<table>
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<th>Stationary patterns for a cooperative model with nonlinear diffusion (Nonlinear Evolution Equations and Mathematical Modeling)</th>
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<tr>
<td>Author(s)</td>
<td>Oeda, Kazuhiro</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2008), 1588: 87-98</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/81556">http://hdl.handle.net/2433/81556</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Stationary patterns for a cooperative model with nonlinear diffusion

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1 Introduction

In this article we study positive steady-state solutions of the following strongly coupled reaction-diffusion system:

\[
\begin{cases}
  u_t = \Delta \left[ \left( 1 + \frac{\alpha}{\mu + v} \right) u \right] + u(a - u + cv) \quad \text{in} \quad \Omega \times (0,T), \\
  v_t = \Delta v + v(-b + dv - v) \quad \text{in} \quad \Omega \times (0,T), \\
  \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial \Omega \times (0,T), \\
  u(\cdot,0) = u_0(\cdot), \quad v(\cdot,0) = v_0(\cdot) \quad \text{in} \quad \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \); \( \partial/\partial n \) denotes the outward normal derivative on \( \partial \Omega \); \( a, b, c, d, \) and \( \mu \) are all positive constants; \( \alpha \) is a non-negative constant; \( u_0 \) and \( v_0 \) are given non-negative functions which are not identically zero. System \( (P) \) is a Lotka-Volterra cooperative model with a density-dependent diffusion term of a fractional type; unknown functions \( u \) and \( v \) represent population densities of two cooperative species, respectively; \( a \) and \( -b \) denote the intrinsic growth rates of the respective species; \( c \) and \( d \) denote interaction coefficients. When \( \alpha = 0, \) \( (P) \) is reduced to a classical Lotka-Volterra cooperative model with diffusion. See [6] and [13] for such a cooperative model.

In the first equation of \( (P) \), the nonlinear diffusion term \( \alpha \Delta \{ u/(\mu + v) \} \) describes a situation where species \( u \) tends to leave low-density areas of species \( v \). This situation is natural because relations between \( u \) and \( v \) are cooperative. A population model with density-dependent diffusion was first proposed by Shigesada, Kawasaki and Teramoto [14] to investigate the habitat segregation phenomena between two competing species. Since their work, many mathematicians have studied population models with density-dependent diffusion. However, population models including
density-dependent diffusion terms of a fractional type have appeared in recent years; for example, see [5], [16] for cooperative models with Dirichlet boundary conditions; [2], [3] for prey-predator models with Dirichlet boundary conditions; [12], [15] for three-species prey-predator models with Neumann boundary conditions. See also the monograph of Okubo and Levin [11] for the biological background.

The stationary problem associated with (P) is

\[
\begin{align*}
\Delta \left[ \left( 1 + \frac{\alpha}{\mu + v} \right) u \right] + u(a - u + cv) &= 0 \quad \text{in } \Omega, \\
\Delta v + v(-b + du - v) &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Our main purpose is to study the existence of stationary patterns (i.e. positive non-constant solutions) for (SP) with the weak cooperative condition

\[
a \frac{b}{d} > c.
\]

From now on, we will always assume (1.1). It is well known that, if \( \alpha = 0 \), then every solution of (P) converges to a unique positive constant steady-state

\[
(u^*, v^*) := \left( \frac{a-bc}{1-cd}, \frac{ad-b}{1-cd} \right)
\]

uniformly as \( t \to \infty \); see [6]. This implies the following proposition.

**Proposition 1.1.** Let \( \alpha = 0 \). Then \((u^*, v^*)\) is a unique positive solution of (SP).

Proposition 1.1 means that no stationary pattern exists in the linear diffusion case. However, the presence of density-dependent diffusion enables us to construct stationary patterns of (SP). We focus on \( \alpha \) to show the emergence of stationary patterns for (SP).

Let \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \) denote eigenvalues of \(-\Delta\) with the homogeneous Neumann boundary condition on \( \partial \Omega \) and let \( m_i \) denote the algebraic multiplicity of \( \lambda_i \). Then we have the following theorem.

**Theorem 1.1.** Suppose that \( \{v^*(b - \mu)\}/(\mu + v^*) \in (\lambda_l, \lambda_{l+1}) \) for some \( l \geq 1 \) and that \( \sum_{i=1}^l m_i \) is odd. Then there exists a positive constant \( \alpha^* = \alpha^*(a, b, c, d, \mu) \) such that (SP) has at least one positive non-constant solution for each \( \alpha > \alpha^* \).

We are also interested in the limiting patterns of (SP) as \( \alpha \to \infty \). Under the restriction \( N \leq 3 \), we obtain the following limiting system as \( \alpha \to \infty \).
Theorem 1.2. Suppose $N \leq 3$ and $b > \mu$. Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^{\infty}$ be any sequence such that $\lim_{i \to \infty} \alpha_i = \infty$ and positive functions $(u_i, v_i)$ satisfy (SP) with $\alpha = \alpha_i$. Then, by passing to a subsequence if necessary, it holds that

$$\lim_{i \to \infty} (u_i, v_i) = (\tau(\mu + \overline{v}), \overline{v}) \quad \text{in} \ C^{1}(\overline{\Omega}) \times C^{1}(\overline{\Omega}),$$

where $\tau$ is a positive constant satisfying $1 < d\tau < b/\mu$, $\overline{v}$ is a positive function in $\Omega$ and $(\tau, \overline{v})$ satisfies

$$\begin{cases}
\Delta \overline{v} + \overline{v}\{-b + d\tau \mu + (d\tau - 1)\overline{v}\} = 0 \quad \text{in} \ \Omega, \\
\overline{v} = 0 \quad \text{on} \ \partial\Omega, \\
\int_{\Omega} (\mu + \overline{v})\{a - \tau \mu + (c - \tau)\overline{v}\}dx = 0.
\end{cases}$$

We expect that the limiting system (1.2) may give much information on profiles of stationary patterns of (SP) for large $\alpha$. We will give some remarks about (1.2) in the last section.

Throughout the article, the usual norms of $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\overline{\Omega})$ are defined by

$$\|\psi\|_p := \left(\int_{\Omega} |\psi(x)|^p dx\right)^{1/p} \quad \text{and} \quad \|\psi\|_\infty := \max_{x \in \Omega} |\psi(x)|,$$

respectively.

2 Stability of the constant solution $(u^*, v^*)$

In this section, we will analyze the linearized stability of the constant stationary solution $(u^*, v^*)$ for (P).

The linearized eigenvalue problem of (P) at $(u^*, v^*)$ is given by

$$\begin{cases}
-\left(1 + \frac{\alpha}{\mu + v^*}\right) \Delta h + \frac{\alpha u^*}{(\mu + v^*)^2} \Delta k + u^* h - cu^* k = \eta h \quad \text{in} \ \Omega, \\
-\Delta k - dv^* h + v^* k = \eta k \quad \text{in} \ \Omega, \\
\frac{\partial h}{\partial n} = \frac{\partial k}{\partial n} = 0 \quad \text{on} \ \partial\Omega.
\end{cases}$$

We know that $(u^*, v^*)$ is linearly stable when $\alpha = 0$. Using the expansions of $h$ and $k$ in terms of eigenfunctions of $-\Delta$, one can see that $\eta$ is an eigenvalue of (2.1) if and only if

$$\det \begin{pmatrix}
-\eta + \left(1 + \frac{\alpha}{\mu + v^*}\right) & u^* & -\frac{\alpha u^*}{(\mu + v^*)^2} \\
-dv^* & \lambda_i & -\eta + \lambda_i + v^* \\
& & \eta + \lambda_i + v^*
\end{pmatrix} = 0$$
for some \( i \geq 0 \). In particular, \( \eta = 0 \) is an eigenvalue of (2.1) if and only if
\[
\frac{\lambda_i}{(\mu + v^*)^2} \{(\mu + v^*)(\lambda_i + v^*) - du^*v^*\} \alpha + (\lambda_i + u^*)(\lambda_i + v^*) - cdu^*v^* = 0
\]
for some \( i \geq 0 \). Note that \((\lambda_i + u^*)(\lambda_i + v^*) - cdu^*v^* > 0\) for all \( i \geq 0 \) because of (1.1). Thus it is easy to see that the linearized stability of \((u^*, v^*)\) changes as \( \alpha \) increases in (P) if and only if
\[
(\mu + v^*)(\lambda_1 + v^*) - du^*v^* = (\mu + v^*)\lambda_1 + v^*(\mu + v^* - du^*) = (\mu + v^*)\lambda_1 + v^*(\mu - b) < 0.
\]
Therefore, \( b > \mu \) is necessary for the linearized stability of \((u^*, v^*)\) to change (and so we do not discuss the case \( b \leq \mu \), especially, \(-b \geq 0\)). This means that the difference in the intrinsic growth rates between two species \( u \) and \( v \) contributes to creating stationary patterns in (SP).

3 Proof of Theorem 1.1

3.1 Reduction to the semilinear system

Our method of the proof of Theorem 1.1 will be based on the Leray-Schauder degree theory (see e.g., [9]) and some a priori estimates. We first introduce a new unknown function \( U \) by
\[
U = \left(1 + \frac{\alpha}{\mu + v}\right) u.
\]
(3.1)

Clearly, there exists a one-to-one correspondence between \((u, v) > 0\) and \((U, v) > 0\). As far as we discuss positive solutions, (SP) is rewritten in the following equivalent form:

\[
\begin{align*}
\Delta U + \frac{\mu + v}{\mu + v + \alpha} U \left(a - \frac{\mu + v}{\mu + v + \alpha} U + cv\right) &= 0 \quad \text{in } \Omega, \\
\Delta v + v \left(-b + d\frac{\mu + v}{\mu + v + \alpha} U - v\right) &= 0 \quad \text{in } \Omega, \\
\frac{\partial U}{\partial n} = \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

Thus, it is sufficient to show the existence of positive non-constant solutions of (EP).
3.2 A priori estimates

In this subsection, we will give some a priori estimates for positive solutions of (EP). Before stating the a priori estimates, we recall the following maximum principle due to Lou and Ni [7].

Lemma 3.1. Suppose that $g \in C(\bar{\Omega} \times \mathbb{R})$.

(i) If $w \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ satisfies

$$
\Delta w(x) + g(x, w(x)) \geq 0 \text{ in } \Omega, \quad \frac{\partial w}{\partial n} \leq 0 \text{ on } \partial\Omega,
$$

and $w(x_0) = \max_{\Omega} w$, then $g(x_0, w(x_0)) \geq 0$.

(ii) If $w \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ satisfies

$$
\Delta w(x) + g(x, w(x)) \leq 0 \text{ in } \Omega, \quad \frac{\partial w}{\partial n} \geq 0 \text{ on } \partial\Omega,
$$

and $w(x_0) = \min_{\Omega} w$, then $g(x_0, w(x_0)) \leq 0$.

Now we can derive the following a priori estimates.

Lemma 3.2. Let $\zeta$ be any fixed positive number. Then there exist two positive constants $C_{*}(\zeta) = C_{*}(\zeta, a, b, c, d, \mu) < C^{*}(\zeta) = C^{*}(\zeta, a, b, c, d, \mu)$ such that, if $\alpha \leq \zeta$, then any positive solution $(U, v)$ of (EP) satisfies

$$
a \leq U(x) \leq C^{*}(\zeta) \text{ and } C_{*}(\zeta) \leq v(x) \leq C^{*}(\zeta) \text{ for all } x \in \bar{\Omega}.
$$

Proof. Let $U(x_0) = \max_{\Omega} U$ and $v(y_0) = \max_{\Omega} v$ with $x_0, y_0 \in \bar{\Omega}$. Applying Lemma 3.1 to (EP), we have

$$
\max_{\bar{\Omega}} U \leq \frac{\mu + v(x_0) + \alpha}{\mu + v(x_0)} (a + cv(x_0))
$$

and

$$
\max_{\bar{\Omega}} v \leq -b + d\frac{\mu + v(y_0)}{\mu + v(y_0) + \alpha} U(y_0) \leq -b + d \max_{\bar{\Omega}} U.
$$

Thus

$$
\max_{\bar{\Omega}} U \leq a + cv(x_0) + \zeta \frac{a + cv(x_0)}{\mu + v(x_0)}
$$

$$
\leq a + c(-b + d \max_{\bar{\Omega}} U) + \zeta \max \left\{ \frac{a}{\mu}, c \right\}.
$$

Therefore, we see

$$
\max_{\bar{\Omega}} U \leq \frac{a - bc + \zeta \max \{a/\mu, c\}}{1 - cd}.
$$

(3.3)
It follows from (3.2) and (3.3) that
\[ \max_{\overline{\Omega}} v \leq -b + \frac{d}{1-cd} a - b + \frac{\zeta}{1-cd} \max\{a/\mu, c\} = \frac{ad - b + \zeta d \max\{a/\mu, c\}}{1-cd}. \] (3.4)

Hence we have obtained the desired upper bound of \((U, v)\).

Let \(U(z_0) = \min_{\overline{\Omega}} U\) with some \(z_0 \in \overline{\Omega}\). Using Lemma 3.1 to the first equation of (EP), we get
\[ \min_{\overline{\Omega}} U \geq \frac{\mu + v(z_0) + \alpha}{\mu + v(z_0)} (a + cv(z_0)) \geq a. \] (3.5)

Thus we have obtained the desired lower bound of \(U\).

Finally, we derive a lower bound of \(v\) by contradiction. Suppose that there exist a certain positive constant \(\zeta_0\) and a sequence \([(U_i, v_i, \alpha_i)]_{i=1}^{\infty}\) such that \(\alpha_i \leq \zeta_0\) for all \(i \in \mathbb{N}\), \(\lim_{i \to \infty} \alpha_i = \alpha_\infty\) for some non-negative constant \(\alpha_\infty\),
\[ \lim_{i \to \infty} \min_{\overline{\Omega}} v_i = 0 \] (3.6)
and positive functions \((U_i, v_i)\) satisfy
\[
\begin{align*}
\Delta U_i + \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i \left( a - \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i + cv_i \right) &= 0 \quad \text{in } \Omega, \\
\Delta v_i + v_i \left( -b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) &= 0 \quad \text{in } \Omega, \\
\frac{\partial U_i}{\partial n} = \frac{\partial v_i}{\partial n} &= 0 \quad \text{on } \partial \Omega. 
\end{align*}
\] (3.7)

By using the regularity theory for elliptic equations (see e.g., [1]) to the second equation of (3.7), it follows from (3.3) and (3.4) that
\[ \|v_i\|_{W^{2,p}(\Omega)} \leq C(\zeta_0) \]
with some positive constant \(C(\zeta_0) = C(\zeta_0, a, b, c, d, \mu)\) independent of \(i\). If \(p > N\), then Sobolev's embedding theorem implies \(\{v_i\}_{i=1}^{\infty}\) is compact in \(C^1(\overline{\Omega})\). Consequently, there exists a subsequence, which is still denoted by \(\{v_i\}_{i=1}^{\infty}\), such that
\[ \lim_{i \to \infty} v_i = v_\infty \quad \text{in } C^1(\overline{\Omega}) \] (3.8)
with some non-negative function \(v_\infty \in C^1(\overline{\Omega})\). Similarly, there exists a non-negative function \(U_\infty \in C^1(\overline{\Omega})\) such that
\[ \lim_{i \to \infty} U_i = U_\infty \quad \text{in } C^1(\overline{\Omega}). \] (3.9)

Therefore, \(v_\infty\) satisfies
\[ \Delta v_\infty + v_\infty \left( -b + d \frac{\mu + v_\infty}{\mu + v_\infty + \alpha_\infty} U_\infty - v_\infty \right) = 0 \quad \text{in } \Omega, \quad \frac{\partial v_\infty}{\partial n} = 0 \quad \text{on } \partial \Omega \]
in a weak sense. By standard elliptic regularity theory we have $v_{\infty} \in C^{2}(\bar{\Omega})$ and thus $v_{\infty}$ is a classical solution of the above equation. Then it follows from (3.6),(3.8) and the strong maximum principle that $v_{\infty} \equiv 0$ in $\bar{\Omega}$. We can easily see from the above argument that $U_{\infty}$ satisfies

$$\Delta U_{\infty} + \frac{\mu}{\mu + \alpha_{\infty}} U_{\infty} \left( a - \frac{\mu}{\mu + \alpha_{\infty}} U_{\infty} \right) = 0 \quad \text{in } \Omega, \quad \frac{\partial U_{\infty}}{\partial n} = 0 \quad \text{on } \partial \Omega$$

in the classical sense. Then by the strong maximum principle and Lemma 3.1, either $U_{\infty} \equiv a(\mu + \alpha_{\infty})/\mu$ or $U_{\infty} \equiv 0$ in $\bar{\Omega}$. Combining (3.5) and (3.9), we can conclude $U_{\infty} \equiv a(\mu + \alpha_{\infty})/\mu$ in $\bar{\Omega}$. Hence

$$\lim_{i \to \infty} \left( -b + d\frac{\mu + v_{i}}{\mu + v_{i} + \alpha_{i}} U_{i} - v_{i} \right) = ad - b > 0 \quad \text{uniformly in } \Omega$$

by (1.1) and this means

$$v_{i} \left( -b + d\frac{\mu + v_{i}}{\mu + v_{i} + \alpha_{i}} U_{i} - v_{i} \right) > 0 \quad \text{in } \Omega$$

for sufficiently large $i \in \mathbb{N}$ because $v_{i} > 0$ in $\Omega$. On the other hand, from the second equation of (3.7), we have

$$\int_{\Omega} v_{i} \left( -b + d\frac{\mu + v_{i}}{\mu + v_{i} + \alpha_{i}} U_{i} - v_{i} \right) dx = -\int_{\Omega} \Delta v_{i} dx = -\int_{\partial \Omega} \frac{\partial v_{i}}{\partial n} d\sigma = 0$$

for all $i \in \mathbb{N}$. This is a contradiction; thus our proof is complete. \qed

### 3.3 Completion of the proof of Theorem 1.1

Set $X = C(\bar{\Omega}) \times C(\bar{\Omega})$. For each $\alpha \geq 0$, define an operator $F_{\alpha}$ by

$$F_{\alpha} \begin{pmatrix} U \\ v \end{pmatrix} = \begin{pmatrix} (-\Delta + I)^{-1} \left[ U + \frac{\mu + v}{\mu + v + \alpha} U \left( a - \frac{\mu + v}{\mu + v + \alpha} U + cv \right) \right] \\ (-\Delta + I)^{-1} \left[ v + v \left( -b + d\frac{\mu + v}{\mu + v + \alpha} U - v \right) \right] \end{pmatrix},$$

where $I$ is the identity map from $C(\bar{\Omega})$ into itself, and $(-\Delta + I)^{-1}$ is the inverse operator of $-\Delta + I$ subject to the homogeneous Neumann boundary condition on $\partial \Omega$. It is easy to see that $F_{\alpha} : X \to X$ is well-defined, and that by elliptic regularity theory and Sobolev's embedding theorem, $F_{\alpha}$ is a continuous and compact operator for each $\alpha \geq 0$. From these observations, one can define the Leray-Schauder degree of $I - F_{\alpha}$ at 0 in a suitable open set. Furthermore, $(U, v)$ is a positive solution of $(I - F_{\alpha})(U, v) = 0$ if and only if $(U, v)$ is a positive solution of (EP).

In view of (3.1), we set

$$U_{\alpha}^{*} = \left( 1 + \frac{\alpha}{\mu + v^{*}} \right) u^{*}.$$
Hence $(U_\alpha^*, v^*)$ is a zero point of $I - F_\alpha$. Then we can calculate the index of $I - F_0$ at $(u^*, v^*)$ and the index of $I - F_\alpha$ at $(U_\alpha^*, v^*)$ for sufficiently large $\alpha$, which are denoted by $\text{index}(I - F_0, (u^*, v^*))$ and $\text{index}(I - F_\alpha, (U_\alpha^*, v^*))$, respectively. We refer to [10] for the proofs of Lemmas 3.3 and 3.4.

**Lemma 3.3.** It holds that $\text{index}(I - F_0, (u^*, v^*)) = 1$.

**Lemma 3.4.** Suppose that $\{v^*(b - \mu)/(\mu + v^*)\} \in (\lambda_l, \lambda_{l+1})$ for some $l \geq 1$. Then there exists a positive constant $\alpha^* = \alpha^*(a, b, c, d, \mu)$ such that, if $\alpha > \alpha^*$, then

$$\text{index}(I - F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^{l} m_i},$$

where $m_i$ denotes the algebraic multiplicity of $\lambda_i$ defined in Section 1.

By virtue of Lemmas 3.3 and 3.4, we are ready to prove Theorem 1.1. In the proof of Theorem 1.1, we represent $(EP)$ as $(EP)_\alpha$ to indicate the dependence on $\alpha$.

**Proof of Theorem 1.1.** Fix any $\alpha > \alpha^*$, where $\alpha^*$ is a constant given in Lemma 3.4. It follows from Lemma 3.2 that there exist two positive constants $C_*(\alpha) = C_*(\alpha, a, b, c, d, \mu) < C^*(\alpha) = C^*(\alpha, a, b, c, d, \mu)$ such that

$$a \leq U(x) \leq C^*(\alpha) \quad \text{and} \quad C_*(\alpha) \leq v(x) \leq C^*(\alpha) \quad \text{for all} \quad x \in \overline{\Omega}$$

for any positive solution $(U, v)$ of $(EP)_\nu$ with any $\nu \in [0, \alpha]$. We define

$$S = \left\{ (U, v) \in X | \frac{a}{2} \leq U \leq 2C^*(\alpha), \frac{C_*(\alpha)}{2} \leq v \leq 2C^*(\alpha) \quad \text{in} \quad \overline{\Omega} \right\};$$

so that $I - F_\nu$ has no zero point on the boundary of $S$ for any $\nu \in [0, \alpha]$. Note that $I - F_0$ has a unique zero point $(u^*, v^*)$ in $S$. On account of the homotopy invariance of the Leray-Schauder degree and Lemma 3.3, we have

$$\text{deg}(I - F_\alpha, S, 0) = \text{deg}(I - F_0, S, 0) = \text{index}(I - F_0, (u^*, v^*)) = 1. \quad (3.10)$$

Suppose that $(EP)_\alpha$ has no positive non-constant solution, i.e. $I - F_\alpha$ has a unique zero point $(U_\alpha^*, v^*)$ in $S$. Then from the assumption $\sum_{i=1}^{l} m_i$ being odd and Lemma 3.4, it follows that

$$\text{deg}(I - F_\alpha, S, 0) = \text{index}(I - F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^{l} m_i} = -1,$$

which contradicts (3.10). Thus we complete the proof. \qed
4 Proof of Theorem 1.2

We first state some a priori estimates independent of $\alpha$.

**Lemma 4.1.** Suppose that $N \leq 3$. Then there exists a positive constant $C_0 = C_0(a, b, c, d, \mu)$ independent of $\alpha$ such that any positive solution $(u, v)$ of (SP) satisfies

$$
\|u\|_\infty \leq C_0 \quad \text{and} \quad \|v\|_\infty \leq C_0.
$$

Lemma 4.1 can be proved by combining the $L^2$-estimates for positive solutions of (SP) (independent of $\alpha$ and $N$) with Harnack inequality (due to Lin, Ni and Takagi [4], and Lou and Ni [8]). We refer to [10] for the proof of Lemma 4.1.

We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^\infty$ be any sequence such that $\lim_{i \to \infty} \alpha_i = \infty$ and positive functions $(u_i, v_i)$ satisfy (SP) with $\alpha = \alpha_i$. Set

$$
\psi_i = \left(\frac{1}{\alpha_i} + \frac{1}{\mu + v_i}\right) u_i.
$$

Note that positive functions $(\psi_i, v_i)$ satisfy

$$
\begin{cases}
\Delta \psi_i + \frac{u_i(a - u_i + cv_i)}{\alpha_i} = 0 \quad \text{in} \quad \Omega, \\
\Delta v_i + v_i(-b + du_i - v_i) = 0 \quad \text{in} \quad \Omega, \\
\frac{\partial \psi_i}{\partial n} = \frac{\partial v_i}{\partial n} = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
$$

and that $\{\psi_i\}_{i=1}^\infty$ is bounded independently of $i$ by Lemma 4.1. Then by the compactness argument as in the proof of (3.8), there exists a subsequence, which is still denoted by $\{\psi_i\}_{i=1}^\infty$, such that

$$
\lim_{i \to \infty} \psi_i = \tau \quad \text{in} \quad C^1(\bar{\Omega})
$$

for a non-negative function $\tau \in C^1(\bar{\Omega})$. Similarly, we see

$$
\lim_{i \to \infty} v_i = \bar{v} \quad \text{in} \quad C^1(\bar{\Omega}) \tag{4.1}
$$

for a non-negative function $\bar{v} \in C^1(\bar{\Omega})$. Therefore, we obtain

$$
\lim_{i \to \infty} u_i = \lim_{i \to \infty} \frac{\psi_i}{1/\alpha_i + 1/(\mu + v_i)} = \tau(\mu + \bar{v}) \quad \text{in} \quad C^1(\bar{\Omega}). \tag{4.2}
$$
We will show that \( \tau \) is a positive constant. Observe that \( \tau \) satisfies

\[
\Delta \tau = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \tau}{\partial n} = 0 \quad \text{on} \quad \partial \Omega
\]

in a weak sense. A standard elliptic regularity theory ensures \( \tau \in C^2(\overline{\Omega}) \); so that \( \tau \) must be a non-negative constant. Let \( v_i(x_i) = \max_{\Omega} v_i \) with some \( x_i \in \overline{\Omega} \). It follows from Lemma 3.1 that

\[
u_i(x_i) \geq \frac{b + v_i(x_i)}{d} > \frac{b}{d} (> 0)
\]

for all \( i \in \mathbb{N} \). This fact, together with (4.2), yields \( \tau > 0 \).

We next prove \( (\tau, \bar{v}) \) satisfies (1.2). Note that \( \bar{v} \) satisfies

\[
\Delta \bar{v} + \bar{v}(-b + d\tau \mu + (d\tau - 1)\bar{v}) = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \bar{v}}{\partial n} = 0 \quad \text{on} \quad \partial \Omega
\]  \hspace{1cm} (4.3)

in a weak sense. In the standard manner, one can see that \( \bar{v} \in C^2(\overline{\Omega}) \) and \( \bar{v} \) is a classical nonnegative solution of (4.3). It follows from the strong maximum principle that either \( \bar{v} \equiv 0 \) or \( \bar{v} > 0 \) in \( \Omega \). We show \( \bar{v} > 0 \) in \( \Omega \) by contradiction. Suppose that \( \bar{v} \equiv 0 \) in \( \Omega \). Then it follows from (4.1) and (4.2) that

\[
\lim_{i \to \infty} a - u_i + cv_i = a - \tau \mu \quad \text{and} \quad \lim_{i \to \infty} -b + du_i - v_i = -b + d\tau \mu
\]

uniformly in \( \Omega \). On the other hand,

\[
\int_{\Omega} u_i(a - u_i + cv_i)dx = \int_{\Omega} v_i(-b + du_i - v_i)dx = 0 \]  \hspace{1cm} (4.4)

for all \( i \in \mathbb{N} \). Consequently, \( a - \tau \mu = -b + d\tau \mu = 0 \) because of \( u_i > 0 \) and \( v_i > 0 \) in \( \Omega \) and thus \( ad - b = 0 \). This contradicts (1.1). Therefore \( \bar{v} > 0 \) in \( \Omega \).

By (4.1), (4.2) and (4.4), it is clear that

\[
\int_{\Omega} (\mu + \bar{v})(a - \tau \mu + (c - \tau)\bar{v})dx = \int_{\Omega} (\mu + \bar{v})(a - \tau (\mu + \bar{v}) + c\bar{v})dx = 0
\]

Hence it only remains to show \( 1 < d\tau < b/\mu \). By the assumption of Theorem 1.2,

\[
-b + d\tau \mu < -\mu + d\tau \mu = \mu(d\tau - 1).
\]

It thus follows from Lemma 3.1 and (4.3) that if \( d\tau - 1 \leq 0 \), then \( \max_{\overline{\Omega}} \bar{v} \leq 0 \) and this contradicts \( \bar{v} > 0 \) in \( \Omega \). Therefore, \( d\tau > 1 \). Using Lemma 3.1 and \( \bar{v} > 0 \) in \( \Omega \) again, we obtain \( d\tau < b/\mu \). Hence we complete the proof. \( \square \)
5 Remarks about the limiting system (1.2)

We easily see that \((\tau, \overline{v}) = (u^*/(\mu + v^*), v^*)\) is the only positive constant solution of (1.2). So our concern is about positive non-constant solutions of (1.2). We discuss the differential equations without the integral constraint in (1.2) under the restriction \(N \leq 3\):

\[
\begin{aligned}
\Delta \overline{v} + \overline{v}\{-b + d\tau \mu + (d\tau - 1)\overline{v}\} &= 0 \quad \text{in } \Omega, \\
\frac{\partial \overline{v}}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(5.1)

Set

\[
w = \frac{d\tau - 1}{b - d\tau \mu} \overline{v},
\]

where \(1 < d\tau < b/\mu\). Then (5.1) is rewritten in the following equivalent form:

\[
\begin{aligned}
\frac{1}{b - d\tau \mu} \Delta w - w + w^2 &= 0 \quad \text{in } \Omega, \\
\frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(5.2)

We note that, if \((0 <) b - d\tau \mu \ll 1\), then (5.2) has no positive non-constant solution (see [4]). Therefore, \(b \gg 1\) is necessary for (1.2) to have positive non-constant solutions. We will study (1.2) in detail in the future.

Acknowledgement. The author would like to express his gratitude to Professor Yoshio Yamada for his useful advice.

References


