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Author(s): SUGITA, Hiroshi

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Kyoto University
Generalization and randomization of some number-theoretic special functions

Hiroshi SUGITA

Department of Mathematics, Graduate School of Science, Osaka University

1 Introduction

There are many special number-theoretic functions around the Riemann zeta function
\[ \zeta(s) = \sum_{k=1}^{\infty} k^{-s}, \quad \text{Re } s > 1, \]
which are denoted as \( \zeta(s, x) = \sum_{k=1}^{\infty} (k+x)^{-s} \), \( \text{Re } s > 1, \quad x > -1 \), (Hurwitz zeta function)\(^1\)

\[ \Gamma(1+x)^{-1} = \exp \left( \zeta'(0) - \zeta'(0, x) \right) \quad \left[ \zeta'(0, x) := \frac{\partial}{\partial s} \zeta(0, x) \right] \quad (1) \]

\[ \lim_{n \to \infty} n^{x} \prod_{k=1}^{n} \left( 1 + \frac{x}{k} \right) \]

\[ \psi(x+1) = \left( \log \Gamma(x+1) \right)' = \frac{\Gamma'(x+1)}{\Gamma(x+1)} \]

\[ = - \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k + x} - \log n \right) \quad (\text{digamma function}) \quad (2) \]

As we see in the above infinite sum or infinite product formulas, these special functions are related to the sequence of natural numbers \( \{k\}_{k=1}^{\infty} \). In this article, we study what we get when \( \{k\}_{k=1}^{\infty} \) is replaced with other positive increasing sequences, including random sequences.

The most popular method for generalization of number-theoretic special functions is the so-called zeta regularization.

**Definition 1** ([4, 6]) Let a positive sequence \( a = \{a_k\}_{k=1}^{\infty} \) satisfy \( \sum_{k=1}^{\infty} a_k^{-\alpha} < \infty \) for some \( \alpha > 0 \). Then we define the zeta function

\[ z(s) := \sum_{k=1}^{\infty} a_k^{-s}, \]

\(^1\)Slightly different from the traditional definition.

\(^2\)This notation will be used for any functions of two variables in this article.
which is holomorphic in $\text{Re } s > \alpha$. If $z(s)$ is analytically continued to a meromorphic function which is holomorphic at $s = 0$, $a$ is said to be zeta regularizable. Then we write

$$z \prod_{k=1}^{\infty} a_k := \exp(-z'(0))$$

and call it the zeta regularized product of $\prod_{k=1}^{\infty} a_k$.

But, for our purpose, this notion is too strong, indeed, it is quite unlikely that random sequences become zeta regularizable. We therefore assume a rather mild condition (Assumption 1 below) which random sequences can satisfy.

This work is somewhat an experimental one. We are not sure that it is a promising research. However, we think that some of results, such as Example 2, Theorem 6, Theorem 7, and their extensions in § 4.1 are fully interesting by themselves.

2 Deterministic generalization

2.1 Zeta regularized product

In this article, we consider real sequences which satisfy the following condition.

**Assumption 1** (i) $a = \{a_k\}_{k=1}^{\infty}$ is a positive non-decreasing sequence diverging to $\infty$.

(ii) $a$ is uniformly distributed in the half line $(0, \infty)$ with the same density as $N$ in the following sense: Setting

$$F(x) := \#\{k \in N; a_k \leq x\},$$

there exists some $\delta > 0$ such that

$$F(x)x^{-1} = 1 + O(x^{-\delta}), \quad x \to \infty.$$  \hfill (3)

**Remark 1** As we will see later, Assumption 1 alone does not assure $a = \{a_k\}_{k=1}^{\infty}$ to be zeta regularizable.

Throughout this section § 2 (except Remark 4), we consider everything under Assumption 1.

**Lemma 1** For any $\epsilon > 0$, we have $\sum_{k=1}^{\infty} a_k^{1-\epsilon} < \infty$.

**Proof.** Since $k \leq F(a_k)$, we see that $ka_k^{-1} \leq F(a_k)a_k^{-1} \to 1$ as $k \to \infty$, which implies\(^3\) $\lim \sup_{k \to \infty} ka_k^{-1} \leq 1$. From this, the assertion of the lemma easily follows. Q.E.D.

**Lemma 2** The following limit exists:

$$\lim_{z \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right] = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} a_k^{-1} - \log n \right] =: q. \hfill (4)$$

\(^3\)In fact, we have $\lim_{k \to \infty} ka_k^{-1} = 1$ ([5]).
Proof. Take $0 < \epsilon < a_1$, and note that $F(\epsilon) = 0$. By integration by parts formula,
\[
\int_{\epsilon}^{x} (F(t)t^{-1} - 1)t^{-1}dt = \int_{\epsilon}^{x} F(t)t^{-2}dt - \int_{\epsilon}^{x} t^{-1}dt
\]
\[
= -F(x)x^{-1} + F(\epsilon)\epsilon^{-1} + \int_{\epsilon}^{x} t^{-1}dt - (\log x - \log \epsilon)
\]
\[
= -F(x)x^{-1} + \left( \sum_{a_k \leq x} a_k^{-1} - \log x \right) + \log \epsilon.
\]
Since Assumption 1 implies $\int_{\epsilon}^{\infty} |F(t)t^{-1} - 1|t^{-1}dt < \infty$ and that $\lim_{x \to \infty} F(x)x^{-1} = 1$, the term $\lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right]$ of the last right-hand side of the above also has a limit as $x \to \infty$. We thus have
\[
\int_{\epsilon}^{\infty} (F(t)t^{-1} - 1)t^{-1}dt = -1 + \log \epsilon + \lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right]. \tag{5}
\]
Since we have
\[
\lim_{x \to \infty} \inf \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right] \leq \lim_{n \to \infty} \inf \left[ \sum_{k=1}^{n} a_k^{-1} - \log n \right]
\]
\[
\leq \lim_{n \to \infty} \sup \left[ \sum_{k=1}^{n} a_k^{-1} - \log n \right]
\]
\[
\leq \lim_{x \to \infty} \sup \left[ \sum_{a_k \leq x} a_k^{-1} - \log F(x) \right],
\]
and since (3) implies
\[
\lim_{x \to \infty} \inf \left[ \sum_{a_k \leq x} a_k^{-1} - \log F(x) \right] = \lim_{x \to \infty} \sup \left[ \sum_{a_k \leq x} a_k^{-1} - \log F(x) \right]
\]
\[
= \lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right],
\]
we know (4) is valid. Q.E.D.

Proposition 1 (cf. [4] Theorem 2) $z(s)$ is analytically continued to a meromorphic function in $\Re s > 1 - \delta$ with a unique single pole at $s = 1$, whose residue is 1. In addition, the `finite part' of $z(s)$ at the pole is equal to $q$, i.e.,
\[
\lim_{s \to 1} \left[ z(s) - \frac{1}{s-1} \right] = q. \tag{6}
\]
Proof. Let $\sigma := \Re s > 1$ and let $0 < \epsilon < a_1$. By integration by parts,
\[
\sum_{a_n \leq x} a_n^{-s} = \int_{\epsilon}^{x} t^{-s}dF(t) = F(x)x^{-s} + s \int_{\epsilon}^{x} F(t)t^{-s-1}dt
\]
\[
= O(x^{1-\sigma}) + s \int_{\epsilon}^{x} (F(t) - t)t^{-s-1}dt + s \int_{\epsilon}^{x} t^{-s}dt
\]
\[
= O(x^{1-\sigma}) + s \int_{\epsilon}^{x} (F(t) - t)t^{-s-1}dt + \frac{sx^{-s+1}}{s-1} - \frac{sx^{-s+1}}{s-1}.
\]
Letting \(x \to \infty\), we have
\[
    z(s) = \frac{se^{-s+1}}{s-1} + s \int_{\epsilon}^{\infty} (F(t) - t)t^{-s-1} dt
    = \frac{1}{s-1} + \frac{se^{-s+1} - 1}{s-1} + s \int_{\epsilon}^{\infty} (F(t)t^{-1} - 1)t^{-s} dt.
\]

This expression and Assumption 1 implies that \(z(s)\) is analytically continued to a meromorphic function in \(\Re s > 1 - \delta\) with a unique single pole at \(s = 1\), whose residue is 1. Moreover
\[
    \lim_{s \to 1} \left[ z(s) - \frac{1}{s-1} \right] = 1 - \log \epsilon + \int_{\epsilon}^{\infty} (F(t)t^{-1} - 1)t^{-1} dt.
\]

Then (5) shows that
\[
    \lim_{s \to 1} \left[ z(s) - \frac{1}{s-1} \right] = \lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right] = q.
\]

Q.E.D.

It is easy to see that the corresponding Hurwitz zeta function
\[
    z(s, x) := \sum_{k=1}^{\infty} \frac{1}{(a_k + x)^{s}}, \quad x > -a_1,
\]
is also analytically continued to a meromorphic function in \(\Re s > 1 - \delta\) with a unique single pole at \(s = 1\), whose residue is 1 (cf. [4] Theorem 1).

However, in general, \(z(s)\) and \(z(s, x)\) do not necessarily become holomorphic at \(s = 0\). Indeed, for the existence of \(z'(0)\), the integral \(\int_{\epsilon}^{\infty} (F(t) - t)t^{-1} dt\) should be convergent, which Assumption 1 does not assure. Nevertheless their difference becomes holomorphic at \(s = 0\).

**Proposition 2** For each \(x > -a_1\), the difference function \(g(s, x) := z(s) - z(s, x)\) is analytically continued to a holomorphic function in \(\Re s > -\delta\).

**Proof.** Since Proposition 1 implies that \(sz(s+1)\) is holomorphic in \(\Re s > -\delta\), it is enough to show that
\[
    h(s) := g(s, x) - sz(s+1)x
\]
is holomorphic in \(\Re s > -\delta\).

First, \(h(s)\) is expressed in the following series in \(\Re s > 1\).
\[
    h(s) = \sum_{k=1}^{\infty} a_k^{-s} - \sum_{k=1}^{\infty} (a_k + x)^{-s} - s \sum_{k=1}^{\infty} a_k^{-s-1} x.
\]

Suppose \(|x| < a_{k_0}\). Then applying the Taylor expansion (negative binomial theorem)
\[
    (a_k + x)^{-s} = a_k^{-s} \sum_{j=0}^{\infty} \binom{s + j - 1}{j} \left(\frac{-x}{a_k}\right)^j
    = a_k^{-s} + \lambda s a_k^{-s-1} + a_k^{-s} \sum_{j=2}^{\infty} \binom{s + j - 1}{j} \left(\frac{-x}{a_k}\right)^j, \quad k \geq k_0,
\]

(7)
which converges absolutely, we see

$$h(s) = -s \sum_{k=1}^{k_0-1} (a_k + x)^{-s} - s \sum_{k=k_0}^{\infty} a_k^{s} \sum_{j=2}^{\infty} \frac{(s+1)(s+2)\cdots(s+j-1)}{j!} \left(\frac{-x}{a_1}\right)^j \left(\frac{a_1}{a_k}\right)^j.$$

(8)

Since

$$\sum_{k=1}^{\infty} a_k^{s} \sum_{j=2}^{\infty} \frac{(s+1)(s+2)\cdots(s+j-1)}{j!} \left(\frac{-x}{a_1}\right)^j \left(\frac{a_1}{a_k}\right)^j \leq \sum_{k=k_0}^{\infty} a_k^{-\Re s} \sum_{j=2}^{\infty} \left|\frac{(s+1)(s+2)\cdots(s+j-1)}{j!}\right| \left(\frac{-x}{a_1}\right)^j a_1^{2} \sum_{k=k_0}^{\infty} a_k^{-\Re s-2} \sum_{j=2}^{\infty} \left|\frac{(s+1)(s+2)\cdots(s+j-1)}{j!}\right| \left(\frac{-x}{a_1}\right)^j$$

is finite in $\Re s > -1$ by Lemma 1, $h(s)$ becomes holomorphic in $\Re s > -1$. Q.E.D.

**Definition 2** We define the zeta regularized product of $\prod_{k=1}^{\infty} (1 + \frac{x}{a_k})$ by

$$z \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) := \exp \left(g'(0, x)\right).$$

(9)

**Remark 2** If $a = \{a_k\}_{k=1}^{\infty}$ is zeta regularizable, we have

$$z \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) = \frac{z - \prod_{k=1}^{\infty} (a_k + x)}{z - \prod_{k=1}^{\infty} a_k} = \exp \left(z'(0) - z'(0, x)\right).$$

2.2 Generalized Wallis formula

**Proposition 3** (Weierstrass' infinite product formula, [4] Theorem 2, [6])

$$z \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) = e^{qx} \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) \exp \left(-\frac{x}{a_k}\right).$$

**Proof.** Noting $\lim_{s \to 0} sz(s + 1) = 1$, we first calculate $h'(0)$.

$$h'(0) = g'(0, x) - \lim_{s \to 0} s \frac{z(s+1) - 1}{x} \lambda$$

$$= g'(0, x) - \lim_{s \to 0} \left[ z(s+1) - \frac{1}{s} x \right]$$

$$= g'(0, x) - qx \quad (\text{cf. } (6)).$$

(10)

On the other hand, (8) implies $h(0) = 0$ and so that $h'(0) = \lim_{s \to 0} h(s)/s$. Therefore

$$h'(0) = - \sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \frac{(j-1)!}{j!} \left(\frac{-x}{a_k}\right)^j = \sum_{k=1}^{\infty} \left[ - \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{-x}{a_k}\right)^j - \frac{x}{a_k} \right]$$

$$= \sum_{k=1}^{\infty} \left[ \log \left(1 + \frac{x}{a_k}\right) - \frac{x}{a_k}\right].$$
This and (10) imply that
\[ g'(0, x) = qx + \sum_{k=1}^{\infty} \left[ \log \left( 1 + \frac{x}{a_k} \right) - \frac{x}{a_k} \right]. \quad (11) \]

Plugging this into the exponential function, we finally obtain
\[ \zeta \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) = e^{qx} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) e^{-\frac{x}{a_k}}. \]

Q.E.D.

**Theorem 1 (Generalized Wallis formula)**
\[ \zeta \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) = \lim_{n \to \infty} n^{-x} \prod_{k=1}^{n} \left( 1 + \frac{x}{a_k} \right). \quad (12) \]

**Remark 3** For the special case where \( a_k = k, k = 1, 2, \ldots, \) and \( x = -1/2, \) we have
\[ \zeta \prod_{k=1}^{\infty} \left( 1 - \frac{1}{2k} \right) = \Gamma(1/2)^{-1} = \pi^{-1/2}, \]
\[ n^{1/2} \prod_{k=1}^{n} \left( 1 - \frac{1}{2k} \right) = \pi^{1/2} \left( \frac{2n}{n} \right)^{2-2n}. \]

So (12) implies now the classical Wallis formula.

**Proof of Theorem 1.** From (4) and Proposition 3, it follows that
\[ \zeta \prod_{k=1}^{\infty} \left( 1 + \frac{\lambda}{a_k} \right) = \lim_{n \to \infty} \exp \left( a_1^{-1} + a_2^{-1} + \cdots + a_n^{-1} - \log n \right) \prod_{k=1}^{n} \left( 1 + \frac{x}{a_k} \right) e^{-\frac{x}{a_k}} \]
\[ = \lim_{n \to \infty} n^{-x} \prod_{k=1}^{n} \left( 1 + \frac{x}{a_k} \right). \]

Q.E.D.

By definition, \( \zeta \prod \left( 1 + \frac{x}{a_k} \right) \) is neither 0 nor infinite. Consequently, Proposition 3 and Theorem 1 have substantial meaning.

**Example 1** The square of the classical Wallis formula is in fact a zeta regularized product:
\[ \pi^{-1} = \zeta \prod_{k=1}^{\infty} \left( 1 - \frac{1}{2k} \right)^2 = \zeta \prod_{k=1}^{\infty} \left( 1 - \frac{1}{a_k} \right), \]
where \( \frac{1}{a_k} = \frac{1}{k} - \frac{1}{4k^2} \) or
\[ a_k = \frac{1}{\frac{1}{k} - \frac{1}{4k^2}} = k + \frac{1}{4} + \frac{1}{4(4k-1)}, \quad k = 1, 2, \ldots, \]
which satisfies Assumption 1. Then let us show that
\[ q = \lim_{s \to 1} \left( x(s) - \frac{1}{s-1} \right) = \gamma - \frac{\pi^2}{24}. \]
Since
\[
\pi^{-1} = \lim_{n \to \infty} n \prod_{k=1}^{n} \left(1 - \frac{1}{2k}\right)^2
= \prod_{k=1}^{\infty} \left(1 - \frac{1}{2k}\right)^2 \exp \left(\frac{1}{k} - \frac{1}{4k^2}\right) \cdot \lim_{n \to \infty} n \prod_{k=1}^{n} \exp \left(-\frac{1}{k} + \frac{1}{4k^2}\right),
\]
we must have
\[
e^{-q} = \lim_{n \to \infty} n \prod_{k=1}^{n} \exp \left(-\frac{1}{k} + \frac{1}{4k^2}\right),
\]
namely,
\[
q = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{4k^2}\right) - \log n \right] = \gamma - \sum_{k=1}^{\infty} \frac{1}{4k^2} = \gamma - \frac{\pi^2}{24}.
\]

**Remark 4** In case \(\sum_{k=1}^{\infty} a_k^{-1} < \infty\), \(z(s)\) becomes finite at \(s = 1\), so that its 'finite part' \(q\) at \(s = 1\) is, of course, \(\sum_{k=1}^{\infty} a_k^{-1}\). Then it holds that \(z(1) = \sum_{k=1}^{\infty} \frac{x}{a_k} = \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right)\). Let us show it.

(i) For a finite sequence \(a = \{a_k\}_{k=1}^{N}\),
\[
z(s) := \sum_{k=1}^{N} a_k^{-s}, \quad z(s) := \sum_{k=1}^{N} (a_k + x)^{-s}, \quad 0 \leq \lambda < a_1,
\]
which are entire functions, it is easy to see that \(\exp(z'(0) - z'(0, x)) = \prod_{k=1}^{N} \left(1 + \frac{x}{a_k}\right)\).

(ii) For an infinite sequence \(a = \{a_k\}_{k=1}^{\infty}\) such that \(\sum_{k=1}^{\infty} a_k^{-1} < \infty\),
\[
z(s) := \sum_{k=1}^{\infty} a_k^{-s}, \quad z(s) := \sum_{k=1}^{\infty} (a_k + x)^{-s}, \quad 0 \leq \lambda < a_1,
\]
are finite at \(s = 1\), but we do not know whether they are analytically continued beyond \(\text{Re}\ s > 1\). Nevertheless their difference \(g(s, x) := z(s) - z(s, x)\) is analytically continued to a holomorphic function in \(\text{Re}\ s > -1\), which is shown in a similar way as Proposition 3. Indeed, by (7),
\[
g(s, x) = -\sum_{k=1}^{\infty} a_k^{-s} \sum_{j=1}^{\infty} \frac{(s+j-1)!}{j!} \left(\frac{-x}{a_k}\right)^{j},
\]
from which it follows that
\[
g'(0, x) = -\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{(j-1)!}{j!} \left(\frac{-x}{a_k}\right)^{j} = -\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{-x}{a_k}\right)^{j} = \sum_{k=1}^{\infty} \log \left(1 + \frac{x}{a_k}\right).
\]
Thus
\[
\exp(g'(0, x)) = \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right).
\]
2.3 Generalized digamma function

If \( a = \{a_k\}_{k=1}^{\infty} \) satisfies Assumption 1, so does \( \{a_k + x\}_{k=1}^{\infty} \) for each \( x > 0 \), and hence we can define

\[
q(x) := \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{x + a_k} - \log n \right), \quad x > 0.
\]

Comparing with (2), we can say that \(-q(x)\) is a generalized digamma function.

Suppose that unit electric charges are located at each point of \( \{a_k\}_{k=1}^{\infty} \) on the real line \( \mathbb{R} \). Then \( q(x) \) can be regarded as the renormalized Coulomb potential at \(-x\) caused by those electric charges. Indeed, we see

\[
q(x) = \frac{d}{dx} \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{x + a_k} - \log n \right)
= \frac{d}{dx} \sum_{k=1}^{\infty} \left( \frac{1}{x + a_k} - \log \frac{k+1}{k} \right)
= -\sum_{k=1}^{\infty} \frac{1}{(x + a_k)^2} = -\psi(2, x).
\]

By (11), we have

\[
q(x) = -\frac{d}{ds} (\zeta(s, x) - \zeta(s, x-1)) \bigg|_{s=0} + \sum_{k=1}^{\infty} \left[ \frac{1}{x + a_k} + \log \left( 1 - \frac{1}{x + a_k} \right) \right]. \tag{13}
\]

Applying this formula to the sequence \( \{a_k = k\}_{k=1}^{\infty} \), we have

\[
-\psi(x+1) = -\log x + \sum_{k=1}^{\infty} \left[ \frac{1}{x + k} + \log \left( 1 - \frac{1}{x + k} \right) \right], \quad (x > 0), \tag{14}
\]

because

\[
\frac{d}{ds} (\zeta(s, x) - \zeta(s, x-1)) \bigg|_{s=0} = \frac{d}{ds} (-x^{-s}) \bigg|_{s=0} = \log x.
\]

**Theorem 2** For any sequence \( a = \{a_k\}_{k=1}^{\infty} \) satisfying Assumption 1 and \( a_k^k^{-1} = 1 + O(k^{-\delta'}), k \to \infty, \quad \delta' > 0 \), we have

\[
q(x) = -\log x + O(x^{-1}), \quad x \to \infty.
\]

**Proof.** From (14) it follows that

\[
-\psi(x+1) = -\log x + O(x^{-\min(1,\delta')}), \quad x \to \infty.
\]

On the other hand, for \( x > 0 \), we have

\[
q(x) + \psi(x+1) = \sum_{k=1}^{\infty} \left( \frac{1}{x + k} - \frac{1}{x + a_k} \right)
= \sum_{k=1}^{\infty} \frac{a_k - k}{(x + k)(x + a_k)}
= \sum_{k=1}^{\infty} \frac{O(k^{1-\delta})}{(x + k)(x + a_k)} = O(x^{-\delta'}), \quad x \to \infty.
\]
2.4 Generalized Gamma functions

The following lemma is easily derived from Theorem 1.

Lemma 3 For each $n \in \mathbb{N}$,

$$z^{-\sum_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right)} = \prod_{k=1}^{n} \left(1 + \frac{x}{a_k}\right) z^{-\sum_{k=n+1}^{\infty} \left(1 + \frac{x}{a_k}\right)}.$$

Now, recalling $z^{-\sum_{k=1}^{\infty} \left(1 + \frac{x}{k}\right)} = \Gamma(1+x)^{-1}$, Lemma 3 implies

$$\Gamma(n+1+x) = \Gamma(1+x) \prod_{k=1}^{n} (k+x) = \frac{\prod_{k=1}^{n} (k+x)}{z^{-\sum_{k=n+1}^{\infty} \left(1 + \frac{x}{k}\right)}}.$$

Therefore

$$\Gamma(x) = \frac{n!}{z^{-\sum_{k=n+1}^{\infty} \left(1 + \frac{x-n-1}{k}\right)}}. \quad (15)$$

We consider an analogy of this.

Definition 3 For each $n \in \mathbb{N}$, we define

$$G^{(n+1)}(x) := \frac{\prod_{k=1}^{n} a_k}{z^{-\sum_{k=n+1}^{\infty} \left(1 + \frac{x-a_{n+1}}{a_k}\right)}}. \quad (16)$$

Obviously, we have

$$G^{(n+1)}(a_{n+1}) = \prod_{k=1}^{n} a_k \quad (17)$$

$$z^{-\sum_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right)} = \frac{\prod_{k=1}^{n} (a_k + x)}{G^{(n+1)}(a_{n+1} + x)}, \quad n = 1, 2, \ldots \quad (18)$$

By (15), when $a_k = k$ for each $k \in \mathbb{N}$, $G^{(n+1)}(x) = \Gamma(x)$ holds for any $n \in \mathbb{N}$. In general, for $a = \{a_k\}_{k=1}^{\infty}$ satisfying the following assumption, the corresponding $G^{(n+1)}$ has a Gamma function-like property.
**Assumption 2** There exists some $\alpha > 0$ such that $a_{k+1} - a_k = 1 + O(k^{-\alpha})$, $k \to \infty$.

**Theorem 3** If $a = \{a_k\}_{k=1}^\infty$ satisfies Assumption 1 and Assumption 2, it holds for any $j \in \mathbb{N}$ that

$$G^{(n+1)}(a_{n+1-j}) \sim \prod_{k=1}^{n-j} a_k, \quad n \to \infty. \quad (19)$$

Here "$\sim$" indicates that the ratio of the both hand sides tends to 1 in the specified limit.

**Proof.** For $j < n$,

$$G^{(n+1)}(a_{n+1-j}) = \prod_{k=1}^{n} a_k \overset{z-\infty}{\prod_{k=n+1}^{\infty}} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)$$

$$= \prod_{k=n+1-j}^{n} a_k^{-1} z-\infty \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right).$$

Therefore it is sufficient to show that

$$\lim_{n \to \infty} \prod_{k=n+1-j}^{n} a_k^{-1} z-\infty \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) = 1. \quad (20)$$

By Proposition 3, we have

$$z-\infty \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)$$

$$= \exp \left(-q_{n+1}(a_{n+1} - a_{n+1-j})\right) \times \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \exp \left(\frac{a_{n+1} - a_{n+1-j}}{a_k}\right),$$

where

$$q_{n+1} := \lim_{N \to \infty} \left[ \sum_{k=n+1}^{N} a_k^{-1} - \log(N - n + 1) \right]$$

$$= q - \sum_{k=1}^{n} a_k^{-1} = -\log n + o(1), \quad n \to \infty.$$

Then Assumption 2 implies that

$$\exp \left(-q_{n+1}(a_{n+1} - a_{n+1-j})\right) = n^{j-O(n^{-\alpha})} e^{o(1)(-j+O(n^{-\alpha}))} \sim n^j, \quad n \to \infty.$$

The following is obvious.

$$\prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \exp \left(\frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \to 1, \quad n \to \infty.$$
From these, it follows that
\[
\prod_{n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \sim n^j, \quad n \to \infty.
\]

And hence
\[
\lim_{n \to \infty} \prod_{k=n+1-j}^{n} a_k \sim \prod_{n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) = \lim_{n \to \infty} \left(\prod_{k=0}^{j-1} \frac{1}{a_{n-k}} \right) = 1.
\]

Q.E.D.

If \( a = \{a_k\}_{k=1}^{\infty} \) satisfies Assumption 1 and Assumption 2, the expression (18) and Theorem 3 can be used for numerical evaluation of \( \prod_{k=1}^{\infty} (1 + \frac{x}{a_k}) \) in some cases. The method is as follows: First, for a suitably large \( n \) and \( j_0 < n \), construct a Lagrange's polynomial \( h_a^{(n,j_0)}(x) \) of degree \( (j_0 - 1) \) that interpolates the points 
\[
(x_j, y_j) = \left(a_{n+1-j}, \sum_{k=1}^{n-j} \log a_k\right), \quad j = 0, 1, \ldots, j_0 - 1.
\]

Substituting \( h_a^{(n,j_0)}(x) \) for \( \log G^{(n+1)}(x) \) in (18), we calculate
\[
c^{(n,n',j_0)}(x) := \frac{\prod_{k=1}^{n'} (a_k + x)}{\exp \left(h_a^{(n,j_0)}(a_{n'} + x)\right)}
\]
as an approximated value of \( \prod_{k=1}^{\infty} (1 + \frac{x}{a_k}) \). (In doing this, to prevent overflow or underflow, all calculations should be done by taking logarithm, i.e., we calculate
\[
\sum_{k=1}^{n'-1} \log(a_k + x) - h_a^{(n,j_0)}(a_{n'} + x)
\]
then plug the result into the exponential function.) Here \( n' \) is a suitable integer between \( n-j_0 \) and \( n \). Probably, it is better to pick up \( n' \) from the middle of the interval \([n+1-j_0, n]\).

Example 2 Let us consider the square of the Wallis formula again. The sequence dealt in Example 1, i.e., \( a_k = k + \frac{1}{4} + \frac{1}{4(4k-1)} \) satisfies Assumption 2 for \( \alpha = 2 \), so that we can apply the above method to get an approximated value of \( \prod_{k=1}^{\infty} (1 - \frac{1}{a_k}) \).

For \( n = 30, 300, 3000 \), we constructed Lagrange polynomials \( h_a^{(n,5)} \), and calculated \( c^{(n,n-2,5)}(1) \), which are listed in the table below. Since the true value is
\[
1/\pi = 1/3.14159265 \ldots,
\]
roughly speaking, the error decreases at the rate of \( O(n^{-2}) \).

For comparison, we also calculated \( v(n) := n \prod_{k=1}^{n} (1 - \frac{1}{a_k}) \) as approximated values due to the Wallis formula. This time, the error decreases at the rate of \( O(n^{-1}) \).
In this way, $c^{(n,n-2,5)}(1)$ is much better than $w(n)$. But this example may be a special case, and since we have not established a precise error estimate, we do not know if our method is valid for general cases.

3 Randomized special functions

By randomizing the objects in the previous sections, we can find a new type of limit theorems in probability theory.

3.1 In the case of Poisson process

Let $\{\xi_i\}_{i=0}^\infty$ be a positive i.i.d. random variables whose common distribution is the exponential distribution with parameter 1, i.e.,

$$P(\xi_i \leq x) = \int_0^x e^{-t} \, dt = 1 - e^{-x}, \quad x \geq 0,$$

and set

$$X = \{X_k\}_{k=1}^\infty := \{\xi_1 + \cdots + \xi_k\}_{k=1}^\infty.$$

Then by virtue of the strong law of large numbers, the sequence $\{b + X_k\}_{k=1}^\infty$, $b > 0$, satisfies Assumption 1 almost surely. Note that

$$\eta(t) := F_X(t) = \#\{k | X_k \leq t\}, \quad t \geq 0,$$

is a standard Poisson process.

3.1.1 Randomization of the Wallis formula

First let us calculate the distribution of the following random variable.

$$W(b, \lambda) := z \prod_{k=1}^\infty \left(1 + \frac{\lambda}{b + X_k}\right), \quad \lambda \geq -b, \, b > 0.$$

**Theorem 4** The $n$-th moment of $z(b, \lambda)$ is calculated as follows.\(^5\)

$$E[W(b, \lambda)^n] = \begin{cases} b^{-\lambda}, & n = 1, \\ b^{-n\lambda} \exp \left( \sum_{r=2}^{n} \binom{n}{r} \frac{\lambda^r}{(r-1)b^{r-1}} \right), & n = 2, 3, \ldots \end{cases}$$

**Lemma 4** (Durrett[1], (5.1) Theorem, Chapt.3.) Under the conditional probability measure $P(\cdot | \eta(t) = N)$, $t > 0$, the distribution of $\{X_k\}_{k=1}^N$ coincides with that of the order statistics of $N$ independent uniformly distributed random variables in $[0, t]$.

\(^4\)P stands for probability.

\(^5\)E stands for expectation.
Sketch of Proof of Theorem 4.

Step 1. Let \( \{X_{t,k}\}_{k=1}^{\infty} \) be independent uniformly distributed random variables in \([0,t]\).

First, we define a random variable

\[
W_{t,N,\lambda} := \prod_{k=1}^{N} \left( 1 + \frac{\lambda}{b + X_{t,k}} \right), \quad N \in \mathbb{N}.
\]

and calculate its moments.

\[
\mathbb{E}[(W_{t,N,\lambda})^{n}] = \prod_{k=1}^{N} \mathbb{E} \left[ \left( 1 + \frac{\lambda}{b + X_{t,k}} \right)^{n} \right] = \left( \int_{0}^{t} \left( 1 + \frac{\lambda}{b + y} \right)^{n} \frac{dy}{t} \right)^{N} = \left( \sum_{r=0}^{n} \binom{n}{r} \lambda^{r} \frac{1}{t} \int_{0}^{t} \frac{(b + y)^{-r}}{(b + y)^{-r}} dy \right)^{N} = \left( 1 + \frac{n\lambda}{t} \left( \log(b + t) - \log(b) + C \right) + O(t^{-2}) \right)^{N} = \left( 1 + \frac{n\lambda}{t} \left( \log(b + t) - \log(b) \right) + O(t^{-2}) \right)^{N}, \quad t \to \infty,
\]

where

\[
C := \sum_{r=2}^{n} \binom{n}{r} \frac{\lambda^{r}}{(r-1)b^{r-1}}.
\]

Step 2. Since \( P(\eta(t) = N) = t^{N}e^{-t}/N! \), Lemma 4 implies that

\[
\mathbb{E} \left[ \eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left( 1 + \frac{\lambda}{1 + X_{k}} \right)^{n} \right] = \sum_{N=0}^{\infty} \mathbb{E} \left[ N^{-n\lambda} \prod_{k=1}^{N} \left( 1 + \frac{\lambda}{1 + X_{k}} \right)^{n} ; \eta(t) = N \right] = \sum_{N=0}^{\infty} \mathbb{E} \left[ N^{-n\lambda} \prod_{k=1}^{N} \left( 1 + \frac{\lambda}{1 + X_{0}} \right)^{n} \right] \frac{t^{N}e^{-t}}{N!} = \sum_{N=0}^{\infty} N^{-n\lambda} \mathbb{E} \left[ (W_{t,N,\lambda})^{n} \right] \frac{t^{N}e^{-t}}{N!}.
\]

A change of variables \( u := N/t \) shows

\[
\mathbb{E} \left[ \eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left( 1 + \frac{\lambda}{1 + X_{k}} \right)^{n} \right] = \sum_{u \in \mathbb{N}} (tu)^{-n\lambda} \mathbb{E} \left[ ((W_{t,u,\lambda})^{n})^{\frac{tu}{t}}e^{-tu} \right].
\]

Since the distribution of \( \eta(t)/t \) is convergent to the Dirac measure \( \delta_{1}(du) \), we see

\[
\mathbb{E}[W(b, \lambda)^{n}] = \mathbb{E} \left[ \lim_{t \to \infty} \eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left( 1 + \frac{\lambda}{1 + X_{k}} \right)^{n} \right].
\]
\[
\lim_{t \to \infty} \mathbb{E}\left[ \eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left(1 + \frac{\lambda}{1 + X_k}\right)^n \right]
= \lim_{t \to \infty} \sum_{u \in (tN)} (tu)^{-n\lambda} \mathbb{E}\left[ (W_{t, tu, \lambda})^n \right] \frac{tu^t e^{-t}}{(tu)!} u \in \frac{\sum_{1}}{t} N
= \lim_{t \to \infty} t^{-n\lambda} \mathbb{E}\left[ (W_{t, t, \lambda})^n \right] = b^{-n\lambda} e^C.
\]

The above calculation can be rigorously justified. Q.E.D.

In case \( b = 1 \), if \( |\lambda| \ll 1 \), then the \( n \)-th moments of \( W(1, \lambda) \) is approximately equal to \( \exp\left(\frac{n(n-1)}{2}\lambda^2\right) \), which is nothing but the \( n \)-th moment of a log normal distribution, more precisely, the distribution of the random variable \( e^Y \) where \( Y \) is distributed as \( \mathcal{N}\left(-\frac{\lambda^2}{2}, \lambda^2\right) \). Hence when \( |\lambda| \ll 1 \), the distribution of \( W(1, \lambda) \) is close to that of \( e^Y \).

3.1.2 Random digamma function

Proposition 3 implies that

\[
\log W(1, \lambda) = Q\lambda + \sum_{k=1}^{\infty} \left[ \log \left(1 + \frac{\lambda}{1 + X_k}\right) - \frac{\lambda}{1 + X_k}\right],
\]

where

\[
Q = \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \frac{1}{1 + X_k} - \log N \right].
\]

This limit exists a.s.

Now, we have

\[
\mathbb{E}[Q] = 0, \quad \mathbb{E}[Q^2] = 1,
\]

which is shown in the following way. First, it is easy to see that

\[
\sum_{k=1}^{\infty} \left[ \log \left(1 + \frac{\lambda}{1 + X_k}\right) - \frac{\lambda}{1 + X_k}\right] = O(\lambda^2), \quad \lambda \to 0.
\]

Hence we see

\[
\mathbb{E}[\log W(1, \lambda)] = -\mathbb{E}[Q]\lambda + O(\lambda^2), \quad \lambda \to 0.
\]

On the other hand, since the mean of \( \log W(1, \lambda) \) is approximately equal to \(-\lambda^2/2\), we see \( \mathbb{E}[Q] = 0 \). And the 2-nd moment of \( \log W(1, \lambda) \) is approximately equal to \( \lambda^2 \), so we see \( \mathbb{E}[Q^2] = 1 \).

Suppose \( |\lambda| \ll 1 \). Since \(-Q\lambda\) is the main part of \( \log W(1, \lambda) \), and \( \log W(1, \lambda) \) is approximately distributed as \( \mathcal{N}\left(-\frac{\lambda^2}{2}, \lambda^2\right) \), one may well expect that \( Q \) is distributed as \( \mathcal{N}(0, 1) \). But although its distribution is close to \( \mathcal{N}(0, 1) \), it is not exactly distribute as \( \mathcal{N}(0, 1) \). In deed we have \( \mathbb{E}[Q^2] = 1/2 \).

Let us investigate a little bit more general case. Let

\[
Q(x) := \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \frac{1}{x + X_k} - \log N \right], \quad x > 0.
\]

Comparing with (2), we can say that \(-Q(x)\) is a random digamma function.

---

This may hold in any case where \( \{X_k\}_{k=1}^{\infty} \) is the partial sum of positive i.i.d. random variables with mean 1.
Theorem 5  The mean of $Q(x)$ is $E[Q(x)] = -\log x$, and the centered moments of $Q(x)$ are

$$E[(Q(x) + \log x)^n] = n! \sum_{p=1}^{\lfloor \sqrt{2n} \rfloor} \sum_{2 \leq n_1 \cdots < n_p \atop k_1 n_1 + \cdots + k_p n_p = n} \frac{1}{x^{n-(k_1+\cdots+k_p)}} \times \prod_{j=1}^{p} \frac{1}{k_j!(n_j!)^{k_j}(n_j-1)^{k_j}}, \quad n = 2, 3, \ldots \quad (21)$$

More concretely, the centered moments of order 1, 2, . . . , 6 are

$$0, \quad \frac{1}{x}, \quad \frac{1}{2x^2}, \quad \frac{3}{x^2} + \frac{1}{3x^3}, \quad \frac{5}{x^3} + \frac{1}{4x^4}, \quad \frac{15}{x^3} + \frac{15}{2x^4} + \frac{1}{5x^5}. \quad$$

Sketch of Proof.  As in the previous section, let $\{X_{t,k}\}_{k=1}^{N}$ be i.i.d. uniform random variables in $[0, t]$. Define

$$Y_{t,N} := \sum_{k=1}^{N} \left( \frac{1}{x+X_{t,k}} - c(t) \right), \quad c(t) = \frac{1}{t}(\log(x+t) - \log x).$$

Then, when $t \to \infty$, we have

$$E \left[ \left( \frac{1}{x+X_{t,k}} - c(t) \right)^n \right] = \begin{cases} 0 & (n = 1) \\ \frac{1}{x^{n-1}(n-1)t} + O(t^{-2+\varepsilon}) & (n \geq 2) \end{cases} \quad (22)$$

Here $\varepsilon > 0$ can be arbitrarily small. Indeed, if we write the L.H.S. of the above expression by integrals,

$$= \int_0^t \left( \frac{1}{x+y} - c(t) \right)^n \frac{dy}{t}$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-c(t))^{n-r} \frac{1}{r} \int_0^t (x+y)^{-r} \frac{dy}{t}$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-c(t))^{n-r} \times \begin{cases} 0 & (r = 0) \\ \frac{1}{c(t)} & (r = 1) \\ \left( \frac{1}{(r-1)t} \right) \left( x^{-r+1} - (x+t)^{-r+1} \right) & (r \geq 2) \end{cases}$$

$$= c(t)^n - nc(t)^n + \sum_{r=2}^{n} \binom{n}{r} (-c(t))^{n-r} \frac{1}{(r-1)t} \left( \frac{1}{x^{r-1}} - \frac{1}{(x+t)^{r-1}} \right).$$

Here we have $c(t) = O(t^{-1+\varepsilon})$, $t \to \infty$, and hence we obtain (22).

Under these preparations, we see

$$E \left[ Y_{t,N}^n \right]$$

$$= \sum_{n_1+n_2+\cdots+n_N=n} E \left[ \prod_{k=1}^{N} \left( \frac{1}{x+X_{t,k}} - c(t) \right)^{n_k} \right]$$

$$= \sum_{n_1+n_2+\cdots+n_N=n} \prod_{k=1}^{N} E \left[ \left( \frac{1}{x+X_{t,k}} - c(t) \right)^{n_k} \right]$$
\[ = \sum_{p=1}^{N} \frac{N!}{(N-p)!} \sum_{\begin{array}{c} 2 \leq n_1 < \cdots < n_p \vspace{1pt} \\ k_1 n_1 + k_2 n_2 + \cdots + k_p n_p = n \end{array}} \frac{1}{k_1! k_2! \cdots k_p!} \cdot \frac{n!}{(n_1!)^{k_1} (n_2!)^{k_2} \cdots (n_p!)^{k_p}} \times \prod_{j=1}^{p} E \left[ \left( \frac{1}{x + X_{t,j}} - c(t) \right)^{n_j} \right]^{k_j} \]

\[ = \sum_{p=1}^{N} \sum_{\begin{array}{c} 2 \leq n_1 < \cdots < n_p \vspace{1pt} \\ k_1 n_1 + k_2 n_2 + \cdots + k_p n_p = n \end{array}} \frac{1}{k_1! k_2! \cdots k_p!} \cdot \frac{n!}{(n_1!)^{k_1} (n_2!)^{k_2} \cdots (n_p!)^{k_p}} \times \prod_{j=1}^{p} \left( \frac{1}{x^{n_j-1} (n_j-1) t} + O(t^{-2+\epsilon}) \right)^{k_j} \]

From this, it follows similarly to the previous theorem that

\[ E[(Q(x) + \log x)^n] = E \left[ \lim_{t \to \infty} \left( \sum_{k=1}^{\eta(t)} \left( \frac{1}{x + X_k} - c(t) \right) \right)^n \right] \]

\[ \lim_{t \to \infty} E \left[ \left( \sum_{k=1}^{\eta(t)} \left( \frac{1}{x + X_k} - c(t) \right) \right)^n \right] \]

\[ \lim_{t \to \infty} \sum_{N=0}^{\infty} E \left[ Y_{t,N}^{n} \right] \frac{t^N e^{-t}}{N!} \]

\[ \lim_{t \to \infty} \sum_{u \in T \cap \mathbb{N}} E \left[ Y_{t,u}^{n} \right] \frac{tt u e^{-t}}{(tu)!} \left( = \lim_{t \to \infty} E \left[ Y_{t,u}^{n} \right] \right) \]

\[ = \sum_{p=1}^{n} \sum_{\begin{array}{c} 2 \leq n_1 < \cdots < n_p \vspace{1pt} \\ k_1 n_1 + k_2 n_2 + \cdots + k_p n_p = n \end{array}} \frac{1}{x^{n-(k_1+k_2+\cdots+k_p)}} \times \frac{n!}{k_1 k_2 \cdots k_p} \cdot \frac{1}{(n_1!)^{k_1} (n_2!)^{k_2} \cdots (n_p!)^{k_p}} \prod_{j=1}^{p} \left( \frac{1}{n_j - 1} + O(t^{-1+\epsilon}) \right)^{k_j}. \]

Now it is easy to get (21). Q.E.D.

The following theorem is an easy consequence of the law of iterated logarithm and Theorem 2.

**Theorem 6** For any \( \varepsilon > 0 \), \( Q(x) = -\log x + O(x^{-(1/2)+\varepsilon}) \), \( x \to \infty \), a.s.
Regarding this theorem as a law of large numbers, the following corresponds to the central limit theorem.

**Theorem 7** The distribution of $\sqrt{x}(Q(x) + \log x)$ converges to $\mathcal{N}(0, 1)$ as $x \to \infty$.

**Proof.** The $n$-th moment of $\sqrt{x}(Q(x) + \log x)$ is

$$
\mathbb{E} \left[ (\sqrt{x}(Q(x) + \log x))^n \right] = n! \sum_{p=1}^{\lfloor \sqrt{2n} \rfloor} \sum_{k_1 + \cdots + k_p = n} \frac{1}{x^{n/2 - (k_1 + \cdots + k_p)}} \prod_{j=1}^{p} \frac{1}{k_j! (n_j!)^{k_j}}.
$$

In the sum $\sum_{p=1}^{\lfloor \sqrt{2n} \rfloor}$ of the R.H.S., the sum for $p$ such that $n/2 > k_1 + \cdots + k_p$ converge to 0 as $x \to \infty$. Therefore as $x \to \infty$, what survive are the terms for $p$ such that $n/2 = k_1 + \cdots + k_p$ (from this $n$ must be even), that is, for $p = 1, k_1 = n/2, n_1 = 2$. Hence we see that

$$
\lim_{x \to \infty} \mathbb{E} \left[ (\sqrt{x}(Q(x) + \log x))^n \right] = \frac{n!}{2^{n/2} (n/2)!}, \quad (n: \text{ even}).
$$

This is nothing but the $n$-th moment of $\mathcal{N}(0, 1)$. Q.E.D.

Suppose that unit electric charges are located at each random point of $\{X_k\}_{k=1}^\infty$, the renormalized Coulomb potential $Q(x)$ at the location $-x$ is distributed approximately as $\mathcal{N}(-\log x, 1/x)$ if $x$ is large.

**Another proof of Theorem 7.** Since the distribution function $F_{\{x+X_k\}}(t)$ corresponding to the sequence $\{x+X_k\}_{k=1}^\infty$ is exactly $\eta(t - x)$, (5) implies that

$$
Q(x) = -\log x + \int_x^\infty (\eta(t - x) - t) \frac{dt}{t^2} = -\log x + \int_0^\infty (\eta(t) - t) \frac{dt}{(t + x)^2}.
$$

Since $\mathbb{E}[\eta(t) - t] = 0$, we readily see $\mathbb{E}[Q(x)] = -\log x$. From the following expression

$$
\sqrt{x} (Q(x) + \log x) = \int_0^\infty (\eta(t) - t) \frac{\sqrt{x}}{(t + x)^2} dt,
$$

let us derive Theorem 7 by using the Lindeberg - Feller theorem ([2] Chapt.2 (4.5) Theorem).

First, note that $\tilde{\eta}(t) := \eta(t) - t$ is a martingale with mean 0. The Fubini theorem (or integration by parts formula) implies

$$
\int_s^T \tilde{\eta}(t) \frac{\sqrt{x}}{(t + x)^2} dt = \int_s^T \left( \tilde{\eta}(S) + \int_s^t d\tilde{\eta}(s) \right) \frac{\sqrt{x}}{(t + x)^2} dt
$$

$$
= \int_s^T \left( \int_s^T \frac{\sqrt{x}}{(t + x)^2} dt \right) d\tilde{\eta}(s) + \tilde{\eta}(S) \int_s^T \frac{\sqrt{x}}{(t + x)^2} dt
$$

$$
= \int_s^T \frac{\sqrt{x}}{s + x} - \tilde{\eta}(s) + \frac{\sqrt{x}}{s + x} - \frac{\sqrt{x}}{T + x} d\tilde{\eta}(s) + \left( \frac{\sqrt{x}}{S + x} - \frac{\sqrt{x}}{T + x} \right) \tilde{\eta}(S)
$$

$$
= \int_s^T \frac{\sqrt{x}}{s + x} d\tilde{\eta}(s) - \frac{\sqrt{x}}{T + x} \tilde{\eta}(T) + \frac{\sqrt{x}}{S + x} \tilde{\eta}(S).
$$

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Letting $S \to 0$, $T \to \infty$, we have the following expression.

\[
\sqrt{x} (Q(x) + \log x) = \int_0^\infty \tilde{\eta}(t) \frac{\sqrt{x}}{(t + x)^2} dt = \int_0^\infty \frac{\sqrt{x}}{t + x} d\tilde{\eta}(t), \quad \text{a.s.}
\]

Now put

\[
\left\{ \begin{array}{ll}
U_{n,m} := \int_{(m-1)^2}^{m^2} \frac{\sqrt{n}}{t+n} d\tilde{\eta}(t), & m = 1, 2, \ldots, n-1, \\
U_{n,n} := \int_{(n-1)^2}^\infty \frac{\sqrt{n}}{t+n} d\tilde{\eta}(t).
\end{array} \right.
\]

Let us the triangular array \( \{U_{n,m}\}_{1 \leq m \leq n, 1 \leq n} \) satisfies the Lindeberg-Feller's conditions.

**Step 1.** Since \( \{\eta(t)\}_{t \geq 0} \) is an independent increment process, \( \{U_{n,m}\}_{m=1}^{n} \) is independent sequence for each \( n \).

**Step 2.** It holds (without letting \( n \to \infty \)) that

\[
\sum_{m=1}^{n} \mathbb{E} [U_{n,m}^2] = \sum_{m=1}^{n-1} \int_{(m-1)^2}^{m^2} \left( \frac{\sqrt{n}}{t+n} \right)^2 dt + \int_{(n-1)^2}^{\infty} \left( \frac{\sqrt{n}}{t+n} \right)^2 dt
\]

\[
= \int_{0}^{\infty} \frac{n}{(t+n)^2} dt = 1.
\]

**Step 3.** Now, to prove the theorem, it is sufficient to show the last Lindeberg-Feller's condition: for any \( \varepsilon > 0 \) we have

\[
\lim_{n \to \infty} \sum_{m=1}^{n} \mathbb{E} [U_{n,m}^2; |U_{n,m}| > \varepsilon] = 0. \quad (24)
\]

**Lemma 5** If a random variable \( U \) has the 4-th moment, then

\[
\mathbb{E} [U^2; |U| > \varepsilon] \leq \varepsilon^{-2} \mathbb{E}[U^4].
\]

**Proof.**

\[
\mathbb{E} [U^2; |U| > \varepsilon] \leq \mathbb{E} \left[ U^2 \cdot \frac{U^2}{\varepsilon^2}; |U| > \varepsilon \right] \leq \frac{1}{\varepsilon^2} \mathbb{E} [U^4].
\]

Q.E.D.

Since

\[
\mathbb{E}[U_{n,n}^2] = \int_{(n-1)^2}^{\infty} \left( \frac{\sqrt{n}}{t+n} \right)^2 dt = \frac{n}{(n-1)^2 + n} \to 0, \quad n \to \infty,
\]

it is sufficient to prove

\[
\lim_{n \to \infty} \sum_{m=1}^{n-1} \mathbb{E}[U_{n,m}^4] = 0 \quad (25)
\]

to show (24) by Lemma 5.

Let us estimate each of \( \mathbb{E}[U_{n,m}^4] \). Putting

\[
U_m(t) := \int_{(m-1)^2}^{t} \frac{\sqrt{n}}{s+n} d\tilde{\eta}(s), \quad m \leq t,
\]
and applying the Itô formula, we have

\[
E[U_{n,m}^4] = E \left[ \int_{(m-1)^2}^{m^2} \left( \left( U_m(t) + \frac{\sqrt{n}}{t+n} \right)^4 - U_m(t)^4 - 4U_m(t)^3 \frac{\sqrt{n}}{t+n} \right) dt \right]
\]

\[
= \int_{(m-1)^2}^{m^2} \left( 6E[U_m(t)^2] \left( \frac{\sqrt{n}}{t+n} \right)^2 + \left( \frac{\sqrt{n}}{t+n} \right)^4 \right) dt
\]

\[
= \int_{(m-1)^2}^{m^2} \left( 6 \left( \frac{n}{(m-1)^2 + n} - \frac{n}{t+n} \right) \frac{n}{(t+n)^2} + \frac{n^2}{(t+n)^4} \right) dt
\]

\[
= n^2 \int_{(m-1)^2}^{m^2} \left( 6 \cdot \frac{2m-1}{((m-1)^2 + n)^4} + \frac{1}{((m-1)^2 + n)^4} \right) dt
\]

\[
= 6n^2 \cdot \frac{(2m-1)^2}{((m-1)^2 + n)^4} + n^2 \cdot \frac{2m-1}{((m-1)^2 + n)^4}
\]

\[
= n^2 \cdot \frac{6(2m-1)^2 + 2m-1}{((m-1)^2 + n)^4}.
\]

From this, we derive

\[
\sum_{m=1}^{n-1} E[U_{n,m}^4] < n^2 \sum_{m=1}^{n-1} \frac{6(2m-1)^2 + 2m-1}{((m-1)^2 + n)^4} = O(n^{-1/2}), \quad n \to \infty,
\]

thus (25) holds. Now the proof of Theorem 7 is complete. Q.E.D.

3.2 In the case of random walk

We next consider the case where \(\{\xi_i\}_{i=1}^\infty\) is a Bernoulli sequence with \(P(\xi_i = 0) = P(\xi_i = 2) = 1/2\), and \(X_n := \sum_{i=1}^n \xi_i\). Again by the strong law of large numbers, \(\{x + X_k\}_{k=1}^\infty\), \(x > 0\), satisfies Assumption 1 almost surely.

3.2.1 Random digamma function

Defining \(G_k\) by

\[
G_k := \#\{ n \in \mathbb{N} ; X_n = 2k \}, \quad k = 1, 2, \ldots
\]

\(\{G_k\}_{k=1}^\infty\) is an i.i.d. sequence with a geometric distribution \(P(G_k = n) = 2^{-n}, n \in \mathbb{N}\), and we have

\[
\sum_{k=1}^{G_1+\cdots+G_N} \frac{1}{x + X_k} = \sum_{k=0}^{N} \frac{G_k}{x + 2k}.
\]

Let us first look at the law of large numbers. Since \(E[G_k] = 2\), for sufficiently large \(N\),

\[
\log(G_1 + G_2 + \cdots + G_N) = \log \left( \frac{1}{N} \sum_{k=1}^{N} G_k \right) + \log N \approx \log 2 + \log N.
\]

\(^7\)For this estimate, use \(n^2 \int_0^\infty x^2/(x^2 + n)^4 dx = \pi/(32\sqrt{n})\).
Therefore, with probability 1,
\[ Q(x) = \lim_{N \to \infty} \left[ \sum_{k=0}^{N} \frac{G_k}{x + 2k} - (\log 2 + \log N) \right] \]
converges. Theorem 2 implies that with probability 1,
\[ Q(x) = -\log x + O(x^{-1}), \quad x \to \infty. \]

The mean of \( Q(x) \) is computed as follows.

\[
\mathbb{E}[Q(x)] = \lim_{N \to \infty} \mathbb{E} \left[ \sum_{k=0}^{N} \frac{G_k}{x + 2k} - (\log 2 + \log N) \right] = \lim_{N \to \infty} \left[ \sum_{k=0}^{N} \frac{2}{x + 2k} - (\log 2 + \log N) \right] = \lim_{N \to \infty} \left[ \sum_{k=0}^{N} \frac{1}{(x/2) + k} - (\log 2 + \log N) \right] = -\psi((x/2) + 1) - \log 2 = -\log x + \sum_{k=0}^{\infty} \left[ \frac{1}{(x/2) + k} + \log \left( 1 - \frac{1}{(x/2) + k} \right) \right].
\]

Next, let us look at the central limit theorem. We put
\[ Q_N(x) := \sum_{k=0}^{N} \frac{G_k}{x + 2k} - (\log 2 + \log N), \]
and calculate its characteristic function (Fourier transform). Noting that
\[ \mathbb{E}[\exp(itG_k)] = \frac{\frac{1}{2} e^{it}}{1 - \frac{1}{2} e^{it}}, \]
we have
\[
\mathbb{E} \left[ e^{itQ_N(x)} \right] = \prod_{k=0}^{N} \mathbb{E} \left[ \exp \left( it \cdot \frac{G_k}{x + 2k} \right) \right] \exp \left( -it(\log 2 + \log N) \right) = \prod_{k=0}^{N} \left[ \frac{\frac{1}{2} \exp \left( it \cdot \frac{1}{x+2k} \right)}{1 - \frac{1}{2} \exp \left( it \cdot \frac{1}{x+2k} \right)} \right] \exp \left( -it(\log 2 + \log N) \right) = \prod_{k=0}^{N} \left[ \frac{\frac{1}{2} \exp \left( -it \cdot \frac{1}{x+2k} \right)}{1 - \frac{1}{2} \exp \left( it \cdot \frac{1}{x+2k} \right)} \right] \exp \left[ -it \left( \sum_{k=0}^{N} \frac{2}{x + 2k} + \log 2 + \log N \right) \right].
\]
Thus
\[
\mathbb{E} \left[ e^{itQ(x)} \right] = \prod_{k=0}^{\infty} \left[ \frac{\exp \left( -it \cdot \frac{1}{x+2k} \right)}{2 - \exp \left( it \cdot \frac{1}{x+2k} \right)} \right] \exp \left[ -it \left( -\psi((x/2) + 1) + \log 2 \right) \right].
\]
Let us take the limit \( x \to \infty \) of the infinite product
\[
\prod_{k=0}^{\infty} \left[ \frac{\exp(-it\cdot\frac{1}{x+2k})}{2 - \exp(it\cdot\frac{1}{x+2k})} \right] = \prod_{k=0}^{\infty} \left[ \frac{1}{2\exp(it\cdot\frac{1}{x+2k}) - \exp(it\cdot\frac{2}{x+2k})} \right].
\]
Developing the denominator, we see
\[
2\exp(it\cdot\frac{1}{x+2k}) - \exp(it\cdot\frac{2}{x+2k}) = 2\left(1 + \frac{it}{x+2k} - \frac{1}{2} \cdot \frac{t^2}{(x+2k)^2} + \cdots\right) - \left(1 + \frac{2it}{x+2k} - \frac{1}{2} \cdot \frac{4t^2}{(x+2k)^2} + \cdots\right)
\]
and hence
\[
\prod_{k=0}^{\infty} \left[ \frac{\exp(-it\cdot\frac{1}{x+2k})}{2 - \exp(it\cdot\frac{1}{x+2k})} \right] = \prod_{k=0}^{\infty} \left(1 + \frac{t^2}{(x+2k)^2} + \cdots\right)^{-1} = \prod_{k=0}^{\infty} \exp\left(-\frac{t^2}{2} \cdot \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{((x/2)+k)^2}\right), \quad x \to \infty.
\]
From the above, when \( x \gg 1 \), the distribution of \( Q(x) \) is close to the normal distribution with mean \(-\psi((x/2) + 1) + \log 2\) and variance \( \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{((x/2)+k)^2} \). Since we have
\[
-\psi((x/2) + 1) + \log 2 = -\log x + O(x^{-1}),
\]
and
\[
\frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{((x/2)+k)^2} = x^{-1} + O(x^{-2}),
\]
as \( x \to \infty \), in the long run, we proved the following convergence in distribution.
\[
\sqrt{x}(Q(x) + \log x) \rightarrow \mathcal{N}(1,0), \quad x \to \infty.
\]
That is, the assertion of Theorem 7 holds in this case, too.

4 Further discussions

4.1 Extension of Theorem 6 and Theorem 7

Recently, Theorem 6 and Theorem 7 have been much more generalized by S.Takanobu.

**Theorem 8** ([5]) Let \( \{\xi_i\}_{i=1}^{\infty} \) be an i.i.d. sequence with \( \xi_i > 0 \), \( \mathbb{E}[\xi_i] = 1 \), and \( \mathbb{E}[\xi_i^\beta] < \infty \) for some \( \beta > 1 \). Then we have \( Q(x) = -\log x + O(x^{-1}) \), \( x \to \infty \), a.s.

**Theorem 9** ([5]) Let \( \{\xi_i\}_{i=1}^{\infty} \) be an i.i.d. sequence with \( \xi_i > 0 \), \( \mathbb{E}[\xi_i] = 1 \), and \( v := \mathbb{V}[\xi_i^2] < \infty \). Then, for \( X = \{X_k\}_{k=1}^{\infty} = \{\xi_1 + \cdots + \xi_k\}_{k=1}^{\infty} \), it holds that the distribution of \( \sqrt{x}(Q_X(x) + \log x) \) converges to \( \mathcal{N}(0,v) \) as \( x \to \infty \).
For i.i.d. sequences \( \{ \xi_i \}_{i=1}^{\infty} \) with \( \xi_i > 0, \, \mathbb{E}[\xi_i] = 1 \), but \( \mathbb{V}[\xi_i^2] = \infty \), we have the following limit theorem.

**Theorem 10** ([5]) (i) If \( [0, \infty) \ni s \mapsto \mathbb{E}[\xi^2_1; \xi_1 \leq s] \in [0, \infty) \) is slowly varying at \( \infty \), there exists a positive sequence \( \{ B_n \}_{n=1}^{\infty} \) such that

\[
\frac{x}{B[x]} (Q_X(x) + \log x) \rightarrow \mathcal{N}(0,1), \quad x \rightarrow \infty, \quad \text{in distribution.}
\]

(ii) If there exist a \( \beta \in (1,2) \) and an \( L(\cdot) \) which is slowly varying at \( \infty \) such that

\[
P(\xi_1 > x) \sim L(x)x^{-\beta}, \quad x \rightarrow \infty,
\]

then there exists \( \{ B_n \}_{n=1}^{\infty} \) such that

\[
\lim_{x \rightarrow \infty} \mathbb{E} \left[ \exp \left( \sqrt{-1}t(\beta-1)^{1/\beta} \frac{x}{B[x]} (Q_X(x) + \log x) \right) \right] = \exp \left( \beta \int_{0}^{\infty} \left( e^{\sqrt{-1}by} - 1 - \sqrt{-1}by \right) \frac{dy}{y^{\beta+1}} \right).
\]

These results with proofs will be written in a paper in near future.

### 4.2 The case of two dimensional random array of electric charges

We mentioned about the electro-static interpretation of random digamma function in § 3.1.2. In this context, a natural question arises: Suppose that unit electrical charges are located at random in an unbounded domain of \( \mathbb{R}^2 \). Then, can we define a renormalized Coulomb potential as a random variable?

**Example 3** Suppose that the distribution of the unit electrical charges are described by a Poisson random measure on the outside of centered circle \( B(O,x)^c \) with the Lebesgue measure as the intensity. Then the Coulomb potential at \( O \) will be expressed as

\[
\int_{x}^{\infty} \frac{dN(\pi t^2)}{t} = \sqrt{\pi} \int_{x^2}^{\infty} \frac{dN(t)}{\sqrt{t}}
\]

by a standard Poisson process \( N(t) \), which is of course divergent. The renormalized potential would be

\[
\sqrt{\pi} \int_{x^2}^{\infty} \frac{d\tilde{N}(t)}{\sqrt{t}}, \quad \tilde{N}(t) := N(t) - t,
\]

but it is not well-defined because

\[
\mathbb{E} \left[ \left( \int_{x^2}^{\infty} \frac{d\tilde{N}(t)}{\sqrt{t}} \right)^2 \right] = \int_{x^2}^{\infty} \frac{dt}{t} = \infty.
\]

To look at the situation closely, let us observe the following deterministic case. The sequence \( a = \{ \sqrt{k} \}_{k=1}^{\infty} \) is zeta regularizable, because the corresponding zeta function is \( z(s) = \zeta(s/2) \). Hence by Theorem 2 in [4] and Theorem 1.8 in [3], we have

\[
z \prod_{k=1}^{\infty} \left( 1 + \frac{x}{\sqrt{k}} \right) = \exp \left( \zeta \left( \frac{1}{2} \right) x - \frac{\gamma x^2}{2} \right) \prod_{k=1}^{\infty} \left( 1 + \frac{x}{\sqrt{k}} \right) \exp \left( -\frac{x}{\sqrt{k}} + \frac{x^2}{2k} \right),
\]
where
\[ \zeta\left(\frac{1}{2}\right) = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n} \right] = -1.46035\ldots \]

For \( X = \{X_k\}_{k=1}^{\infty} := \{\xi_1 + \cdots + \xi_k\}_{k=1}^{\infty} \), partial sums of i.i.d. random variables, Assumption 1 is satisfied with \( \delta < 1/2 \), and then the corresponding zeta function
\[ Z(s) = \sum_{k=1}^{\infty} X_k^{-s} \]
will become meromorphic in \( \text{Re} \ s > 1/2 \), but \( Z(1/2) \) may not be defined. This fact has something to do with the non-existence of the limit
\[ \int_{x^2}^{\infty} \frac{d\tilde{N}(t)}{\sqrt{t}} = \lim_{y \to \infty} \left[ \sum_{x^2 \leq X_k \leq y} \frac{1}{\sqrt{X_k}} - 2\sqrt{y} + 2x \right] . \]

References


