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Solutions having boundary layers to a nonlinear elliptic equation on a spherical cap

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1 Introduction

In this paper, we consider the nonlinear elliptic equation

$$\Lambda u + \lambda(-u + u_+^p) = 0 \quad \text{in } \Omega \subset \mathbb{S}^n \tag{1.1}$$

under the homogeneous Dirichlet boundary condition. Here Λ denotes the Laplace-Beltrami operator on the standard unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. We assume that $n \geq 3$, $p > 1$, $\lambda > 0$ and that $\Omega \subset \mathbb{S}^n$ is a geodesic open ball, called a "spherical cap", centered at the North Pole $(0, \ldots, 0, 1)$. To start our analysis, we express Ω in polar coordinates in order to make our setting clear.

Let $(y_1, y_2, \ldots, y_{n+1})$ be the Cartesian coordinates in $\mathbb{R}^{n+1}$. We express

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the points of $\mathbb{S}^n$ in terms of polar coordinates:

$$\begin{cases} y_k = \Big(\prod_{j=1}^{k} \sin\theta_j\Big)\cos\theta_{k+1}, \quad k = 1,2,\dots,n-2, \\ y_{n-1} = \Big(\prod_{j=1}^{n-1} \sin\theta_j\Big)\cos\phi, \\ y_n = \Big(\prod_{j=1}^{n-1} \sin\theta_j\Big)\sin\phi, \\ y_{n+1} = \cos\theta_1. \end{cases}$$

Then Ω can be expressed in polar coordinates as

$$\Omega = \Omega_\varepsilon = \Big\{(\theta_1,\theta_2,\dots,\theta_{n-1},\phi)\Big|\ 0 \le \theta_1 \le \pi - \varepsilon, \\ 0 \le \theta_i \le \pi, (i = 2,3,\dots,n-1),\ 0 \le \phi \le 2\pi\Big\}.$$

We shall consider how solutions behave as $\varepsilon \to 0$, i.e., what will happen to solutions if Ω becomes closer to the full sphere $\mathbb{S}^n$.

In polar coordinates, Λ becomes

$$\Lambda u = \sum_{k=1}^{n-1}(\sin\theta_1 \dots \sin\theta_{k-1})^{-2}(\sin\theta_k)^{k-n}\frac{\partial}{\partial\theta_k}\Big\{(\sin\theta_k)^{n-k}\frac{\partial u}{\partial\theta_k}\Big\} \\ +\Big(\prod_{k=1}^{n-1}\sin\theta_k\Big)^{-2}\frac{\partial^2 u}{\partial\phi^2}.$$

Here we can consider (1.1) in the class of "radial" functions, that is, functions depending only on the azimuthal angle θ_1 (="latitude"). For such a function v, Λ reads as

$$\Lambda v = \frac{1}{\sin^{n-1}\theta_1}\frac{\partial}{\partial\theta_1}\Big\{\Big(\sin^{n-1}\theta_1\Big)\frac{\partial v}{\partial\theta_1}\Big\}, \tag{1.2}$$

which will be denoted by Λ_{θ_1}. Then (1.1) becomes

$$\Lambda_{\theta_1} v + \lambda(-v + v_+^p) = 0. \tag{1.3}$$

Thus (1.1) is reduced to an ordinary differential equation of the degenerate Sturm-Liouville type.

As for the precedent works, Stingelin [18] considered (1.1) for a fixed spherical cap, containing the upper hemi-sphere and for large $\lambda > 0$ and the homogeneous Dirichlet condition, and showed numerically the bifurcation diagram, which seemed as if imperfect bifurcations occur. His diagrams look very like the one obtained in Kabeya, Morishita and Ninomiya [13] for the problem

$$\Delta u + \lambda(u^p - u) = 0 \quad \text{in} \quad \{|y| < 1\} \subset \mathbb{R}^n, \quad \frac{\partial u}{\partial \nu} + \epsilon u = 0 \quad \text{on} \quad \{|y| = 1\},$$

where $\partial/\partial\nu$ denotes the outer normal derivative.

Inspired by [18], we will determine the asymptotic behavior of the solutions as $\varepsilon \to 0$.

Recently, Brezis and Peletier [5] studied (1.3) for $n = 3$ and $p = 5$. They confirmed Stingelin's results [18] did and they showed several properties of the bifurcation diagram for large λ (and necessarily with small $u(0) > 0$). Moreover, very recently, Bandle and Wei [6, 7, 8] studied intensively this subject from the singular perturbation point of view, as in Ambrosetti, Malchiodi and Ni [1, 2] and Malchiodi, Ni and Wei [15]. Various concentration phenomena have been observed in [6, 7, 8] for large λ with a fixed domain.

Notice that (1.1) on $\mathbb{S}^n$ has a constant solution $v \equiv 1$ for any $\lambda > 0$. Although this constant is never a solution to the Dirichlet problem, as in Section 4 of [5], the analysis of the corresponding linearized problem is important. More precisely, for given $\varepsilon > 0$, we consider the problem

$$\begin{cases} \Lambda_{\theta_1} v + (p-1)\lambda v = 0 & \text{in} \quad (0, \pi - \epsilon) \\ v_{\theta_1}(0) = 0, \ v(\pi - \varepsilon), \end{cases} \tag{1.4}$$

and look for azimuthal eigenvalues $\lambda^D_{k,\varepsilon} > 0$ and the corresponding eigenfunctions $v = \varphi^D_{k,\varepsilon}$ with $k-1$ zeros in $(0, \pi - \varepsilon)$ $(k = 1, 2, 3, \dots)$.

As a comparison, we also consider properties of a solution to the following eigenvalue problems

$$\begin{cases} \Lambda_{\theta_1} v + (p-1)\lambda v = 0, \\ v_{\theta_1}(\pi) = v_{\theta_1}(0) = 0. \end{cases} \tag{1.5}$$

We say that λ_k $(k \in \mathbb{N})$ is the k-th eigenvalue if a nontrivial solution $v = \varphi_k$ to (1.5) changes its sign $(k-1)$ times in $[0, \pi)$. The first eigenvalue λ_1 is zero and the corresponding eigenfunction is a constant.

The eigenvalues and the eigenfunctions will play important roles to the analysis.

Taking the consideration above into account, we investigate the following Neumann-Dirichlet boundary value problem of the ordinary differential equation

$$\begin{cases} \Lambda_{\theta_1} v + \lambda(-v + v^p) = 0, & 0 < \theta_1 < \pi - \varepsilon, \\ v(\pi - \varepsilon) = 0, \quad v_{\theta_1}(0) = 0. \end{cases} \tag{1.6}$$

We should note that treating (1.6) as in Yanagida and Yotsutani [20, 21] or in Kabeya, Yanagida and Yotsutani [14] does not seem to work well. We analyze (1.6) as it is.

Also, we mention that we need not restrict the exponent p to sub-Sobolev critical or critical one. We introduce an exponent q and a Banach space.

For $n \geq 3$, choose a fixed q satisfying

$$\max\left\{\frac{n}{2}, \left(1 - \frac{1}{p}\right) n\right\} \leq q < n,$$

and set

$$\mathcal{W} := W^{1,q}_{0,\mathrm{az}}(\Omega_\varepsilon),$$

where $\mathcal{W}$ is the completion of $C_0^\infty(\Omega_\varepsilon)$-functions depending only on θ_1, with respect to the norm

$$\|\Phi\|_{\mathcal{W}} := \left(\int_{\Omega_\varepsilon} |\Phi_{\theta_1}|^q \, dS + \int_{\Omega_\varepsilon} |\Phi|^q \, dS\right)^{\frac{1}{q}}.$$

Also we define

$$L^p_{\mathrm{az}}(\Omega_\varepsilon) := \{f \in L^p(\Omega_\varepsilon) \mid f \text{ depends only on } \theta_1\}.$$

Note that for a function f depending only on θ_1, we have

$$\int_{\Omega_\varepsilon} f(\theta_1) \, dS = |\mathbb{S}^{n-1}| \int_0^{\pi-\varepsilon} f(\theta_1) \sin^{n-1}\theta_1 \, d\theta_1.$$

Because of the particular choice of q, Sobolev's embedding

$$W^{1,q}(\Omega_\varepsilon) \hookrightarrow L^{pq}(\Omega_\varepsilon)$$

holds. Moreover, we denote the orthogonal projection with respect to $L^2_{\mathrm{az}}(\Omega_\varepsilon)$ into the linear space $\langle \varphi^D_{j,\varepsilon} \rangle$, by $P_{j,\varepsilon}$ and the projection into its orthogonal space $\langle \varphi^D_{j,\varepsilon} \rangle^\perp$ by $Q_{j,\varepsilon}$. More precisely,

$$P_{j,\varepsilon} u := \left(\int_{\Omega_\varepsilon} \varphi^D_{j,\varepsilon} u \, dS\right) \varphi^D_{j,\varepsilon}$$

and

$$Q_{j,\varepsilon} u := u - \left(\int_{\Omega_\varepsilon} \varphi_{j,\varepsilon}^D u \, dS \right) \varphi_{j\varepsilon}^D,$$

where $\varphi_{j,\varepsilon}^D$ is normalized such that $\|\varphi_{j,\varepsilon}^D\|_{L^2} = 1$.

Note that the orthogonal decomposition is possible even for the Banach space $\mathcal{W}$ since the space $\langle \varphi_{j,\varepsilon} \rangle$ is one dimensional.

Now, we are in a position to state our main result.

Theorem 1.1 *Let $p > 1$, $n \geq 3$, $j \geq 2$. Suppose that $\varepsilon_* > 0$ and $\zeta_* > 0$ be sufficiently small. Then there exist a set $\mathcal{S}_\varepsilon(j) \in (\lambda_j - \zeta_*, \lambda_j + \zeta_*) \times \mathcal{W}$ for any $\varepsilon \in (0, \varepsilon_*)$, which satisfies the following:*

(1) *there exist a positive constant s_* (depending only on $\varepsilon_* > 0$), functions w_ε, $h(s) \in \mathcal{W}$ and a map $H_\varepsilon(s, \lambda) : \mathbb{R}^2 \mapsto \mathbb{R}$ such that*

$$\begin{aligned} \mathcal{S}_\varepsilon(j) = \{ (\lambda, v) \in (\lambda_j - \delta^*, \lambda_j + \delta_*) \times \mathcal{W} \mid \\ v(\theta_1; \varepsilon) = 1 + w_\varepsilon + s\varphi_{j,\varepsilon} + h(s), \\ \textit{is a solution to } (1.1) \textit{ and } H_\varepsilon(s, \lambda) = 0, \textit{for } |s| < s_* \}. \end{aligned}$$

(2) *$h(s) \perp \varphi_{j,\varepsilon}$ in $\mathcal{W}$.*

(3) *$\|w_\varepsilon\|_{\mathcal{W}} = O(\varepsilon^{(n-q)/q})$ and $w_\varepsilon(\theta_1) \to 1$ locally uniformly on $[0, \pi)$ as $\varepsilon \to 0$.*

(4) *The equation $H_\varepsilon(s, \lambda) = 0$ is asymptotically expressed as*

$$s\kappa + a_1 s^2 + \eta(\varepsilon) + O(\varepsilon^{(n-q)/q} |s|^{\min\{2,p\}} + \varepsilon^{2(n-q)/q} |s| + |s|^{\min\{2,p\}+1}) = 0 \tag{1.7}$$

with $\kappa = (p-1)(\lambda - \lambda_{k,\varepsilon})$, where φ_j is the j-th eigenfunction of the whole sphere case, a_1 is defined as

$$a_1 = \frac{p(p-1)}{2} \int_{\Omega_\varepsilon} (\varphi_{j,\varepsilon}^D)^3 \, dS$$

and $\eta(\varepsilon)$ is depending only on ε and satisfies $|\eta| \geq O(\varepsilon^n)$. Moreover, if $n = 3$, then

$$\eta(\varepsilon) = \frac{(-1)^{j+1}}{\lambda_j} j\varepsilon + O(\varepsilon^2).$$

Remark 1.1 *The leading three terms of (1.7) indicate that two bifurcation curves are close but they are disconnected with each other.*

Remark 1.2 *Our proof is indeed valid for $\varepsilon = 0$ (the whole sphere case with the homogeneous Neumann boundary condition). In this case, we can regard $w_\varepsilon \equiv 0$ and (1.7) is expressed as*

$$s\kappa + a_1 s^2 + O(|s|^{\min\{2,p\}+1}) = 0 \tag{1.8}$$

with

$$a_1 = \frac{p(p-1)}{2} \int_{\Omega_\varepsilon} (\varphi_j)^3 \, dS.$$

(1.8) represents two connected curves, that is, the local bifurcation at $\lambda = \lambda_j$ is ensured.

The organization of this paper is as follows. Analysis on the linear problem will be done by using the Legendre associate functions in Section 2. Sketch of a proof of Theorem 1.1 will be given in Section 3 with two key lemmas.

2 Analysis of the linearized equation

In this section, we consider the linearized problem. The constant 1 is no longer a solution to (1.1), however, the linearized equation around 1 gives us the first approximation. Moreover, the behavior of a solution to the linearized equation suggests the existence of the layer of the solution near the boundary.

We investigate exact solutions to (1.4) and (1.5) by using the Legendre associate functions. Here, we enumerate important facts and formulae (see for details, Kabeya and Ninomiya [12]).

Letting $t = \cos(\theta) = y_{n+1}$, we have

$$\frac{\partial}{\partial t}\left\{(1-t^2)^{n/2}\frac{\partial \psi}{\partial t}\right\} + (1-t^2)^{n/2-1}(p-1)\lambda\psi = 0, \tag{2.1}$$

and (2.1) is called a "hyper-sphere" equation. Any solution of (2.1) are expressed by the Legendre associate functions P_ν^μ, Q_ν^μ as

$$\psi = c_1(1-t^2)^{-\mu/2}P_\nu^\mu(t) + c_2(1-t^2)^{-\mu/2}Q_\nu^\mu(t), \tag{2.2}$$

where

$$\mu = \frac{n-2}{2}, \quad \nu = \frac{\sqrt{(n-1)^2 + 4(p-1)\lambda} - 1}{2}. \tag{2.3}$$

The Legendre associate functions P_ν^μ and Q_ν^μ are the independent solutions to the associated Legendre equation

$$\frac{d}{dt}\left\{(1-t^2)\frac{dP}{dt}\right\}+\left\{\nu(\nu+1)-\frac{\mu^2}{1-t^2}\right\}P=0. \tag{2.4}$$

In case of $n=2m-1$, P_ν^μ has a singularity at $t=1$ and hence $c_1=0$ must hold. Moreover, if

$$\lim_{t\to-1}(1-t^2)^{-\mu/2}Q_\nu^\mu(t)$$

is finite, then v corresponds to an eigenfunction and λ does to an eigenvalue to the whole sphere problem. Hence, we have

$$\frac{n-2}{2}+\frac{\sqrt{(n-1)^2+4(p-1)\lambda}-1}{2}=\ell$$

for $\ell=n-2,n-1,\dots$ when $n=2m-1$. Thus, we obtain

$$(p-1)\lambda=(\ell+1)(\ell+2-n). \tag{2.5}$$

On the other hand, in case of $n=2m$, then Q_ν^μ has a singularity at $t=1$ and there must hold $c_2=0$. Similarly, if

$$\lim_{t\to-1}(1-t^2)^{-\mu/2}P_\nu^\mu(t)$$

is finite, then v becomes an eigenfunction. In this case, the eigenvalues λ are expressed as

$$(p-1)\lambda=\left(\ell+\frac{n}{2}\right)\left(\ell+1-\frac{n}{2}\right) \tag{2.6}$$

for $\ell=n/2-1,n/2,\dots$ when $n=2m$. In view of (2.5) and (2.6), for the whole sphere case, we see that the eigenvalue λ_k to (1.5) is expressed as

$$\lambda_k=(k-1)(k+n-2)$$

for $k=1,2,\dots,$ well-known eigenvalues for $-\Lambda$. The case $k=1$ corresponds to the constant eigenfunction. The corresponding eigenfunction $\varphi_k(\theta_1)$ is

$$\varphi_k(\theta_1)=\frac{1}{\sin^{(n-2)/2}\theta_1}Q_{k-1+(n-2)/2}^{(n-2)/2}(\cos\theta_1)$$

when n is odd, and is

$$\varphi_k(\theta_1)=\frac{1}{\sin^{(n-2)/2}\theta_1}P_{k-1+(n-2)/2}^{(n-2)/2}(\cos\theta_1)$$

when n is even.

The eigenvalues $\lambda_{k,\varepsilon}^{D}$ for our problem (2.1) with $\psi(-\cos\varepsilon) = 0$ are determined by

$$Q_{\nu}^{(n-2)/2}(-\cos\varepsilon) = 0 \text{ for } n = 2m-1,$$

and by

$$P_{\nu}^{(n-2)/2}(-\cos\varepsilon) = 0 \quad \text{for } n = 2m,$$

with

$$\nu = \frac{\sqrt{(n-1)^2 + 4(p-1)\lambda_{k,\varepsilon}^{D}} - 1}{2}$$

The eigenfunction corresponding to $\lambda_{k,\varepsilon}^{D}$ is denoted by $\varphi_{k,\varepsilon}^{D}$. More precisely, the eigenfunction is expressed in terms of the Legendre associate functions as

$$\varphi_{k,\varepsilon}^{D} = \begin{cases} \dfrac{1}{\sin^{(n-2)/2}\theta_1} Q^{(n-2)/2}_{\{\sqrt{(n-1)^2+4(p-1)\lambda_{k,\varepsilon}^{D}}-1\}/2}(\cos\theta_1), & \text{for } n = 2m-1, \\ \dfrac{1}{\sin^{(n-2)/2}\theta_1} P^{(n-2)/2}_{\{\sqrt{(n-1)^2+4(p-1)\lambda_{k,\varepsilon}^{D}}-1\}/2}(\cos\theta_1), & \text{for } n = 2m. \end{cases} \tag{2.7}$$

By the continuous dependence on the parameter ν, we see that $\lambda_{k,\varepsilon}^{D}$ is close to λ_k, the eigenvalue of $-\Lambda$ on the whole sphere $\mathbb{S}^n$ if $\varepsilon > 0$ is small enough. So is true for eigenfunctions.

Remark 2.1 *Consider the case of $n = 3$. By (2.3), we see that*

$$\nu = \sqrt{(p-1)\lambda + 1} - \frac{1}{2}$$

and that a solution ψ to (2.1) is written as

$$\begin{aligned} \psi &= c_3(1-t^2)^{-1/4} Q_{\nu}^{1/2}(t) \\ &= \frac{c_4}{\sqrt{\sin\theta_1}} P_{\nu}^{-1/2}(\cos\theta_1) = \frac{c_5 \sin\{\sqrt{(p-1)\lambda + 1}\,\theta_1\}}{\sin\theta_1} \end{aligned}$$

with some constants c_j $(j = 3, 4, 5)$. If follows from $\psi(\pi - \varepsilon) = 0$ that we have the eigenvalues

$$\lambda_{k,\varepsilon}^{D} = \frac{1}{p-1}\left\{ \left(\frac{k\pi}{\pi - \varepsilon}\right)^2 - 1 \right\}.$$

Thus, the solution to (1.4) is explicitly expressed as

$$\varphi^D_{k,\varepsilon}(\theta_1) = \frac{c_6}{\sqrt{\sin\theta_1}} P_\nu^{-1/2}(\cos\theta_1) = \frac{c_7}{\sin\theta_1}\sin\frac{k\pi\theta_1}{\pi-\varepsilon}$$

where c_6 and c_7 are normalizing constants. The convergence of λ^D_{k,θ_1} is readily seen. See also [5] for the three dimensional case.

3 Sketch of Proof of Theorem 1.1

In this section, we describe the key steps to prove Theorem 1.1 and a sketch of a proof of Theorem 1.1. An intuitive explanation is the following. First, we construct an auxiliary function ρ_ε, which looks like a cut-off function having a "boundary layer". Secondly, we determine a solution $w_\varepsilon \in Q_{j,\varepsilon}\mathcal{W}$ to the projected equation

$$Q_{j,\varepsilon}\Big[\Lambda(w_\varepsilon+\rho_\varepsilon) + \lambda\left\{(w_\varepsilon+\rho_\varepsilon)^p_+ - (w_\varepsilon+\rho_\varepsilon)\right\}\Big] = 0. \tag{3.1}$$

Thirdly, we seek for a solution $u = s\varphi^D_{j,\varepsilon} + h(s) + \xi_\varepsilon$ to (1.4) with $\xi_\varepsilon := (w_\varepsilon+\rho_\varepsilon)_+$ and $h(s) \in Q_{j,\varepsilon}\mathcal{W}$. Finally, we investigate the relation between s and $\tau := \lambda - \lambda_{j,\varepsilon}$ in order to see how the local imperfect bifurcation occurs. In this final process, we test (1.4) with $\varphi^D_{j,\varepsilon}$. Full proofs of the following lemmas and Theorem 1.1 are written in Bandle, Kabeya and Ninomiya [4].

We define $\rho_\varepsilon \in C^\infty([0,\pi-\varepsilon])$ as follows:

$$\rho_\varepsilon := \begin{cases} 1, & 0 \le \theta \le \pi - 2\varepsilon, \\ \rho\left(\dfrac{\theta-(\pi-2\varepsilon)}{\varepsilon}\right), & \pi-2\varepsilon \le \theta \le \pi-\varepsilon, \\ 0, & \pi-\varepsilon \le \theta \le \pi, \end{cases}$$

where $\rho(s) \in C^\infty([0,1])$ is a non-increasing function such that

$$\rho(0) = 1, \rho'(0) = \rho''(0) = \rho(1) = \rho'(1) = \rho''(1) = 0.$$

Next, we shall construct the solution w_ε of (3.1) by means of a contraction principle in $Q_{j,\varepsilon}\mathcal{W}$. For this purpose we rewrite equation (3.1) as follows:

$$\begin{aligned} Q_{j,\varepsilon}\Big[&\{\Lambda + \lambda(p-1)I\}(w_\varepsilon+\rho_\varepsilon-1) \\ &+\lambda\Big(\{w_\varepsilon+(\rho_\varepsilon-1)+1\}^p_+ - p\{w_\varepsilon+\rho_\varepsilon-1\} - 1\Big)\Big] = 0. \end{aligned}$$

Since λ is close to $\lambda_{j,\varepsilon}$, the operator $T_{j,\varepsilon} : Q_{j,\varepsilon}\mathcal{W} \to Q_{j,\varepsilon}\mathcal{W}$ given by

$$T_{j,\varepsilon} := -\left[Q_{j,\varepsilon}(\Lambda + \lambda(p-1)I)\right]^{-1}.$$

is well-defined.

Hence w_ε is a solution of the integral equation

$$\begin{aligned} w_\varepsilon &= \lambda T_{j,\varepsilon} Q_{j,\varepsilon}\Big[\{w_\varepsilon + (\rho_\varepsilon - 1) + 1\}_+^p \\ &\quad -p(w_\varepsilon + \rho_\varepsilon - 1) - 1\Big] - Q_{j,\varepsilon}(\rho_\varepsilon - 1). \end{aligned} \tag{3.2}$$

Thus we define $K_{1,\varepsilon}(w_\varepsilon)$ by the right hand of the above equation:

$$K_{1,\varepsilon}(w) := \lambda T_{j,\varepsilon} Q_{j,\varepsilon}\left[\{w + (\rho_\varepsilon - 1) + 1\}_+^p - p(w + \rho_\varepsilon - 1) - 1\right] - Q_{j,\varepsilon}(\rho_\varepsilon - 1).$$

Remark 3.1 Note that $\operatorname{supp}(\rho_\varepsilon - 1) \subset [\pi - 2\varepsilon, \pi - \varepsilon]$. Consequently the term $Q_{j,\varepsilon}(\rho_\varepsilon - 1)$ can be regarded as "small" in the topology of $\mathcal{W}$.

Lemma 3.1 *There exist a positive constant M_1 (independent of ε and λ) and a positive constant ε_* such that $K_{1,\varepsilon}$ is a contraction mapping from*

$$B_{1,\varepsilon} = \left\{U \in Q_{j,\varepsilon}\mathcal{W} \mid \|U\|_{\mathcal{W}} \le M_1 \varepsilon^{(n-q)/q}\right\}$$

into itself for any $\varepsilon \in (0, \varepsilon_*)$ and any $\lambda \in J_j := (\lambda_{j,\varepsilon} - \varepsilon_*, \lambda_{j,\varepsilon} + \varepsilon_*)$. That is, there exists a fixed point w_ε to (3.2) in $B_{1,\varepsilon}$ and $\xi_\varepsilon = w_{j,\varepsilon} + \rho_\varepsilon$ is a solution to (3.1). Moreover, $w_{j,\varepsilon}$ is continuously differentiable in λ and continuous in ε.

Next, we construct $h(s)$ in $Q_{j,\varepsilon}\mathcal{W}$ so that $u = s\varphi_{j,\varepsilon}^D + h(s) + \xi_\varepsilon$ is a solution to (1.1). Substituting $u = s\varphi_{j,\varepsilon}^D + h(s) + \xi_\varepsilon$ to (1.1), we have

$$\begin{aligned} &s\Lambda\varphi_{j,\varepsilon}^D + (\Lambda + (p-1)\lambda)h + \Lambda\xi_\varepsilon \\ &\qquad + \lambda\left\{(\xi_\varepsilon^p - \xi_\varepsilon) + p(\xi_{j,\varepsilon}^{p-1} - 1)(s\varphi_{j,\varepsilon}^D + h) + R(s,\varepsilon;h)\right\} = 0 \end{aligned} \tag{3.3}$$

where

$$R(s,\varepsilon;h) = (s\varphi_{j,\varepsilon}^D + h + \xi)^p - \xi_\varepsilon^p - p\xi_\varepsilon^{p-1}(s\varphi_{j,\varepsilon}^D + h). \tag{3.4}$$

We decompose (3.1) into $P_{j,\varepsilon}\mathcal{W}$-space and $Q_{j,\varepsilon}\mathcal{W}$-space. We will ensure that $h(s)$ exists in $Q_{j,\varepsilon}\mathcal{W}$ for any s near $s = 0$. Since ξ_ε satisfies

$$Q_{j,\varepsilon}\left[\left\{\Lambda\xi_{j,\varepsilon} + \lambda(\xi_{j,\varepsilon}^p - \xi_{j,\varepsilon})\right\}\right] = 0,$$

we see that $h(s)$ satisfies

$$Q_{j,\varepsilon}\left[\{\Lambda+(p-1)\lambda\}h+\lambda\left\{p(\xi_{j,\varepsilon}^{p-1}-1)(s\varphi_{j,\varepsilon}^{D}+h)+R(s,\varepsilon;h)\right\}\right]=0.$$

Again, we will find h by the contraction mapping principle. Let us define

$$K_{2,\varepsilon}(s)[h]:=\lambda T_{j,\varepsilon}Q_{j,\varepsilon}\left[p(\xi_{j,\varepsilon}^{p-1}-1)(s\varphi_{j,\varepsilon}^{D}+h)+R(s,\varepsilon;h)\right]$$

and

$$B_{2,\varepsilon,s}:=\left\{h\in Q_{j,\varepsilon}\mathcal{W}\,|\,\|h\|_{\mathcal{W}}\le M_2(\varepsilon^{(n-q)/p}|s|+s^{\min\{p,2\}})\right\}.$$

Lemma 3.2 *There exist $s^*>0$, $M_2>0$ and $\varepsilon^*>0$ such that for any s and ε ($|s|<s^*$, $0<\varepsilon<\varepsilon^*$), $K_{2,\varepsilon}(s)$ is a contraction map from $B_{2,\varepsilon,s}$ into itself. That is, there exists a fixed point $h(s)=h_{j,\varepsilon}(s)\in Q_{j,\varepsilon}\mathcal{W}$ of $K_{2,\varepsilon}(s)$ satisfying (3.3). Moreover, $h_{j,\varepsilon}(s)$ is continuous in ε and differentiable in s and λ.*

The final step to prove Theorem 1.1 is to take the inner product of $w_\varepsilon=\xi_\varepsilon+s\varphi_{j,\varepsilon}^{D}+h(s)$ with $\varphi_{j,\varepsilon}^{D}$ to determine the relation between s and $\kappa=(p-1)(\lambda-\lambda_j)$ for fixed $\varepsilon>0$. Then we have

$$\begin{aligned}
&H_\varepsilon(s,\lambda)\\
&:=s\int_{\Omega_\varepsilon}(\Lambda\varphi_{j,\varepsilon}^{D})\varphi_{j,\varepsilon}^{D}\,dS+\int_{\Omega_\varepsilon}\left\{(\Lambda+(p-1)\lambda)h\right\}\varphi_{j,\varepsilon}^{D}\,dS\\
&+\int_{\Omega_\varepsilon}\left\{\Lambda\xi_\varepsilon+\lambda(p-1)\xi_\varepsilon\right\}\varphi_{j,\varepsilon}^{D}\,dS\\
&+\lambda\int_{\Omega_\varepsilon}\left\{\xi_\varepsilon^{p}-p(\xi_\varepsilon-1)-1\right\}\varphi_{j,\varepsilon}^{D}\,dS+\lambda s\int_{\Omega_\varepsilon}\left\{p\xi_\varepsilon^{p-1}-1\right\}(\varphi_{j,\varepsilon}^{D})^2\,dS\\
&+\int_{\Omega_\varepsilon}\left\{p(\xi_\varepsilon^{p-1}-1)(s\varphi_{j,\varepsilon}^{D}+h)+\lambda R(s,\varepsilon;h)\right\}\varphi_{j,\varepsilon}^{D}\,dS=0.
\end{aligned}$$

Noting that $h(s)\in Q_{j,\varepsilon}\mathcal{W}$ and $\Lambda h(s)+\lambda(p-1)h\in Q_{j,\varepsilon}\mathcal{W}$, we see that

$$\int_{\Omega_\varepsilon}\{\Lambda h+\lambda(p-1)h\}\,\varphi_{j,\varepsilon}^{D}\,dS=0.$$

Moreover, using the Lemmas above, we obtain the desired relation. □

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