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Kyoto University
Stationary patterns for a cooperative model with nonlinear diffusion

Kazuhiro Oeda
Graduate School of Fundamental Science and Technology, Waseda University

1 Introduction

In this article we study positive steady-state solutions of the following strongly coupled reaction-diffusion system:

\[
\begin{aligned}
u_t &= \Delta \left[ \frac{\alpha}{\mu + v} u \right] + u(a - u + cv) \quad \text{in } \Omega \times (0,T), \\
v_t &= \Delta v + v(-b + du - v) \quad \text{in } \Omega \times (0,T), \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega \times (0,T), \\
u(\cdot, 0) &= u_0(\cdot), \quad v(\cdot, 0) = v_0(\cdot) \quad \text{in } \Omega,
\end{aligned}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\); \(\partial/\partial n\) denotes the outward normal derivative on \(\partial \Omega\); \(a, b, c, d, \mu\) are all positive constants; \(\alpha\) is a non-negative constant; \(u_0\) and \(v_0\) are given non-negative functions which are not identically zero. System (P) is a Lotka-Volterra cooperative model with a density-dependent diffusion term of a fractional type; unknown functions \(u\) and \(v\) represent population densities of two cooperative species, respectively; \(a\) and \(b\) denote the intrinsic growth rates of the respective species; \(c\) and \(d\) denote interaction coefficients. When \(\alpha = 0\), (P) is reduced to a classical Lotka-Volterra cooperative model with diffusion. See [6] and [13] for such a cooperative model.

In the first equation of (P), the nonlinear diffusion term \(\alpha \Delta \{u/(\mu + v)\}\) describes a situation where species \(u\) tends to leave low-density areas of species \(v\). This situation is natural because relations between \(u\) and \(v\) are cooperative. A population model with density-dependent diffusion was first proposed by Shigesada, Kawasaki and Teramoto [14] to investigate the habitat segregation phenomena between two competing species. Since their work, many mathematicians have studied population models with density-dependent diffusion. However, population models including
density-dependent diffusion terms of a fractional type have appeared in recent years; for example, see [5], [16] for cooperative models with Dirichlet boundary conditions; [2], [3] for prey-predator models with Dirichlet boundary conditions; [12], [15] for three-species prey-predator models with Neumann boundary conditions. See also the monograph of Okubo and Levin [11] for the biological background.

The stationary problem associated with (P) is

$$
\begin{align*}
\Delta \left[ \left(1 + \frac{\alpha}{\mu + v} \right) u \right] + u(a - u + cv) &= 0 \quad \text{in } \Omega, \\
\Delta v + v(-b + du - v) &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

Our main purpose is to study the existence of stationary patterns (i.e. positive non-constant solutions) for (SP) with the weak cooperative condition

$$
\frac{a}{b} > \frac{1}{d} > c.
$$

(1.1)

From now on, we will always assume (1.1). It is well known that, if $\alpha = 0$, then every solution of (P) converges to a unique positive constant steady-state

$$(u^*, v^*) := \left( \frac{a - bc}{1 - cd}, \frac{ad - b}{1 - cd} \right)$$

uniformly as $t \to \infty$; see [6]. This implies the following proposition.

**Proposition 1.1.** Let $\alpha = 0$. Then $(u^*, v^*)$ is a unique positive solution of (SP).

Proposition 1.1 means that no stationary pattern exists in the linear diffusion case. However, the presence of density-dependent diffusion enables us to construct stationary patterns of (SP). We focus on $\alpha$ to show the emergence of stationary patterns for (SP).

Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ denote eigenvalues of $-\Delta$ with the homogeneous Neumann boundary condition on $\partial \Omega$ and let $m_i$ denote the algebraic multiplicity of $\lambda_i$. Then we have the following theorem.

**Theorem 1.1.** Suppose that $\{v^*(b - \mu)\}/(\mu + v^*) \in (\lambda_l, \lambda_{l+1})$ for some $l \geq 1$ and that $\sum_{i=1}^l m_i$ is odd. Then there exists a positive constant $\alpha^* = \alpha^*(a, b, c, d, \mu)$ such that (SP) has at least one positive non-constant solution for each $\alpha > \alpha^*$.

We are also interested in the limiting patterns of (SP) as $\alpha \to \infty$. Under the restriction $N \leq 3$, we obtain the following limiting system as $\alpha \to \infty$. 
Theorem 1.2. Suppose $N \leq 3$ and $b > \mu$. Let $\{(u_{i}, v_{i}, \alpha_{i})\}_{i=1}^{\infty}$ be any sequence such that $\lim_{i \to \infty} \alpha_{i} = \infty$ and positive functions $(u_{i}, v_{i})$ satisfy (SP) with $\alpha = \alpha_{i}$. Then, by passing to a subsequence if necessary, it holds that
\[
\lim_{i \to \infty} (u_{i}, v_{i}) = (\tau(\mu + \bar{v}), \bar{v}) \quad \text{in} \quad C^{1}(\Omega) \times C^{1}(\Omega),
\]
where $\tau$ is a positive constant satisfying $1 < d\tau < b/\mu$, $\bar{v}$ is a positive function in $\Omega$ and $(\tau, \bar{v})$ satisfies
\[
\begin{aligned}
\Delta \bar{v} + \bar{v}\{-b + d\tau \mu + (\tau - 1)\bar{v}\} &= 0 \quad \text{in} \quad \Omega, \\
\frac{\partial \bar{v}}{\partial n} &= 0 \quad \text{on} \quad \partial \Omega, \\
\int_{\Omega} (\mu + \bar{v})\{a - \tau \mu + (c - \tau)\bar{v}\}dx &= 0.
\end{aligned}
\]

We expect that the limiting system (1.2) may give much information on profiles of stationary patterns of (SP) for large $\alpha$. We will give some remarks about (1.2) in the last section.

Throughout the article, the usual norms of $L^{p}(\Omega)$ for $p \in [1, \infty)$ and $C(\overline{\Omega})$ are defined by
\[
\|\psi\|_{p} := \left(\int_{\Omega} |\psi(x)|^{p}dx\right)^{1/p} \quad \text{and} \quad \|\psi\|_{\infty} := \max_{x \in \Omega} |\psi(x)|,
\]
respectively.

2 Stability of the constant solution $(u^{*}, v^{*})$

In this section, we will analyze the linearized stability of the constant stationary solution $(u^{*}, v^{*})$ for (P).

The linearized eigenvalue problem of (P) at $(u^{*}, v^{*})$ is given by
\[
\begin{aligned}
\{ -\left(1 + \frac{\alpha}{\mu + v^{*}}\right) \Delta h + \frac{\alpha u^{*}}{(\mu + v^{*})^{2}} \Delta k + u^{*}h - cu^{*}k = \eta h \quad \text{in} \quad \Omega, \\
-\Delta k - dv^{*}h + v^{*}k = \eta k \quad \text{in} \quad \Omega, \\
\frac{\partial h}{\partial n} = \frac{\partial k}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.
\end{aligned}
\]

We know that $(u^{*}, v^{*})$ is linearly stable when $\alpha = 0$. Using the expansions of $h$ and $k$ in terms of eigenfunctions of $-\Delta$, one can see that $\eta$ is an eigenvalue of (2.1) if and only if
\[
\det \left( \begin{array}{cc}
-\eta + \left(1 + \frac{\alpha}{\mu + v^{*}}\right) & \lambda_{i} + u^{*} \\
-dv^{*} & -\frac{\alpha u^{*}}{(\mu + v^{*})^{2}} \lambda_{i} - cu^{*}
\end{array} \right) = 0
\]
for some $i \geq 0$. In particular, $\eta = 0$ is an eigenvalue of (2.1) if and only if
\[
\frac{\lambda_i}{(\mu + v^*)^2}(\mu + v^*)(\lambda_i + v^*) - du^*v^*\alpha + (\lambda_i + u^*)(\lambda_i + v^*) - cdv^*v^* = 0
\]
for some $i \geq 0$. Note that $(\lambda_i + u^*)(\lambda_i + v^*) - cdv^*v^* > 0$ for all $i \geq 0$ because of (1.1). Thus it is easy to see that the linearized stability of $(u^*, v^*)$ changes as $\alpha$ increases in (P) if and only if
\[
(\mu + v^*)(\lambda_1 + v^*) - du^*v^* = (\mu + v^*)\lambda_1 + v^*(\mu + v^* - du^*)
\]
\[
= (\mu + v^*)\lambda_1 + v^*(\mu - b)
\]
\[
< 0.
\]
Therefore, $b > \mu$ is necessary for the linearized stability of $(u^*, v^*)$ to change (and so we do not discuss the case $b \leq \mu$, especially, $-b \geq 0$). This means that the difference in the intrinsic growth rates between two species $u$ and $v$ contributes to creating stationary patterns in (SP).

3 Proof of Theorem 1.1

3.1 Reduction to the semilinear system

Our method of the proof of Theorem 1.1 will be based on the Leray-Schauder degree theory (see e.g., [9]) and some a priori estimates. We first introduce a new unknown function $U$ by
\[
U = \left(1 + \frac{\alpha}{\mu + v}\right)u.
\]
(3.1)
Clearly, there exists a one-to-one correspondence between $(u, v) > 0$ and $(U, v) > 0$. As far as we discuss positive solutions, (SP) is rewritten in the following equivalent form:
\[
(EP) \begin{cases}
\Delta U + \frac{\mu + v}{\mu + v + \alpha}U \left( a - \frac{\mu + v}{\mu + v + \alpha}U + cv \right) = 0 \quad \text{in } \Omega, \\
\Delta v + v \left( -b + d\frac{\mu + v}{\mu + v + \alpha}U - v \right) = 0 \quad \text{in } \Omega, \\
\frac{\partial U}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega.
\end{cases}
\]
Thus, it is sufficient to show the existence of positive non-constant solutions of (EP).
3.2 A priori estimates

In this subsection, we will give some a priori estimates for positive solutions of (EP). Before stating the a priori estimates, we recall the following maximum principle due to Lou and Ni [7].

Lemma 3.1. Suppose that $g \in C(\overline{\Omega} \times \mathbb{R})$.
(i) If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies
\[ \Delta w(x) + g(x, w(x)) \geq 0 \quad \text{in} \quad \Omega, \quad \frac{\partial w}{\partial n} \leq 0 \quad \text{on} \quad \partial \Omega, \]
and $w(x_0) = \max_{\Omega} w$, then $g(x_0, w(x_0)) \geq 0$.
(ii) If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies
\[ \Delta w(x) + g(x, w(x)) \leq 0 \quad \text{in} \quad \Omega, \quad \frac{\partial w}{\partial n} \geq 0 \quad \text{on} \quad \partial \Omega, \]
and $w(x_0) = \min_{\Omega} w$, then $g(x_0, w(x_0)) \leq 0$.

Now we can derive the following a priori estimates.

Lemma 3.2. Let $\zeta$ be any fixed positive number. Then there exist two positive constants $C_* (\zeta) = C_*(\zeta, a, b, c, d, \mu) < C^* (\zeta) = C^*(\zeta, a, b, c, d, \mu)$ such that, if $\alpha \leq \zeta$, then any positive solution $(U, v)$ of (EP) satisfies
\[ a \leq U(x) \leq C^*(\zeta) \quad \text{and} \quad C_* (\zeta) \leq v(x) \leq C^* (\zeta) \quad \text{for all} \quad x \in \overline{\Omega}. \]

Proof. Let $U(x_0) = \max_{\Omega} U$ and $v(y_0) = \max_{\Omega} v$ with $x_0, y_0 \in \overline{\Omega}$. Applying Lemma 3.1 to (EP), we have
\[ \max_{\overline{\Omega}} U \leq \frac{\mu + v(x_0) + \alpha}{\mu + v(x_0)} (a + cv(x_0)) \]
and
\[ \max_{\overline{\Omega}} v \leq -b + d \frac{\mu + v(y_0)}{\mu + v(y_0) + \alpha} U(y_0) \leq -b + d \max_{\Omega} U. \quad (3.2) \]
Thus
\[ \max_{\overline{\Omega}} U \leq a + cv(x_0) + \zeta \frac{a + cv(x_0)}{\mu + v(x_0)} \leq a + c(-b + d \max_{\overline{\Omega}} U) + \zeta \max_{\overline{\Omega}} \left\{ \frac{a}{\mu}, c \right\}. \]
Therefore, we see
\[ \max_{\overline{\Omega}} U \leq \frac{a - bc + \zeta \max \{a/\mu, c\}}{1 - cd}. \quad (3.3) \]
It follows from (3.2) and (3.3) that
\[
\max_{\overline{\Omega}} v \leq -b + \frac{d(a-bc + \zeta \max\{a/\mu, c\})}{1-cd} = \frac{ad-b + \zeta d \max\{a/\mu, c\}}{1-cd}.
\] (3.4)
Hence we have obtained the desired upper bound of \((U, v)\).

Let \(U(z_0) = \min_{\overline{\Omega}} U\) with some \(z_0 \in \overline{\Omega}\). Using Lemma 3.1 to the first equation of (EP), we get
\[
\min_{\overline{\Omega}} U \geq \frac{\mu + v(z_0) + \alpha}{\mu + v(z_0)} (a + cv(z_0)) \geq a.
\] (3.5)
Thus we have obtained the desired lower bound of \(U\).

Finally, we derive a lower bound of \(v\) by contradiction. Suppose that there exist a certain positive constant \(\zeta_0\) and a sequence \(\{(U_i, v_i, \alpha_i)\}_{i=1}^\infty\) such that \(\alpha_i \leq \zeta_0\) for all \(i \in \mathbb{N}\), \(\lim_{i \to \infty} \alpha_i = \alpha_\infty\) for some non-negative constant \(\alpha_\infty\),
\[
\lim_{i \to \infty} \min_{\overline{\Omega}} v_i = 0
\] (3.6)
and positive functions \((U_i, v_i)\) satisfy
\[
\begin{align*}
\Delta U_i + \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i \left( a - \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i + cv_i \right) &= 0 \text{ in } \Omega, \\
\Delta v_i + v_i \left( -b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) &= 0 \text{ in } \Omega, \\
\frac{\partial U_i}{\partial n} = \frac{\partial v_i}{\partial n} &= 0 \text{ on } \partial\Omega.
\end{align*}
\] (3.7)
By using the regularity theory for elliptic equations (see e.g., [1]) to the second equation of (3.7), it follows from (3.3) and (3.4) that
\[
||v_i||_{W^{2,p}(\Omega)} \leq C(\zeta_0)
\]
with some positive constant \(C(\zeta_0) = C(\zeta_0, a, b, c, d, \mu)\) independent of \(i\). If \(p > N\), then Sobolev's embedding theorem implies \(\{v_i\}_{i=1}^\infty\) is compact in \(C^1(\overline{\Omega})\). Consequently, there exists a subsequence, which is still denoted by \(\{v_i\}_{i=1}^\infty\), such that
\[
\lim_{i \to \infty} v_i = v_\infty \text{ in } C^1(\overline{\Omega})
\] (3.8)
with some non-negative function \(v_\infty \in C^1(\overline{\Omega})\). Similarly, there exists a non-negative function \(U_\infty \in C^1(\overline{\Omega})\) such that
\[
\lim_{i \to \infty} U_i = U_\infty \text{ in } C^1(\overline{\Omega}).
\] (3.9)
Therefore, \(v_\infty\) satisfies
\[
\Delta v_\infty + v_\infty \left( -b + d \frac{\mu + v_\infty}{\mu + v_\infty + \alpha_\infty} U_\infty - v_\infty \right) = 0 \text{ in } \Omega, \quad \frac{\partial v_\infty}{\partial n} = 0 \text{ on } \partial\Omega
\]
in a weak sense. By standard elliptic regularity theory we have $v_\infty \in C^2(\bar{\Omega})$ and thus $v_\infty$ is a classical solution of the above equation. Then it follows from (3.6),(3.8) and the strong maximum principle that $v_\infty \equiv 0$ in $\bar{\Omega}$. We can easily see from the above argument that $U_\infty$ satisfies

$$\Delta U_\infty + \frac{\mu}{\mu + \alpha_\infty} U_\infty \left( a - \frac{\mu}{\mu + \alpha_\infty} U_\infty \right) = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial U_\infty}{\partial n} = 0 \quad \text{on} \quad \partial \Omega$$

in the classical sense. Then by the strong maximum principle and Lemma 3.1, either $U_\infty \equiv a(\mu + \alpha_\infty)/\mu$ or $U_\infty \equiv 0$ in $\bar{\Omega}$. Combining (3.5) and (3.9), we can conclude $U_\infty \equiv a(\mu + \alpha_\infty)/\mu$ in $\bar{\Omega}$. Hence

$$\lim_{i \to \infty} \left( -b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) = ad - b > 0 \quad \text{uniformly in} \quad \Omega$$

by (1.1) and this means

$$v_i \left( -b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) > 0 \quad \text{in} \quad \Omega$$

for sufficiently large $i \in \mathbb{N}$ because $v_i > 0$ in $\Omega$. On the other hand, from the second equation of (3.7), we have

$$\int_{\Omega} v_i \left( -b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) dx = - \int_{\Omega} \Delta v_i dx = - \int_{\partial \Omega} \frac{\partial v_i}{\partial n} d\sigma = 0$$

for all $i \in \mathbb{N}$. This is a contradiction; thus our proof is complete. \qed

### 3.3 Completion of the proof of Theorem 1.1

Set $X = C(\bar{\Omega}) \times C(\bar{\Omega})$. For each $\alpha \geq 0$, define an operator $F_\alpha$ by

$$F_\alpha \left( \begin{array}{l} U \\ v \end{array} \right) = \left( (-\Delta + I)^{-1} \left[ U + \frac{\mu + v}{\mu + v + \alpha} U \left( a - \frac{\mu + v}{\mu + v + \alpha} U + cv \right) \right] , \right)$$

where $I$ is the identity map from $C(\bar{\Omega})$ into itself, and $(-\Delta + I)^{-1}$ is the inverse operator of $-\Delta + I$ subject to the homogeneous Neumann boundary condition on $\partial \Omega$. It is easy to see that $F_\alpha : X \to X$ is well-defined, and that by elliptic regularity theory and Sobolev's embedding theorem, $F_\alpha$ is a continuous and compact operator for each $\alpha \geq 0$. From these observations, one can define the Leray-Schauder degree of $I - F_\alpha$ at $0$ in a suitable open set. Furthermore, $(U, v)$ is a positive solution of $(I - F_\alpha)(U, v) = 0$ if and only if $(U, v)$ is a positive solution of (EP).

In view of (3.1), we set

$$U_\alpha^* = \left( 1 + \frac{\alpha}{\mu + v^*} \right) u^*.$$
Hence $(U^*_\alpha, v^*)$ is a zero point of $I - F_\alpha$. Then we can calculate the index of $I - F_0$ at $(u^*, v^*)$ and the index of $I - F_\alpha$ at $(U^*_\alpha, v^*)$ for sufficiently large $\alpha$, which are denoted by $\text{index}(I - F_0, (u^*, v^*))$ and $\text{index}(I - F_\alpha, (U^*_\alpha, v^*))$, respectively. We refer to [10] for the proofs of Lemmas 3.3 and 3.4.

Lemma 3.3. It holds that $\text{index}(I - F_0, (u^*, v^*)) = 1$.

Lemma 3.4. Suppose that $\{v^*(b - \mu)/\mu + v^*) \in (\lambda_i, \lambda_{i+1})$ for some $l \geq 1$. Then there exists a positive constant $\alpha^* = \alpha^*(a, b, c, d, \mu)$ such that, if $\alpha > \alpha^*$, then

$$\text{index}(I - F_\alpha, (U^*_\alpha, v^*)) = (-1)^{\sum_{i=1}^{l} m_i},$$

where $m_i$ denotes the algebraic multiplicity of $\lambda_i$ defined in Section 1.

By virtue of Lemmas 3.3 and 3.4, we are ready to prove Theorem 1.1. In the proof of Theorem 1.1, we represent $(EP)$ as $(EP)_\alpha$ to indicate the dependence on $\alpha$.

Proof of Theorem 1.1. Fix any $\alpha > \alpha^*$, where $\alpha^*$ is a constant given in Lemma 3.4. It follows from Lemma 3.2 that there exist two positive constants $C_\ast(\alpha) = C_\ast(\alpha, a, b, c, d, \mu) < C^\ast(\alpha) = C^\ast(\alpha, a, b, c, d, \mu)$ such that

$$a \leq U(x) \leq C^\ast(\alpha) \quad \text{and} \quad C_\ast(\alpha) \leq v(x) \leq C^\ast(\alpha) \quad \text{for all} \quad x \in \overline{\Omega},$$

for any positive solution $(U, v)$ of $(EP)_\nu$ with any $\nu \in [0, \alpha]$. We define

$$S = \left\{(U, v) \in X \mid \frac{a}{2} \leq U \leq 2C^\ast(\alpha), \quad \frac{C_\ast(\alpha)}{2} \leq v \leq 2C^\ast(\alpha) \quad \text{in} \quad \Omega \right\};$$

so that $I - F_\nu$ has no zero point on the boundary of $S$ for any $\nu \in [0, \alpha]$. Note that $I - F_0$ has a unique zero point $(u^*, v^*)$ in $S$. On account of the homotopy invariance of the Leray-Schauder degree and Lemma 3.3, we have

$$\text{deg}(I - F_\alpha, S, 0) = \text{deg}(I - F_0, S, 0) = \text{index}(I - F_0, (u^*, v^*)) = 1. \quad (3.10)$$

Suppose that $(EP)_\alpha$ has no positive non-constant solution, i.e. $I - F_\alpha$ has a unique zero point $(U^*_\alpha, v^*)$ in $S$. Then from the assumption $\sum_{i=1}^{l} m_i$ being odd and Lemma 3.4, it follows that

$$\text{deg}(I - F_\alpha, S, 0) = \text{index}(I - F_\alpha, (U^*_\alpha, v^*)) = (-1)^{\sum_{i=1}^{l} m_i} = -1,$$

which contradicts (3.10). Thus we complete the proof. $\square$
4 Proof of Theorem 1.2

We first state some a priori estimates independent of $\alpha$.

**Lemma 4.1.** Suppose that $N \leq 3$. Then there exists a positive constant $C_0 = C_0(a, b, c, d, \mu)$ independent of $\alpha$ such that any positive solution $(u, v)$ of (SP) satisfies

$$\|u\|_{\infty} \leq C_0 \quad \text{and} \quad \|v\|_{\infty} \leq C_0.$$

Lemma 4.1 can be proved by combining the $L^2$-estimates for positive solutions of (SP) (independent of $\alpha$ and $N$) with Harnack inequality (due to Lin, Ni and Takagi [4], and Lou and Ni [8]). We refer to [10] for the proof of Lemma 4.1.

We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^{\infty}$ be any sequence such that $\lim_{i \to \infty} \alpha_i = \infty$ and positive functions $(u_i, v_i)$ satisfy (SP) with $\alpha = \alpha_i$. Set

$$\psi_i = \left(\frac{1}{\alpha_i} + \frac{1}{\mu + v_i}\right) u_i.$$

Note that positive functions $(\psi_i, v_i)$ satisfy

$$\begin{cases}
\Delta \psi_i + \frac{u_i(a - u_i + cv_i)}{\alpha_i} = 0 \quad \text{in } \Omega, \\
\Delta v_i + v_i(-b + du_i - v_i) = 0 \quad \text{in } \Omega, \\
\frac{\partial \psi_i}{\partial n} = \frac{\partial v_i}{\partial n} = 0 \quad \text{on } \partial \Omega,
\end{cases}$$

and that $\{\psi_i\}_{i=1}^{\infty}$ is bounded independently of $i$ by Lemma 4.1. Then by the compactness argument as in the proof of (3.8), there exists a subsequence, which is still denoted by $\{\psi_i\}_{i=1}^{\infty}$, such that

$$\lim_{i \to \infty} \psi_i = \tau \quad \text{in } C^1(\bar{\Omega})$$

for a non-negative function $\tau \in C^1(\bar{\Omega})$. Similarly, we see

$$\lim_{i \to \infty} v_i = \bar{v} \quad \text{in } C^1(\bar{\Omega})$$

(4.1)

for a non-negative function $\bar{v} \in C^1(\bar{\Omega})$. Therefore, we obtain

$$\lim_{i \to \infty} u_i = \lim_{i \to \infty} \frac{\psi_i}{1/\alpha_i + 1/(\mu + v_i)} = \tau(\mu + \bar{v}) \quad \text{in } C^1(\bar{\Omega}).$$

(4.2)
We will show that $\tau$ is a positive constant. Observe that $\tau$ satisfies

$$\Delta \tau = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \tau}{\partial n} = 0 \quad \text{on} \quad \partial \Omega$$

in a weak sense. A standard elliptic regularity theory ensures $\tau \in C^2(\bar{\Omega})$; so that $\tau$ must be a non-negative constant. Let $v_i(x_i) = \max_{\Omega} v_i$ with some $x_i \in \bar{\Omega}$. It follows from Lemma 3.1 that

$$u_i(x_i) \geq \frac{b + v_i(x_i)}{d} > \frac{b}{d} (> 0)$$

for all $i \in \mathbb{N}$. This fact, together with (4.2), yields $\tau > 0$.

We next prove $(\tau, \bar{v})$ satisfies (1.2). Note that $\bar{v}$ satisfies

$$\Delta \bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \bar{v}}{\partial n} = 0 \quad \text{on} \quad \partial \Omega \quad (4.3)$$

in a weak sense. In the standard manner, one can see that $\bar{v} \in C^2(\bar{\Omega})$ and $\bar{v}$ is a classical nonnegative solution of (4.3). It follows from the strong maximum principle that either $\bar{v} \equiv 0$ or $\bar{v} > 0$ in $\Omega$. We show $\bar{v} > 0$ in $\Omega$ by contradiction. Suppose that $\bar{v} \equiv 0$ in $\Omega$. Then it follows from (4.1) and (4.2) that

$$\lim_{i \to \infty} a - u_i + cv_i = a - \tau\mu \quad \text{and} \quad \lim_{i \to \infty} -b + du_i - v_i = -b + d\tau\mu$$

uniformly in $\Omega$. On the other hand,

$$\int_{\Omega} u_i(a - u_i + cv_i)dx = \int_{\Omega} v_i(-b + du_i - v_i)dx = 0 \quad (4.4)$$

for all $i \in \mathbb{N}$. Consequently, $a - \tau\mu = -b + d\tau\mu = 0$ because of $u_i > 0$ and $v_i > 0$ in $\Omega$ and thus $ad - b = 0$. This contradicts (1.1). Therefore $\bar{v} > 0$ in $\Omega$.

By (4.1), (4.2) and (4.4), it is clear that

$$\int_{\Omega} (\mu + \bar{v})\{a - \tau\mu + (d - \tau)\bar{v}\}dx = \int_{\Omega} (\mu + \bar{v})\{-\tau(\mu + \bar{v}) + c\bar{v}\}dx = 0.$$

Hence it only remains to show $1 < d\tau < b/\mu$. By the assumption of Theorem 1.2,

$$-b + d\tau\mu < -\mu + d\tau\mu = \mu(d\tau - 1).$$

It thus follows from Lemma 3.1 and (4.3) that if $d\tau - 1 \leq 0$, then $\max_{\Omega} \bar{v} \leq 0$ and this contradicts $\bar{v} > 0$ in $\Omega$. Therefore, $d\tau > 1$. Using Lemma 3.1 and $\bar{v} > 0$ in $\Omega$ again, we obtain $d\tau < b/\mu$. Hence we complete the proof. $\square$
5 Remarks about the limiting system (1.2)

We easily see that \((\tau, \overline{v}) = (u^*/(\mu + v^*), v^*)\) is the only positive constant solution of (1.2). So our concern is about positive non-constant solutions of (1.2). We discuss the differential equations without the integral constraint in (1.2) under the restriction \(N \leq 3\):

\[
\begin{aligned}
\Delta \overline{v} + \overline{v}\{-b + d\tau\mu + (d\tau - 1)\overline{v}\} &= 0 \quad \text{in } \Omega, \\
\frac{\partial \overline{v}}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(5.1)

Set

\[w = \frac{d\tau - 1}{b - d\tau\mu} \overline{v},\]

where \(1 < d\tau < b/\mu\). Then (5.1) is rewritten in the following equivalent form:

\[
\begin{aligned}
\frac{1}{b - d\tau\mu} \Delta w - w + w^2 &= 0 \quad \text{in } \Omega, \\
\frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(5.2)

We note that, if \((0 <) b - d\tau\mu \ll 1\), then (5.2) has no positive non-constant solution (see [4]). Therefore, \(b \gg 1\) is necessary for (1.2) to have positive non-constant solutions. We will study (1.2) in detail in the future.

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References


