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Existence of Solutions with Moving Singularities for a Semilinear Parabolic Equation

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Abstract

We study the Cauchy problem for a semilinear parabolic equation with a power nonlinearity. It is known that in some parameter range, the equation has a singular steady state. Our concern is a solution with a moving singularity that is obtained by perturbing the singular steady state. By the formal expansion, it turns out that the correction term must satisfy the heat equation with inverse-square potential near the singular point. From the well-posedness of this equation, we see that there appears a critical exponent. Paying attention to this exponent, given a motion of the singular point and suitable initial data, we establish the time-local existence result.

1 Introduction

We study singular solutions of the semilinear parabolic equation

\[
\begin{align*}
\left\{ \begin{array}{ll}
    u_t = \Delta u + u^p & \text{in } \mathbb{R}^N \times (0, \infty), \\
    u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N,
\end{array} \right.
\end{align*}
\]  

(1.1)

where \( p > 1 \) is a parameter and \( u_0 \in L_{loc}^1(\mathbb{R}^N) \) is a nonnegative function. It is known that for

\[ N \geq 3, \quad p > p_{\text{sing}} := \frac{N}{N-2}, \]

(1.1) has an explicit singular steady state \( \varphi(|x|) \in C^\infty(\mathbb{R}^N \setminus \{0\}) \) with a singular point \( 0 \);

\[ \varphi(|x|) = L|x|^{-m}, \quad m = \frac{2}{p-1}, \quad L^{p-1} = m(N - m - 2). \]
Then \( \varphi(|x|) \) satisfies (1.1) in the distribution sense, and
\[
\varphi_{rr} + \frac{N - 1}{r} \varphi_r + \varphi^p = 0, \quad r = |x| > 0.
\tag{1.2}
\]
Clearly, the spatial singularity of \( u = \varphi(|x|) \) persists for all \( t > 0 \), but the singular point does not move in time.

Our aim of this paper is to discuss the existence of a solution of (1.1) whose spatial singularity moves in time. More precisely, we define a solution with a moving singularity as follows.

Definition 1. The function \( u(x, t) \) is said to be a solution of (1.1) with a moving singularity \( \xi(t) \in \mathbb{R}^N \) for \( t \in (0, T) \), where \( 0 < T \leq \infty \), if the following conditions hold:

(i) \( u, \ u^p \in C([0,T);L_{loc}^1(\mathbb{R}^N)) \) satisfy (1.1) in the distribution sense.

(ii) \( u(x, t) \) is defined on \( \{(x, t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\}, t \in (0, T)\} \), and is twice continuously differentiable with respect to \( x \) and continuously differentiable with respect to \( t \).

(iii) \( u(x, t) \to \infty \) as \( x \to \xi(t) \) for every \( t \in [0, T) \).

In this paper, we study the time-local existence for a solution with a moving singularity of the Cauchy problem (1.1). In order to state our result, we first introduce a critical exponent given by
\[
p_* := \frac{N + 2\sqrt{N - 1}}{N - 4 + 2\sqrt{N - 1}},
\]
which appeared in the papers of Véron [8] and Chen-Lin [3]. It was shown in [8] that \( p_* \) is related to the linearized stability of the singular steady state, while it was shown in [3] that \( p_* \) plays a crucial role for the existence of solutions with a prescribed singular set of the Dirichlet problem
\[
\begin{aligned}
\Delta u + u^p &= 0 \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial\Omega,
\end{aligned}
\]
where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \). In fact, in [3], they proved that if \( N \geq 3 \), \( p_{\text{sing}} < p < p_* \), then for any closed set \( K \subset \Omega \), there exists a singular solution having \( K \) as a singular set. We note that \( p_* \) is larger than \( p_{\text{sing}} \) and is smaller than the Sobolev critical exponent \( p_S := (N+2)/(N-2) \).

We also introduce the important numbers
\[
\lambda_1 := \frac{N - 2 - \sqrt{(N - 2)^2 - 4pL^p - 1}}{2},
\lambda_2 := \frac{N - 2 + \sqrt{(N - 2)^2 - 4pL^p - 1}}{2}.
\]
We note that for $N \geq 3$, $p_{\text{sing}} < p < p_*$, the constants $\lambda_1 < \lambda_2$ are positive roots of
\[ \lambda^2 - (N - 2)\lambda + pI^{p-1} = 0. \]
Finally, for $a \in \mathbb{R}$, $[a]$ denotes the largest integer not greater than $a$.

Our result is concerning the time-local existence of a solution of \eqref{1.1} with a moving singularity.

**Theorem 1.** Let $N \geq 3$ and $p_{\text{sing}} < p < p_*$. Assume the following conditions:

(A1) $\xi(t) \in C^{i+\alpha}([0, \infty); \mathbb{R}^N)$ ($\alpha > 0$) with $i = \left\lfloor \frac{|m-\lambda_2|+1}{2} \right\rfloor + 1$.

(A2) $u_0$ is nonnegative and continuous in $x \in \mathbb{R}^N \setminus \xi(0)$, and is uniformly bounded for $|x - \xi(0)| \geq 1$.

(A3) If $m - \lambda_2$ is not an integer, then
\[
\begin{align*}
u_0(x) &= L|x - \xi(0)|^{-m} \left\{ 1 + \sum_{i=1}^{[m-\lambda_2]} b_i \left( \frac{x - \xi(0)}{|x - \xi(0)|}, 0 \right) |x - \xi(0)|^i ight. \\
&\quad \left. + O(|x - \xi(0)|^{m-\lambda_2+\epsilon}) \right\}
\end{align*}
\]
as $x \to \xi(0)$ for some $\epsilon > 0$, where $b_i(\omega, t)$ are functions on $S^{N-1}$ defined later by \eqref{2.9}-\eqref{2.5}. If $m - \lambda_2$ is an integer, then
\[
\begin{align*}
u_0(x) &= L|x - \xi(0)|^{-m} \left\{ 1 + \sum_{i=1}^{m-\lambda_2} b_i \left( \frac{x - \xi(0)}{|x - \xi(0)|}, 0 \right) |x - \xi(0)|^i ight. \\
&\quad \left. + c(0)|x - \xi(0)|^{m-\lambda_2} \log |x - \xi(0)| + O(|x - \xi(0)|^{m-\lambda_2+\epsilon}) \right\}
\end{align*}
\]
as $x \to \xi(0)$ for some $\epsilon > 0$, where $b_i(\omega, t)$ are functions on $S^{N-1}$ defined later by \eqref{2.9}-\eqref{2.5} and $b_{m-\lambda_2}(\omega, t)$ and $c(t)$ satisfy \eqref{3.1}

Then for some $T > 0$, there exists a solution of \eqref{1.1} with a moving singularity $\xi(t)$.

**Remark 1.** If $N \geq 3$ and
\[
p_{\text{sing}} < p < \min\left\{ p_*, \frac{3N+5}{3N-3} \right\},
\]
then $0 \leq m - \lambda_2 < 1$ so that $[m - \lambda_2] = 0$. In this case, (A1) implies $\xi(t) \in C^{i+\alpha}([0, \infty); \mathbb{R}^N)$ ($\alpha > 0$), and (A3) is simplified as
\[
\begin{align*}
u_0(x) &= L|x - \xi(0)|^{-m} + O(|x - \xi(0)|^{-\lambda_2+\epsilon}) \quad \text{as } x \to \xi(0).
\end{align*}
\[
\text{(1.3)}
\]
In this paper, we consider only the time-local existence of the Cauchy problem with a moving singularity. Needless to say, the existence of time-global solutions are important questions. Also, when the solution with a moving singularity is not time-global, it is interesting to ask what happens at the maximal existence time. These questions will be future works.

This paper is organized as follows: In Section 2 we carry out formal analysis for a solution of (1.1) as a perturbation of the singular steady state. In Section 3 we state the outline of proof of the time-local existence.

2 Formal expansion at a singular point

In this section, we consider the formal expansion of a solution $u(x, t)$ of (1.1) with a moving singularity $\xi(t)$. Assuming that the solution resembles the singular steady state around $\xi(t)$, we may naturally expand $u(x, t)$ as

$$u(x, t) = L r^{-m} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t) r^i + v(y, t)r^m \right\},$$

where

$$y = x - \xi(t), \quad r = |x - \xi(t)|, \quad \omega = \frac{1}{r}(x - \xi) \in S^{N-1}, \quad k = [m],$$

and the remainder term $v$ satisfies

$$v(y, t) = o(|y|^{-m}) \text{ as } |y| \to 0.$$  \hspace{1cm} (2.2)

Substituting (2.1) into (1.1), and using

$$r_i = -\frac{(x - \xi) \cdot \xi_i}{r}, \quad \omega_i = -\frac{1}{r}\xi_i + \frac{\omega \cdot \xi_i}{r}\omega,$$

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}}$$

and the Taylor expansion, we compare the coefficients of $r^{-m+i-2}$ for $i = 0, 1, \ldots, k$. Then we obtain

$$r^{-m-2}; (L r^{-m})_{rr} + \frac{N-1}{r}(L r^{-m})_r + (L r^{-m})_p = 0,$$

$$r^{-m-1}; \Delta_{S^{N-1}} b_1 + \{(-m + 1)(N - m - 1) + pm(N - m - 2)\} b_1$$

$$= mw \cdot \xi_i, \quad (2.3)$$
\begin{align}
& r^{-m}; \Delta_{S^{N-1}} b_2 + \{(-m + 2)(N - m) + pm(N - m - 2)\} b_2 \\
& = (m - 1)b_1 \omega \cdot \xi_t - (\xi_t - (\omega \cdot \xi_t)\omega) \cdot \nabla b_1 + \frac{p(p-1)}{2} m(N - m - 2)b_1^2, \quad (2.4)
\end{align}

\begin{align}
& r^{-m-1}
& = \Delta_{S^{N-1}} b_i + \{(-m + i)(N - m + i - 2) + pm(N - m - 2)\} b_i \\
& = G_i(\omega; b_1, b_2, \ldots, b_{i-1}, \xi) \quad (i = 3, 4, \ldots, k). \quad (2.5)
\end{align}

where \( \Delta_{S^{N-1}} \) is the Laplace-Beltrami operator on \( S^{N-1} \) and the function \( G_i(\omega; b_1, b_2, \ldots, b_{i-1}, \xi) \) on \( S^{N-1} \times [0, \infty) \) is determined by \( (b_1, b_2, \ldots, b_{i-1}, \xi) \).

The equality for \( r^{-m-2} \) always holds by (1.2). From other equations, we have the above system of inhomogeneous elliptic equations for \( b_i \) on \( S^{N-1} \): By these equations, \( b_1, b_2, \ldots \) are determined sequentially.

Let us consider the solvability of (2.3), (2.4) and (2.5). It is well known (see, e.g. [2]) that for every \( j = 0, 1, 2, \ldots \), the eigenvalues of \( -\Delta_{S^{N-1}} \) are given by
\[ \mu_j = j(N + j - 2), \quad j = 0, 1, 2, \ldots \]
and the eigenspace \( E_j \) associated with \( \mu_j \) is given by
\[ E_j = \{ f|_{S^{N-1}} : f \text{ is a harmonic homogeneous polynomial of degree } j \} \]

Therefore, unless
\[ (-m + i)(N - m + i - 2) + pm(N - m - 2) = j(N + j - 2), \quad (2.6) \]
the operators in the left-hand side of (2.3), (2.4) and (2.5) are invertible. We define a set \( \Lambda \) by
\[ \Lambda := \left\{ p > 1 : (2.6) \text{ holds for some } i \in \{1, 2, \ldots, \left[\frac{2}{p-1}\right]\}, \quad j \in \{0, 1, 2, \ldots, i\} \right\} \]

Moreover, we consider \( G_i(\omega; b_1, b_2, \ldots, b_{i-1}, \xi) \) in detail and obtain next lemma.

**Lemma 1.** Suppose that \( \xi(t) \) satisfies (A1). If \( p \notin \Lambda \), then there exist \( b_1(\omega, t), b_2(\omega, t), \ldots, b_k(\omega, t) \in C^{\infty,1}(S^{N-1} \times [0, \infty)) \) such that (2.3), (2.4) and (2.5) hold.

By this lemma, in order to consider the existence of the solution of (1.1) with a moving singularity, it suffices to consider \( v(y, t) \). By taking \( b_i(\omega, t) \) as Lemma 1, (1.1) is satisfied if \( v(y, t) \) satisfies
\begin{equation}
u_t = \Delta v + \xi_t \cdot \nabla v + F(v, y, t) \quad \text{in } \mathbb{R}^N \times (0, \infty). \quad (2.7)
\end{equation}
where $F(v, y, t)$ is determined by $b_1, b_2, \ldots, b_k$ and $\xi$. After tedious computations, we notice that

$$F(v, y, t) = \frac{pL^{p-1}}{r^2} v + o(r^{-2}) \quad \text{as } r \to 0.$$ 

In order to consider the existence of solutions of (2.7), we first consider

$$v_t = \Delta v + \frac{pL^{p-1}}{r^2} v \quad \text{in } \mathbb{R}^N \times (0, \infty). \quad (2.8)$$

This equation has been investigated in [1, 7, 6], and it was shown that (2.8) is well-posed when

$$0 < pL^{p-1} < \frac{(N-2)^2}{4}, \quad (2.9)$$

and

$$|v(y, 0)| \leq Cr^{-\lambda} \quad \text{for some } \lambda_1 < \lambda < \lambda_2, \quad C > 0.$$ 

The inequalities (2.9) hold if and only if $p$ satisfies

$$p_{\text{sing}} < p < p_\ast \quad \text{for } N \geq 3, \quad \text{or } p > p_{\text{JL}} := \frac{N - 2\sqrt{N - 1}}{N - 4 - 2\sqrt{N - 1}} \quad \text{for } N > 10.$$ 

Here the exponent $p_{\text{JL}}$ was first introduced by Joseph-Lundgren [4] and is known to play an important role for the dynamics of solutions of (1.1).

Since the gradient term in (2.7) and the higher order term of $F$ do not affect the well-posedness, we must assume (2.9) for the solvability of (2.7). If $p > p_{\text{JL}}$, then $\lambda_1 < m$ does not hold so that (2.2) may not be true. Hence we exclude the case $p_{\text{JL}} < p$. Based on the above formal analysis, we will focus on the case $p_{\text{sing}} < p < p_\ast$.

### 3 Time-local existence

Taking into account of the formal analysis in the previous section, we will show the existence of a time-local solution with a moving singularity. To this end, we develop the idea of Marchi [6] for the well-posedness of the linear equation (2.8).

The outline of the proof is divided into three steps. Roughly speaking, we construct a suitable supersolution and subsolution with a moving singularity in Subsection 3.1. In Subsection 3.2, we construct a sequence of approximate solutions and find a convergent subsequence. In Subsection 3.3, we show that the limiting function is indeed a solution of (1.1) with a moving singularity.
3.1 Construction of a supersolution and a subsolution

In this subsection, we construct a supersolution and a subsolution of (1.1) that are suitable for our purpose.

First we note that if \( m - \lambda_2 \) is not an integer, then (2.6) does not hold for all \( i = 1, 2, \ldots, [m - \lambda_2], \) \( j = 0, 1, \ldots, i \). Indeed, if (2.6) does not hold for some \( 1 \leq i \leq m - \lambda_2, \) \( j = 1, \ldots, i \), then \( i = -\lambda_2, j = 0 \), contradicting that \( m - \lambda_2 \) is not an integer. Therefore, if \( m - \lambda_2 \) is not an integer, then by Lemma 1 and (A1), we can determine \( b_1(\omega, t), b_2(\omega, t), \ldots, b_{[m-\lambda_2]}(\omega, t) \in C^{2,1}(S^{N-1} \times [0, \infty)) \) by (2.3), (2.4) and (2.5).

On the other hand, if \( m - \lambda_2 \) is an integer, (2.6) holds for \( i = m - \lambda_2, j = 0 \). However, we carry out similar argument by replacing \( b_{[m-\lambda_2]}(\omega, t) r^{[m-\lambda_2]} \) with \( (b_{m-\lambda_2}(\omega, t) + c(t) \log r) r^{m-\lambda_2} \) that satisfies

\[
\Delta_{S^{N-1}} b_{m-\lambda_2} = (I-P_0)G(\omega, t), \quad c(t) = (N-2\lambda_2 - 2)^{-1}P_0G(\omega, t),
\]

where \( P_0 \) is define the projection on \( E_0 \) and \( G(\omega, t) \) is the right-hand side of (2.5) with \( i = m - \lambda_2 \).

Now we fix \( \lambda = \lambda_2 - \epsilon \) satisfying

\[
\min\{\lambda_1, m - [m - \lambda_2] - 1\} < \lambda < \lambda_2
\]

and replace \( k \) defined in Section 2 with \( k := [m - \lambda_2] \). From (A2) and (A3), it follows that \( u_0 \in C(\mathbb{R}^N \setminus \xi(0)) \cap L^\infty(\mathbb{R}^N \setminus B(\xi(0), 1)), \) \( u_0 \geq 0, \) and

\[
u_0(x) = L|x-\xi(0)|^{-m}\left\{1 + \sum_{i=1}^{k} b_i\left(\frac{x-\xi(0)}{|x-\xi(0)|}, 0\right)|x-\xi(0)|^i + O(|x-\xi(0)|^{m-\lambda})\right\}
\]

as \( x \to \xi(0) \).

Then there exist constants \( C > 0 \) and \( R > 0 \) such that

\[
|u_0(x) - L|x-\xi(0)|^{-m}\left\{1 + \sum_{i=1}^{k} b_i(\omega, 0)(\frac{x-\xi(0)}{|x-\xi(0)|})|x-\xi(0)|^i\right\} < CL|x-\xi(0)|^{-\lambda} \text{ in } B(\xi(0), R).
\]

Fix any \( T_1 > 0 \).

First we construct a supersolution and a subsolution of (1.1) in a neighborhood of \( \xi(t) \) by using (2.7). By (2.1), we have

\[
u_t - \Delta u - u^p = L\{v_t - \Delta v - \xi_t \cdot \nabla v - F(v, y, t)\}.
\]
Hence
\[ \overline{u}(x, t) = L r^{-n} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t) r^i + v^1(y, t) r^m \right\} \]
is a supersolution of (1.1) if and only if \( v^+ \) is a supersolution of (2.7). Since it follows from tedious calculation that \( \overline{v} := C r^{-\lambda} \) is a supersolution of (2.7) on \( B_R \times (0, T_1) \) if \( R > 0 \) is sufficiently small,
\[ \overline{u} := L |x - \xi(t)|^{-n} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t) |x - \xi(t)|^i + C |x - \xi(t)|^{m-\lambda} \right\} \]
is a supersolution of (1.1) on \( \bigcup_{0 \leq t \leq T_1} B_R(\xi(t)) \times \{ t \} \) for small \( R > 0 \). Similarly, we can show that
\[ \underline{u} := L |x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t) |x - \xi(t)|^i - C |x - \xi(t)|^{m-\lambda} \right\} \]
is a subsolution of (1.1) on \( \bigcup_{0 \leq t \leq T_1} B_R(\xi(t)) \times \{ t \} \) for small \( R > 0 \).

Next, we construct a supersolution and a subsolution near infinity. By direct calculation, it is shown that
\[ \overline{u} := C_1 (1 - \frac{t}{2T_2})^{-\frac{1}{2(p-1)}} \]
is a supersolution of (1.1) on \( \mathbb{R}^N \setminus B(\xi(t), 1) \times (0, T_2) \), provided that
\[ C_1 > \|u_0\|_{L^\infty(\mathbb{R}^N \setminus B(\xi(0), 1))}, \quad T_2 < 2\sqrt{2(p-1)}C_1^{p-1}. \]
Clearly \( u \equiv 0 \) is a subsolution (1.1).

Finally, connecting these supersolutions and subsolutions in the intermediate region, we obtain a supersolution \( \overline{u} \) and a subsolution \( \underline{u} \) such that \( \overline{u}, \overline{u}^p, \underline{u}, \underline{u}^p \in L^1_{loc}(\mathbb{R}^N \times [0, T]) \) and the following properties hold:

(i) \( \overline{u}(x, t) \) and \( \underline{u}(x, t) \) are defined on \( \{(x, t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\}, \ t \in [0, T] \} \) and are twice continuously differentiable with respect to \( x \) and continuously differentiable with respect to \( t \).

(ii) For every \( t \in [0, T] \), \( \overline{u}(x, t), \underline{u}(x, t) \to \infty \) as \( x \to \xi(t) \). In particular,
\[ \overline{u}(x, t) = L |x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t) |x - \xi(t)|^i + C |x - \xi(t)|^{m-\lambda} \right\}, \]
\[ \underline{u}(x, t) = L |x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^{k} b_i(\omega, t) |x - \xi(t)|^i - C |x - \xi(t)|^{m-\lambda} \right\} \]
for \( |x - \xi(t)| \leq R_0 \) and \( 0 \leq t \leq T \).
(iii) The inequalities
\[ \bar{u}(x, 0) > u_0(x) > \underline{u}(x, 0) \text{ in } \mathbb{R}^N \backslash \{\xi(0)\}, \]
\[ \bar{u}(x, t) > \underline{u}(x, t) \text{ in } \mathbb{R}^N \times [0, T] \backslash \bigcup_{0 \leq t \leq T} (\xi(t), t) \]
hold.

(iv) The inequalities
\[ \bar{u}_t \geq \Delta \bar{u} + \bar{u}^p \text{ in } \mathbb{R}^N \times [0, T] \backslash \bigcup_{0 \leq t \leq T} (\xi(t), t), \]
\[ \underline{u}_t \leq \Delta \underline{u} + \underline{u}^p \text{ in } \mathbb{R}^N \times [0, T] \backslash \bigcup_{0 \leq t \leq T} (\xi(t), t) \]
hold.
for some small \( R_0 \) and \( T \).

3.2 Construction of approximate solutions

In this subsection, by using the supersolution and subsolution given in the previous subsection, we construct a series of approximate solutions that is convergent in an appropriate function space.

Define a sequence of bounded domains
\[ A_n(t) := \{x \in \mathbb{R}^N : |x - \xi(t)| \leq n, |x - \xi(t)| \geq \frac{1}{n} \} \quad (n = 1, 2, \ldots) . \]

For each \( n \), let \( u_n(x, t) \) be a classical solution of
\[
\begin{cases}
  u_{n,t} = \Delta u_n + u_n^p & \text{in } \bigcup_{0 \leq t \leq T} A_n(t) \times \{t\}, \\
  u_n = \underline{u} & \text{on } \bigcup_{0 \leq t \leq T} \partial A_n(t) \times \{t\}, \\
  u_n(x, 0) = u_{0,n}(x) & \text{in } A_n(0),
\end{cases}
\]
where the initial value is assumed to satisfy
\[ \underline{u}(x, 0) \leq u_{0,n}(x) \leq u_{0,n+1}(x) \leq \bar{u}(x, 0) \text{ in } A_n(0), \]
\[ u_{0,n}(x) = \underline{u}(x, 0) \text{ on } \partial A_n(0), u_{0,n} \nearrow u_0 \text{ as } n \to \infty. \]
It is easily seen that \( u \leq u_n \leq \bar{u} \) in \( \bigcup_{0 \leq t \leq T} A_n(t) \times \{ t \} \) by the comparison principle. Furthermore, by the standard parabolic theory [5] and the Ascoli-Arzela theorem, from \( \{ u_n \} \), we can obtain a subsequence \( \{ u_{n(j)} \} \) and some function \( u(x, t) \) such that

\[
u_{n(j)} \to u \text{ locally uniformly in } R^N \times (0, T) \setminus \bigcup_{0 < t < T} (\xi(t), t) \text{ as } n(j) \to \infty
\]

Hence the limiting function \( u(x, t) \) satisfies

\[
u \in C(R^N \times (0, T) \setminus \bigcup_{0 < t < T}(\xi(t), t)),
\]

\[
u \leq u \leq \bar{u} \text{ in } R^N \times (0, T) \setminus \bigcup_{0 < t < T}(\xi(t), t).
\]

3.3 Completion of the proof

In this subsection, we show that the limiting function \( u(x, t) \) obtained in Subsection 3.2 is indeed a solution of (1.1) with a moving singularity \( \xi(t) \) for \( t \in (0, T) \).

First, by \( \underline{u} \leq u \leq \bar{u} \) and the Lebesgue convergence theorem, we can show that the function \( u \) satisfies (1.1) in the distribution sense. Next, by \( \underline{u} \leq u \leq \bar{u} \) and the standard parabolic theory [5], the function \( u \) has the desired properties as stated in Definition 1. Consequently, it is shown that the function \( u \) is a solution of (1.1) with a moving singularity \( \xi(t) \) for \( t \in (0, T) \).

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