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Existence of Solutions with Moving Singularities for a Semilinear Parabolic Equation

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Abstract

We study the Cauchy problem for a semilinear parabolic equation with a power nonlinearity. It is known that in some parameter range, the equation has a singular steady state. Our concern is a solution with a moving singularity that is obtained by perturbing the singular steady state. By the formal expansion, it turns out that the correction term must satisfy the heat equation with inverse-square potential near the singular point. From the well-posedness of this equation, we see that there appears a critical exponent. Paying attention to this exponent, given a motion of the singular point and suitable initial data, we establish the time-local existence result.

1 Introduction

We study singular solutions of the semilinear parabolic equation

$$\begin{cases}
    u_t = \Delta u + u^p & \text{in } \mathbb{R}^N \times (0, \infty), \\
    u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N,
\end{cases}$$

(1.1)

where $p > 1$ is a parameter and $u_0 \in L^1_{loc}(\mathbb{R}^N)$ is a nonnegative function. It is known that for

$$N \geq 3, \quad p > p_{\text{sing}} := \frac{N}{N-2},$$

(1.1) has an explicit singular steady state $\varphi(|x|) \in C^\infty(\mathbb{R}^N \setminus \{0\})$ with a singular point $0$;

$$\varphi(|x|) = L|x|^{-m}, \quad m = \frac{2}{p-1}, \quad L^{p-1} = m(N - m - 2).$$
Then $\varphi(|x|)$ satisfies (1.1) in the distribution sense, and
\[
\varphi_{rr} + \frac{N-1}{r} \varphi_r + \varphi^p = 0, \quad r = |x| > 0. \tag{1.2}
\]
Clearly, the spatial singularity of $u = \varphi(|x|)$ persists for all $t > 0$, but the singular point does not move in time.

Our aim of this paper is to discuss the existence of a solution of (1.1) whose spatial singularity moves in time. More precisely, we define a solution with a moving singularity as follows.

**Definition 1.** The function $u(x, t)$ is said to be a solution of (1.1) with a moving singularity $\xi(t) \in \mathbb{R}^N$ for $t \in (0, T)$, where $0 < T \leq \infty$, if the following conditions hold:

(i) $u$, $u^p \in C([0, T); L_{loc}^1(\mathbb{R}^N))$ satisfy (1.1) in the distribution sense.

(ii) $u(x, t)$ is defined on $\{(x, t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\}, t \in (0, T)\}$, and is twice continuously differentiable with respect to $x$ and continuously differentiable with respect to $t$.

(iii) $u(x, t) \to \infty$ as $x \to \xi(t)$ for every $t \in [0, T)$.

In this paper, we study the time-local existence for a solution with a moving singularity of the Cauchy problem (1.1). In order to state our result, we first introduce a critical exponent given by
\[
p_* := \frac{N + 2\sqrt{N - 1}}{N - 4 + 2\sqrt{N - 1}},
\]
which appeared in the papers of Véron [8] and Chen-Lin [3]. It was shown in [8] that $p_*$ is related to the linearized stability of the singular steady state, while it was shown in [3] that $p_*$ plays a crucial role for the existence of solutions with a prescribed singular set of the Dirichlet problem
\[
\begin{cases}
\Delta u + u^p = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$. In fact, in [3], they proved that if $N \geq 3$, $p_{\text{sing}} < p < p_*$, then for any closed set $K \subset \Omega$, there exists a singular solution having $K$ as a singular set. We note that $p_*$ is larger than $p_{\text{sing}}$ and is smaller than the Sobolev critical exponent $p_S := (N+2)/(N-2)$. We also introduce the important numbers
\[
\lambda_1 := \frac{N - 2 - \sqrt{(N - 2)^2 - 4pL^p - 1}}{2},
\]
\[
\lambda_2 := \frac{N - 2 + \sqrt{(N - 2)^2 - 4pL^p - 1}}{2}.
\]
We note that for $N \geq 3$, $p_{\text{sing}} < p < p_*$, the constants $\lambda_1 < \lambda_2$ are positive roots of
\[
\lambda^2 - (N - 2)\lambda + pI_p^{-1} = 0.
\]
Finally, for $a \in \mathbb{R}$, $[a]$ denotes the largest integer not greater than $a$.

Our result is concerning the time-local existence of a solution of (1.1) with a moving singularity.

**Theorem 1.** Let $N \geq 3$ and $p_{\text{sing}} < p < p_*$. Assume the following conditions:

(A1) $\xi(t) \in C^{i+\alpha}([0, \infty); \mathbb{R}^N)$ ($\alpha > 0$) with $i = \left\lfloor \frac{m-\lambda_2}{2} \right\rfloor + 1$.

(A2) $u_0$ is nonnegative and continuous in $x \in \mathbb{R}^N \setminus \xi(0)$, and is uniformly bounded for $|x - \xi(0)| \geq 1$.

(A3) If $m - \lambda_2$ is not an integer, then
\[
u_0(x) = L|x - \xi(0)|^{-m} \left\{ 1 + \sum_{i=1}^{\left\lfloor \frac{m-\lambda_2}{2} \right\rfloor} b_i \left( \frac{x - \xi(0)}{|x - \xi(0)|}, 0 \right) |x - \xi(0)|^i \right.
\]
\[+ O(|x - \xi(0)|^{m-\lambda_2+\epsilon}) \left. \right\}
\]
as $x \to \xi(0)$ for some $\epsilon > 0$, where $b_i(\omega, t)$ are functions on $S^{N-1}$ defined later by (2.9)-(2.5). If $m - \lambda_2$ is an integer, then
\[
u_0(x) = L|x - \xi(0)|^{-m} \left\{ 1 + \sum_{i=1}^{m-\lambda_2} b_i \left( \frac{x - \xi(0)}{|x - \xi(0)|}, 0 \right) |x - \xi(0)|^i \right.
\]
\[+ c(0)|x - \xi(0)|^{m-\lambda_2} \log |x - \xi(0)| + O(|x - \xi(0)|^{m-\lambda_2+\epsilon}) \left. \right\}
\]
as $x \to \xi(0)$ for some $\epsilon > 0$, where $b_i(\omega, t)$ are functions on $S^{N-1}$ defined later by (2.9)-(2.5) and $b_{m-\lambda_2}(\omega, t)$ and $c(t)$ satisfy (3.1).

Then for some $T > 0$, there exists a solution of (1.1) with a moving singularity $\xi(t)$.

**Remark 1.** If $N \geq 3$ and
\[p_{\text{sing}} < p < \min \left\{ p_*, \frac{3N+5}{3N-3} \right\},
\]
then $0 \leq m - \lambda_2 < 1$ so that $[m - \lambda_2] = 0$. In this case, (A1) implies $\xi(t) \in C^{1+\alpha}([0, \infty); \mathbb{R}^N)$ ($\alpha > 0$), and (A3) is simplified as
\[
u_0(x) = L|x - \xi(0)|^{-m} + O(|x - \xi(0)|^{-\lambda_2+\epsilon}) \quad \text{as} \ x \to \xi(0).
\]
In this paper, we consider only the time-local existence of the Cauchy problem with a moving singularity. Needless to say, the existence of time-global solutions are important questions. Also, when the solution with a moving singularity is not time-global, it is interesting to ask what happens at the maximal existence time. These questions will be future works.

This paper is organized as follows: In Section 2 we carry out formal analysis for a solution of (1.1) as a perturbation of the singular steady state. In Section 3 we state the outline of proof of the time-local existence.

2 Formal expansion at a singular point

In this section, we consider the formal expansion of a solution $u(x, t)$ of (1.1) with a moving singularity $\xi(t)$. Assuming that the solution resembles the singular steady state around $\xi(t)$, we may naturally expand $u(x, t)$ as

$$u(x, t) = Lr^{-m}\left\{1 + \sum_{i=1}^{k} b_i(\omega, t)r^i + v(y, t)r^m\right\}, \quad (2.1)$$

where

$$y = x - \xi(t), \quad r = |x - \xi(t)|, \quad \omega = \frac{1}{r}(x - \xi) \in S^{N-1}, \quad k = [m],$$

and the remainder term $v$ satisfies

$$v(y, t) = o(|y|^{-m}) \text{ as } |y| \rightarrow 0. \quad (2.2)$$

Substituting (2.1) into (1.1), and using

$$r_t = -(x - \xi) \cdot \xi_t, \quad \omega_t = -\frac{1}{r}\xi_t + \frac{\omega \cdot \xi_t}{r},$$

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}}$$

and the Taylor expansion, we compare the coefficients of $r^{-m+i-2}$ for $i = 0, 1, \ldots, k$. Then we obtain

$$r^{-m-2}; (Lr^{-m})_{rr} + \frac{N-1}{r}(Lr^{-m})_r + (Lr^{-m})^p = 0,$$

$$r^{-m-1}; \Delta_{S^{N-1}}b_1 + \{(-m+1)(N-m-1) + pm(N-m-2)\}b_1 = mw \cdot \xi_t, \quad (2.3)$$
$$r^{-m}; \Delta_{S^{N-1}} b_2 + \{(-m + 2)(N - m) + pm(N - m - 2)\} b_2$$

$$= (m - 1)b_1 \omega \cdot \xi_t - (\xi_t - (\omega \cdot \xi_t)\omega) \cdot \nabla b_1 + \frac{p(p-1)}{2}m(N - m - 2)b_1^2, \quad (2.4)$$

$$r^{-m}i^{-2}; \Delta_{S^{N-1}} b_i + \{(-m + i)(N - m + i - 2) + pm(N - m - 2)\} b_i$$

$$= G_i(\omega; b_1, b_2, \ldots, b_{i-1}, \xi) \quad (i = 3, 4, \ldots, k). \quad (2.5)$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on $S^{N-1}$ and the function $G_i(\omega; b_1, b_2, \ldots, b_{i-1}, \xi)$ on $S^{N-1} \times [0, \infty)$ is determined by $(b_1, b_2, \ldots, b_{i-1}, \xi)$.

The equality for $r^{-m-2}$ always holds by (1.2). From other equations, we have the above system of inhomogeneous elliptic equations for $b_i$ on $S^{N-1}$: By these equations, $b_1, b_2, \ldots$ are determined sequentially.

Let us consider the solvability of (2.3), (2.4) and (2.5). It is well known (see, e.g. [2]) that for every $j = 0, 1, 2, \ldots$, the eigenvalues of $-\Delta_{S^{N-1}}$ are given by

$$\mu_j = j(N + j - 2), \quad j = 0, 1, 2, \ldots.$$ and the eigenspace $E_j$ associated with $\mu_j$ is given by

$$E_j = \{ f|_{S^{N-1}} : f \text{ is a harmonic homogeneous polynomial of degree } j \}.$$ Therefore, unless

$$(-m + i)(N - m + i - 2) + pm(N - m - 2) = j(N + j - 2), \quad (2.6)$$ the operators in the left-hand side of (2.3), (2.4) and (2.5) are invertible. We define a set $\Lambda$ by

$$\Lambda := \{ p > 1 : (2.6) \text{ holds for some } i \in \{1, 2, \ldots, \left[\frac{2}{p-1}\right]\}, \quad j \in \{0, 1, 2, \ldots, i\}\}.$$ Moreover, we consider $G_i(\omega; b_1, b_2, \ldots, b_{i-1}, \xi)$ in detail and obtain next lemma.

**Lemma 1.** Suppose that $\xi(t)$ satisfies (A1). If $p \not\in \Lambda$, then there exist $b_1(\omega, t), b_2(\omega, t), \ldots, b_k(\omega, t) \in C^{\infty,1}(S^{N-1} \times [0, \infty))$ such that (2.3), (2.4) and (2.5) hold.

By this lemma, in order to consider the existence of the solution of (1.1) with a moving singularity, it suffices to consider $v(y, t)$. By taking $b_i(\omega, t)$ as Lemma 1, (1.1) is satisfied if $v(y, t)$ satisfies

$$v_t = \Delta v + \xi_t \cdot \nabla v + F(v, y, t) \quad \text{in } \mathbb{R}^N \times (0, \infty). \quad (2.7)$$
where $F(v, y, t)$ is determined by $b_1, b_2, \ldots, b_k$ and $\xi$. After tedious computations, we notice that

$$F(v, y, t) = \frac{pL^{p-1}}{r^2} v + o(r^{-2}) \quad \text{as } r \to 0.$$  

In order to consider the existence of solutions of (2.7), we first consider

$$v_t = \Delta v + \frac{pL^{p-1}}{r^2} v \quad \text{in } \mathbb{R}^N \times (0, \infty). \quad (2.8)$$

This equation has been investigated in [1, 7, 6], and it was shown that (2.8) is well-posed when

$$0 < pL^{p-1} < \frac{(N-2)^2}{4}, \quad (2.9)$$

and

$$|v(y, 0)| \leq Cr^{-\lambda} \quad \text{for some } \lambda_1 < \lambda < \lambda_2, \ C > 0.$$  

The inequalities (2.9) hold if and only if $p$ satisfies

$$p_{\text{sing}} < p < p_* \quad \text{for } N \geq 3, \quad \text{or } p > p_{\text{JL}} := \frac{N - 2\sqrt{N - 1}}{N - 4 - 2\sqrt{N - 1}} \quad \text{for } N > 10.$$  

Here the exponent $p_{\text{JL}}$ was first introduced by Joseph-Lundgren [4] and is known to play an important role for the dynamics of solutions of (1.1).

Since the gradient term in (2.7) and the higher order term of $F$ do not affect the well-posedness, we must assume (2.9) for the solvability of (2.7). If $p > p_{\text{JL}}$, then $\lambda_1 < m$ does not hold so that (2.2) may not be true. Hence we exclude the case $p_{\text{JL}} < p$. Based on the above formal analysis, we will focus on the case $p_{\text{sing}} < p < p_*$.  

3 Time-local existence

Taking into account of the formal analysis in the previous section, we will show the existence of a time-local solution with a moving singularity. To this end, we develop the idea of Marchi [6] for the well-posedness of the linear equation (2.8).

The outline of the proof is divided into three steps. Roughly speaking, we construct a suitable supersolution and subsolution with a moving singularity in Subsection 3.1. In Subsection 3.2, we construct a sequence of approximate solutions and find a convergent subsequence. In Subsection 3.3, we show that the limiting function is indeed a solution of (1.1) with a moving singularity.
3.1 Construction of a supersolution and a subsolution

In this subsection, we construct a supersolution and a subsolution of (1.1) that are suitable for our purpose.

First we note that if \( m - \lambda_2 \) is not an integer, then (2.6) does not hold for all \( i = 1, 2, \ldots, [m - \lambda_2], \ j = 0, 1, \ldots, i \). Indeed, if (2.6) does not hold for some \( 1 \leq i \leq m - \lambda_2, j = 1, \ldots, i \), then \( i = -\lambda_2, j = 0 \), contradicting that \( m - \lambda_2 \) is not an integer. Therefore, if \( m - \lambda_2 \) is not an integer, then by Lemma 1 and (A1), we can determine \( b_1(\omega, t), b_2(\omega, t), \ldots, b_{[m-\lambda_2]}(\omega, t) \in C^{2,1}(S^{N-1} \times [0, \infty)) \) by (2.3), (2.4) and (2.5).

On the other hand, if \( m - \lambda_2 \) is an integer, (2.6) holds for \( i = m - \lambda_2, j = 0 \).

However, we carry out similar argument by replacing \( b_{[m-\lambda_2]}(\omega, t)r^{[m-\lambda_2]} \) with \( (b_{m-\lambda_2}(\omega, t) + c(t) \log r)r^{m-\lambda_2} \) that satisfies

\[
\Delta_{S^{N-1}} b_{m-\lambda_2} = (I - P_0)G(\omega, t), \quad c(t) = (N - 2\lambda_2 - 2)^{-1}P_0G(\omega, t), \quad (3.1)
\]

where \( P_0 \) is defined as the projection on \( E_0 \) and \( G(\omega, t) \) is the right-hand side of (2.5) with \( i = m - \lambda_2 \).

Now we fix \( \lambda = \lambda_2 - \epsilon \) satisfying

\[
\min\{\lambda_1, m - [m - \lambda_2] - 1\} < \lambda < \lambda_2
\]

and replace \( k \) defined in Section 2 with \( k := [m - \lambda_2] \). From (A2) and (A3), it follows that \( u_0 \in C(\mathbb{R}^N \setminus \xi(0)) \cap L^\infty(\mathbb{R}^N \setminus B(\xi(0), 1)), \ u_0 \geq 0, \) and

\[
u_0(x) = L|x - \xi(0)|^{-m}\{1 + \sum_{i=1}^{k} b_i \left( \frac{x - \xi(0)}{|x - \xi(0)|}, 0 \right) |x - \xi(0)|^i
\]

\[+ O(|x - \xi(0)|^{m-\lambda}) \} \text{ as } x \to \xi(0).\]

Then there exist constants \( C > 0 \) and \( R > 0 \) such that

\[
|u_0(x) - L|x - \xi(0)|^{-m}\left\{1 + \sum_{i=1}^{k} b_i(\omega, 0) \left( \frac{x - \xi(0)}{|x - \xi(0)|} \right) |x - \xi(0)|^i \right\}|
\]

\[< CL|x - \xi(0)|^{-\lambda} \text{ in } B(\xi(0), R).\]

Fix any \( T_1 > 0 \).

First we construct a supersolution and a subsolution of (1.1) in a neighborhood of \( \xi(t) \) by using (2.7). By (2.1), we have

\[
u_t - \Delta u - u^p = L\{v_t - \Delta v - \xi_t \cdot \nabla v - F(v, y, t)\}.\]
Hence
\[
\overline{u}(x,t) = L r^{-m} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t)r^i + v^1(y,t)r^m \right\}
\]
is a supersolution of (1.1) if and only if \( v^+ \) is a supersolution of (2.7). Since it follows from tedious calculation that \( \overline{u} := Cr^{-\lambda} \) is a supersolution of (2.7) on \( B_R \times (0, T_1) \) if \( R > 0 \) is sufficiently small,
\[
\overline{u} := L|x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t)|x - \xi(t)|^i + C|x - \xi(t)|^{m-\lambda} \right\}
\]
is a supersolution of (1.1) on \( \bigcup_{0 \leq t \leq T_1} B_R(\xi(t)) \times \{t\} \) for small \( R > 0 \). Similarly, we can show that
\[
\underline{u} := L|x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t)|x - \xi(t)|^i - C|x - \xi(t)|^{m-\lambda} \right\}
\]
is a subsolution of (1.1) on \( \bigcup_{0 \leq t \leq T_1} B_R(\xi(t)) \times \{t\} \) for small \( R > 0 \).

Next, we construct a supersolution and a subsolution near infinity. By direct calculation, it is shown that
\[
\overline{u} := C_1 \left( 1 - \frac{t}{2T_2} \right)^{-\frac{1}{2(\nu - 1)}}
\]
is a supersolution of (1.1) on \( \mathbb{R}^N \setminus B(\xi(t), 1) \times (0, T_2) \), provided that
\[
C_1 > \|u_0\|_{L^\infty(\mathbb{R}^N \setminus B(\xi(0), 1))}, \quad T_2 < 2\sqrt{2}(p - 1)C_1^{p-1}.
\]
Clearly \( u \equiv 0 \) is a subsolution (1.1).

Finally, connecting these supersolutions and subsolutions in the intermediate region, we obtain a supersolution \( \overline{u} \) and a subsolution \( \underline{u} \) such that \( \overline{u}, \overline{u}^p, \underline{u}, \underline{u}^p \in L^1_{loc}(\mathbb{R}^N \times [0, T]) \) and the following properties hold:

(i) \( \overline{u}(x,t) \) and \( \underline{u}(x,t) \) are defined on \( \{(x,t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\}, t \in [0, T]\} \) and are twice continuously differentiable with respect to \( x \) and continuously differentiable with respect to \( t \).

(ii) For every \( t \in [0, T] \), \( \overline{u}(x,t), \underline{u}(x,t) \to \infty \) as \( x \to \xi(t) \). In particular,
\[
\overline{u}(x,t) = L|x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t)|x - \xi(t)|^i + C|x - \xi(t)|^{m-\lambda} \right\},
\]
\[
\underline{u}(x,t) = L|x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t)|x - \xi(t)|^i - C|x - \xi(t)|^{m-\lambda} \right\}
\]
for \( |x - \xi(t)| \leq R_0 \) and \( 0 \leq t \leq T \).
(iii) The inequalities
\[ \overline{u}(x, 0) > u_0(x) > \underline{u}(x, 0) \text{ in } \mathbb{R}^N \setminus \{\xi(0)\}, \]
\[ \overline{u}(x, t) > \underline{u}(x, t) \text{ in } \mathbb{R}^N \times [0, T] \setminus \bigcup_{0 \leq t \leq T} (\xi(t), t) \]
hold.

(iv) The inequalities
\[ \overline{u}_t \geq \Delta \overline{u} + \overline{u} \text{ in } \mathbb{R}^N \times [0, T] \setminus \bigcup_{0 \leq t \leq T} (\xi(t), t), \]
\[ \underline{u}_t \leq \Delta \underline{u} + \underline{u} \text{ in } \mathbb{R}^N \times [0, T] \setminus \bigcup_{0 \leq t \leq T} (\xi(t), t) \]
hold.

for some small $R_0$ and $T$.

### 3.2 Construction of approximate solutions

In this subsection, by using the supersolution and subsolution given in the previous subsection, we construct a series of approximate solutions that is convergent in an appropriate function space.

Define a sequence of bounded domains
\[ \Lambda_n(t) := \{x \in \mathbb{R}^N : |x - \xi(t)| \leq n, |x - \xi(t)| \geq \frac{1}{n}\} \quad (n = 1, 2, \ldots). \]

For each $n$, let $u_n(x, t)$ be a classical solution of
\[
\begin{cases}
  u_{n,t} = \Delta u_n + u_n^p & \text{in } \bigcup_{0 \leq t \leq T} A_n(t) \times \{t\}, \\
  u_n = \overline{u} & \text{on } \bigcup_{0 \leq t \leq T} \partial A_n(t) \times \{t\}, \\
  u_n(x, 0) = u_{0,n}(x) & \text{in } A_n(0),
\end{cases}
\]
where the initial value is assumed to satisfy
\[
\overline{u}(x, 0) \leq u_{0,n}(x) \leq u_{0,n+1}(x) \leq \overline{u}(x, 0) \text{ in } A_n(0),
\]
\[ u_{0,n}(x) = \underline{u}(x, 0) \text{ on } \partial A_n(0), u_{0,n} \nearrow u_0 \text{ as } n \to \infty. \]
It is easily seen that $\underline{u} \leq u_n \leq \overline{u}$ in $\bigcup_{0 \leq t \leq T} A_n(t) \times \{t\}$ by the comparison principle. Furthermore, by the standard parabolic theory [5] and the Ascoli-Arzelà theorem, from $\{u_n\}$, we can obtain a subsequence $\{u_{n(j)}\}_j$ and some function $u(x,t)$ such that

$$u_{n(j)} \to u \text{ locally uniformly in } R^N \times (0,T) \setminus \bigcup_{0< t<T} (\xi(t),t) \text{ as } n(j) \to \infty.$$  

Hence the limiting function $u(x,t)$ satisfies

$$u \in C(R^N \times (0,T) \setminus \bigcup_{0< t<T} (\xi(t),t)),$$

$$\underline{u} \leq u \leq \overline{u} \text{ in } R^N \times (0,T) \setminus \bigcup_{0< t<T} (\xi(t),t).$$

### 3.3 Completion of the proof

In this subsection, we show that the limiting function $u(x,t)$ obtained in Subsection 3.2 is indeed a solution of (1.1) with a moving singularity $\xi(t)$ for $t \in (0,T)$.

First, by $\underline{u} \leq u \leq \overline{u}$ and the Lebesgue convergence theorem, we can show that the function $u$ satisfies (1.1) in the distribution sense. Next, by $\underline{u} \leq u \leq \overline{u}$ and the standard parabolic theory [5], the function $u$ has the desired properties as stated in Definition 1. Consequently, it is shown that the function $u$ is a solution of (1.1) with a moving singularity $\xi(t)$ for $t \in (0,T)$.

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