let \( w \) be the solution of
\[\begin{cases} w_t = \Delta w, \\ w(x, T_0) = \max\{u(x, T_0)/v(T_0), \delta_f\}, \end{cases} \quad x \in \mathbb{R}^n, t \in (T_0, T^*),
\[w(x, T_0) = \max\{u(x, T_0), \delta_f v(T_0)\}, \quad x \in \mathbb{R}^n.\]

Put \( \bar{u} = vw \). Then we have
\[\begin{cases} \bar{u}_t = \Delta \bar{u} + w f(v), \\ \bar{u}(x, T_0) = \max\{u(x, T_0), \delta_f v(T_0)\}, \end{cases} \quad x \in \mathbb{R}^n, t \in (T_0, T^*),
\[\bar{u}(x, T_0) = \max\{u(x, T_0), \delta_f v(T_0)\}, \quad x \in \mathbb{R}^n.\]
Since \( w(x, t) \in [\delta_f, 1) \) and \( v(t) \geq M_f \), we have
\[
wf(v) \geq f(wv) = f(\overline{u})
\]
by (13). This \( \overline{u} \) is supersolution of (1).

Since for any \( x \in \mathbb{R}^n, \sup_{t \in [T_0, T^*)} w(x, t) < 1 \), we can take \( \delta = \delta(a, T_0, u_0, f) \in (0, 1) \) satisfying \( w(x, t) \leq \delta(x, t) \in B_1(a) \times [T_0, T^*) \). Thus, we obtain
\[
u(x, t) \leq \overline{u}(x, t) = w(x, t)v(t) \leq \delta v(t)
\]
and Lemma 3.1 is proved. \( \square \)

For any \( a \in \mathbb{R}^n \), we consider the solution \( \phi = \phi_a \) of the equation:
\[
\begin{cases}
\phi_t = \Delta \phi + f(\phi), & x \in B_1, t \in (t_1, T_v), \\
\phi(x, 0) = \phi_0(x), & x \in B_1, \\
\phi(x, t) = v(t), & x \in \partial B_1, t \in (t_1, T_v),
\end{cases}
\]
(14)

where \( \phi_0(x) = v(t_1)(1 - \epsilon \cos \frac{\pi|x|}{2}) \) with \( \epsilon = \epsilon(u_0, f, a) > 0 \) sufficiently small satisfying
\[
\phi_0(x) \geq u(x + a, t_1)
\]
and \( B_1 \) denotes the open ball of radius 1 and centered at 0. It is easily seen that
\[
\Delta \phi_0(x) + f(\phi_0(x)) \geq 0.
\]

By the maximum principle [10] we have
\[
\phi(x, t) \geq u(x + a, t) \quad \text{and} \quad \phi_t \geq 0 \quad \text{for} \quad x \in B_1, \ t \in [t_1, T_v).
\]
(16)

If \( w \) has no blow-up point in \( \mathbb{R}^n \), the \( u \) has no blow-up point in \( \mathbb{R}^n \), neither. We should show that \( w \) has no blow-up point.

**Lemma 3.2.** Assume the same hypotheses of Lemma 3.1. Let \( \Omega \in B_1 \) be a domain. If \( \partial_t \phi(x, t) \geq 0 \) in \( \Omega \times (t_1, T_v) \) and there exist \( \nu \in S^{n-1} \) and \( \delta > 0 \), such that
\[
\nu \cdot \nabla \phi(x, t) \leq -\delta |\nabla \phi(x, t)| < 0 \quad \text{in} \quad \Omega \times (t_1, T_v),
\]
then \( \phi \) does not uniformly blow-up in \( \Omega \):
\[
\inf_{x \in \Omega} \phi(x, t) \leq L < \infty \quad \text{for} \quad t \in (t_1, T_v).
\]
Proof of Lemma 3.2. This lemma is proved in [9] (See [9, Lemma 4.1]). \qed

Proof of Theorem 2. Put $r \in (0, 1)$. Define

$$
\mu(x, t) = \phi(2r - x_1, x', t) - \phi(x_1, x', t),
$$

where $x = (x_1, x')$ with $x' = (x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}$. Then, we obtain

$$
\begin{cases}
\mu_t \geq \Delta \mu + C(x, t) \mu, & x \in D_r, t \in (t_1, T_v), \\
\mu(x, 0) = \phi_0(2r - x_1, x') - \phi_0(x_1, x') \geq 0, & x \in D_r, \\
\mu(x, t) \geq 0, & x \in \partial D_r, t \in (t_1, T_v),
\end{cases}
$$

where

$$
C(x, t) = \int_0^1 \{\theta \phi(2r - x_1, x', t) + (1 - \theta) \phi(x_1, x', t)\} d\theta
$$

$$
D_r = \{x : x_1 < r\} \cap \{x : (x - 2r)^2 < 1\}.
$$

Thus, by the maximum principle [10] we have

$$
\mu \geq 0 \quad \text{in} \quad D \times [t_1, T_v)
$$

and

$$
\phi(2r - x_1, x', t) \geq \phi(x_1, x', t) \quad \text{in} \quad D \times [t_1, T_v).
$$

Since $r \in (0, 1)$ is arbitrary, we obtain that $\phi_{x_1} \geq 0$ for $x \in \{x|x_1 > 0\}$ and

$$
-e_1 \cdot \nabla \phi \leq -\phi_{x_1} \leq -\frac{\delta x_1}{|x|} |\nabla \phi| \quad \text{in} \quad D \cup \{x|x_1 \geq 0\}
$$

with some $\delta > 0$, where $e_1 = (1, 0, 0, \ldots, 0)$. Since $\phi_t \geq 0$ and $\inf_{x \in B_1} \phi(x, t) = \phi(0, t)$, by Lemma 3.2 we have

$$
\lim_{t \to T_v} \phi(0, t) \leq L \text{ with some } L < \infty.
$$

Thus

$$
\lim_{t \to T_v} u(a, t) \leq L \text{ with same } L.
$$

Since $a \in \mathbb{R}^n$ is arbitrary, $u$ does not blow up at $t = T_v$ in $\mathbb{R}^n$. \qed
4 On blow-up direction

We shall prove Theorem 3 which gives a condition for blow-up direction.

Proof of Theorem 3. We first prove that if $u_0$ satisfies (11), then $\psi$ is a blow-up direction. By assumption we obtain that $u_0(x)$ satisfies (5) with some sequences $\{x_m\}_{m=1}^{\infty}$ satisfying $\lim_{m \to \infty} x_m/|x_m| = \psi$. Then, from the proof of Theorem 1 it follows that

$$\lim_{m \to \infty} u(x_m, t_m) = \infty$$

with the sequence $\{t_m\}_{m=1}^{\infty}$ satisfying $\lim_{m \to \infty} t_m = T_v$. Since $\lim_{m \to \infty} x_m/|x_m| = \psi$ by the assumption we obtain that $\psi$ is a blow-up direction.

We next show that if $\psi$ is a blow-up direction, then there exist $\{x_m\}_{m=0}^{\infty} \subset \mathbb{R}^n$ such that $x_m/|x_m| \to \psi$, $t_m \to T_v$ and $u(x_m, t_m) \to \infty$ as $m \to \infty$. In contrary it says that if for any sequences $\{x_m\}_{m=1}^{\infty} \subset \mathbb{R}^n$ satisfying $\lim_{m \to \infty} x_m/|x_m| = \psi$, $u_0$ does not satisfy (11), then $\psi$ is not a blow-up direction.

Since $\lim_{m \to \infty} u_0(x + x_m) = M$ a.e. in $\mathbb{R}^n$, we have

$$\lim_{m \to \infty} \sup_{x \in B_3(x_m)} \frac{1}{4\pi t} \int_{\mathbb{R}^n} e^{-(x-y)^2/4t} u_0(y) dy < M$$

for $t > 0$. Since the solution of (1) satisfies the integral equation

$$u(x, t) = e^{\Delta t} u_0(x) + \int_0^t e^{\Delta(t-s)} f(u(x, s)) ds,$$

we have

$$u(x, t) \leq e^{\Delta t} u_0(x) + \int_0^t f(v(s)) ds = v(t) - M + e^{\Delta t} u_0(x)$$

for $(x, t) \in \mathbb{R}^n \times [0, T^*)$.

Let $M_f$, $\delta_f$ and $T_0$ be the same as proof of Lemma 3.1. We consider the solution $w$ of

$$\begin{cases}
  w_t = \Delta w, & x \in \mathbb{R}^n, t \in (T_0, T_v), \\
  w(x, T_0) = \max\{v(T_0) - M + e^{\Delta T_0} u_0(x)/v(T_0), \delta_f\}, & x \in \mathbb{R}^n.
\end{cases}$$

We now introduce $\tilde{u} = vw$. From the proof of Lemma 3.1, it follows that $\tilde{u} \geq u$ for $(x, t) \in \mathbb{R}^n \times [T_0, T^*)$. Then we have

$$u(x, t) \leq v(t) e^{\Delta(t-T_0)} \max\{v(T_0) - M + e^{\Delta T_0} u_0(x)/v(T_0), \delta_f\}$$
for \((x, t) \in \mathbb{R}^n \times [T_0, T_v)\).

Put \(U_m = \sup_{x \in B_2(x_m)} e^{T_0} u(x)\). From (17), there exists \(M_0 \in (0, M)\) such that

\[
\lim_{m \to \infty} U_m \leq M_0(< M).
\]

There exists a sequence \(\{V_k\}_{k=1}^\infty\) such that \(V_k = (M_0 + M)/2\), \(\lim_{k \to \infty} V_k = M_0\)
\(V_{k+1} \leq V_k\) and \(V_k \geq U_{m_k}\) with a sequence \(\{m_k\}_{k=1}^\infty\) satisfying \(u_{k+1} > u_k\) for \(k \in \mathbb{N}\). Thus, since \((x - y)^2 \leq 2x^2 + 2y^2\), we obtain

\[
\sup_{x \in B_1(\tilde{x}_k)} w(x, t) \leq W_k(t)
\]

\[
= e^{\Delta(t-T_0)} \max \left\{ \frac{v(T_0) - (M - V_k)e^{-|x|^2/2t} \int_{|y| < 2} e^{-|y|^2/2t} u_0(y) dy}{(4\pi T_0)^{-n/2} v(T_0)}, \delta_f \right\} < 1
\]

for \(t \in [T_0, T_v]\), where \(\tilde{x}_k = x_{m_k}\). By comparison we have \(W_{k+1}(t) \leq W_k(t)\) for \(t \in [T_0, T_v]\) and \(k \in \mathbb{N}\). From Lemma 3.2 and comparison it follows that there exist the sequence \(\{\eta_k\}_{k=1}^\infty\) satisfying \(0 < \eta_{k+1} \leq \eta_k < \infty\) such that

\[
\lim_{t \to T_v} u(x_{m_k}, t) \leq \eta_k.
\]

Since the sequence \(\{x_m\}_{m=1}^\infty\) is arbitrary, we obtain that \(\psi\) is not blow-up direction.

\[\square\]

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References


