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Generalization and randomization of some number-theoretic special functions

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1 Introduction

There are many special number-theoretic functions around the Riemann zeta function \( \zeta(s) = \sum_{k=1}^{\infty} k^{-s} \), Re \( s > 1 \), such as

\[
\zeta(s, x) = \sum_{k=1}^{\infty} (k+x)^{-s}, \quad \text{Re } s > 1, \quad x > -1, \quad \text{(Hurwitz zeta function),}^1
\]

\[
\Gamma(1+x)^{-1} = \exp \left( \zeta'(0) - \zeta'(0, x) \right) \quad \left[ \zeta'(0, x) := \frac{\partial}{\partial s} \zeta(0, x) \right]^{2}
\]

\[
= e^{\gamma x} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{k} \right) e^{-x/k} \quad \left[ \text{where } \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \right]
\]

\[
\psi(x+1) = (\log \Gamma(x+1))' = \frac{\Gamma'(x+1)}{\Gamma(x+1)}
\]

\[
= - \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k+x} - \log n \right) \quad \text{(digamma function)} \quad (2)
\]

As we see in the above infinite sum or infinite product formulas, these special functions are related to the sequence of natural numbers \( \{ k \}_{k=1}^{\infty} \). In this article, we study what we get when \( \{ k \}_{k=1}^{\infty} \) is replaced with other positive increasing sequences, including random sequences.

The most popular method for generalization of number-theoretic special functions is the so-called zeta regularization.

Definition 1 ([4, 6]) Let a positive sequence \( a = \{ a_k \}_{k=1}^{\infty} \) satisfy \( \sum_{k=1}^{\infty} a_k^{-\alpha} < \infty \) for some \( \alpha > 0 \). Then we define the zeta function

\[
z(s) := \sum_{k=1}^{\infty} a_k^{-s},
\]

\(^2\)Slightly different from the traditional definition.

\(^2\)This notation will be used for any functions of two variables in this article.
which is holomorphic in $\text{Re } s > \alpha$. If $z(s)$ is analytically continued to a meromorphic function which is holomorphic at $s = 0$, $a$ is said to be zeta regularizable. Then we write

$$z \prod_{k=1}^{\infty} a_k := \exp(-z'(0))$$

and call it the zeta regularized product of $\prod_{k=1}^{\infty} a_k$.

But, for our purpose, this notion is too strong, indeed, it is quite unlikely that random sequences become zeta regularizable. We therefore assume a rather mild condition (Assumption 1 below) which random sequences can satisfy.

This work is somewhat an experimental one. We are not sure that it is a promising research. However, we think that some of results, such as Example 2, Theorem 6, Theorem 7, and their extensions in § 4.1 are fully interesting by themselves.

2 Deterministic generalization

2.1 Zeta regularized product

In this article, we consider real sequences which satisfy the following condition.

Assumption 1 (i) $a = \{a_k\}_{k=1}^{\infty}$ is a positive non-decreasing sequence diverging to $\infty$.
(ii) $a$ is uniformly distributed in the half line $(0, \infty)$ with the same density as $N$ in the following sense: Setting

$$F(x) := \#\{k \in N; a_k \leq x\},$$

there exists some $\delta > 0$ such that

$$F(x)x^{-1} = 1 + O(x^{-\delta}), \ x \to \infty.$$

(3)

Remark 1 As we will see later, Assumption 1 alone does not assure $a = \{a_k\}_{k=1}^{\infty}$ to be zeta regularizable.

Throughout this section § 2 (except Remark 4), we consider everything under Assumption 1.

Lemma 1 For any $\epsilon > 0$, we have $\sum_{k=1}^{\infty} a_k^{-1-\epsilon} < \infty$.

Proof. Since $k \leq F(a_k)$, we see that $ka_k^{-1} \leq F(a_k)a_k^{-1} \to 1$ as $k \to \infty$, which implies\(^3\) $\limsup_{k \to \infty} ka_k^{-1} \leq 1$. From this, the assertion of the lemma easily follows. Q.E.D.

Lemma 2 The following limit exists:

$$\lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right] = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} a_k^{-1} - \log n \right] =: q.$$  (4)

\(^3\)In fact, we have $\lim_{k \to \infty} ka_k^{-1} = 1$ ([5]).
Proof. Take $0 < \epsilon < a_1$, and note that $F(\epsilon) = 0$. By integration by parts formula,

$$\int_{\epsilon}^{x} (F(t) t^{-1} - 1) t^{-1} dt = \int_{\epsilon}^{x} F(t) t^{-2} dt - \int_{\epsilon}^{x} t^{-1} dt$$

\[= -F(x) x^{-1} + F(\epsilon) \epsilon^{-1} + \int_{\epsilon}^{x} t^{-1} dF(t) - (\log x - \log \epsilon)\]

Since Assumption 1 implies $\int_{\epsilon}^{\infty} |F(t) t^{-1} - 1| t^{-1} dt < \infty$ and that $\lim_{x \to \infty} F(x) x^{-1} = 1$, the term $\lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right]$ of the last right-hand side of the above also has a limit as $x \to \infty$. We thus have

$$\int_{\epsilon}^{\infty} (F(t) t^{-1} - 1) t^{-1} dt = -1 + \log \epsilon + \lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right]. \quad (5)$$

Since we have

$$\lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log F(x) \right] \leq \lim_{n \to \infty} \left[ \sum_{k=1}^{n} a_k^{-1} - \log n \right] \leq \limsup_{n \to \infty} \left[ \sum_{k=1}^{n} a_k^{-1} - \log n \right] \leq \lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log F(x) \right],$$

and since (3) implies

$$\lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log F(x) \right] = \limsup_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log F(x) \right] = \lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right],$$

we know (4) is valid. Q.E.D.

Proposition 1 (cf. [4] Theorem 2) $z(s)$ is analytically continued to a meromorphic function in $\Re s > 1 - \delta$ with a unique single pole at $s = 1$, whose residue is 1. In addition, the 'finite part' of $z(s)$ at the pole is equal to $q$, i.e.,

$$\lim_{s \to 1} \left[ z(s) - \frac{1}{s - 1} \right] = q. \quad (6)$$

Proof. Let $\sigma := \Re s > 1$ and let $0 < \epsilon < a_1$. By integration by parts,

$$\sum_{a_n \leq x} a_n^{-s} = \int_{\epsilon}^{x} t^{-s} dF(t) = F(x) x^{-s} + s \int_{\epsilon}^{x} F(t) t^{-s-1} dt$$

\[= O(x^{1-\sigma}) + s \int_{\epsilon}^{x} (F(t) - t) t^{-s-1} dt + s \int_{\epsilon}^{x} t^{-s} dt\]

\[= O(x^{1-\sigma}) + s \int_{\epsilon}^{x} (F(t) - t) t^{-s-1} dt + \frac{s \epsilon^{-s+1}}{s - 1} - \frac{s x^{-s+1}}{s - 1}.\]
Letting $x \to \infty$, we have

$$z(s) = \frac{s\epsilon^{-s+1}}{s-1} + s\int_{\epsilon}^{\infty}(F(t) - t)t^{-s-1}dt$$

This expression and Assumption 1 implies that $z(s)$ is analytically continued to a meromorphic function in $\text{Re} \ s > 1 - \delta$ with a unique single pole at $s = 1$, whose residue is 1. Moreover

$$\lim_{s \to 1} \left[ z(s) - \frac{1}{s-1} \right] = 1 - \log \epsilon + \int_{\epsilon}^{\infty}(F(t)t^{-1} - 1)t^{-1}dt.$$ 

Then (5) shows that

$$\lim_{s \to 1} \left[ z(s) - \frac{1}{s-1} \right] = \lim_{x \to \infty} \left[ \sum_{a_k \leq x} a_k^{-1} - \log x \right] = q.$$ 

Q.E.D.

It is easy to see that the corresponding Hurwitz zeta function

$$z(s, x) := \sum_{k=1}^{\infty} (a_k + x)^{-s}, \quad x > -a_1,$$

is also analytically continued to a meromorphic function in $\text{Re} \ s > 1 - \delta$ with a unique single pole at $s = 1$, whose residue is 1 (cf. [4] Theorem 1).

However, in general, $z(s)$ and $z(s, x)$ do not necessarily become holomorphic at $s = 0$. Indeed, for the existence of $z'(0)$, the integral $\int_{\epsilon}^{\infty}(F(t) - t)t^{-1}dt$ should be convergent, which Assumption 1 does not assure. Nevertheless their difference becomes holomorphic at $s = 0$.

**Proposition 2** For each $x > -a_1$, the difference function $g(s, x) := z(s) - z(s, x)$ is analytically continued to a holomorphic function in $\text{Re} \ s > -\delta$.

**Proof.** Since Proposition 1 implies that $sz(s+1)$ is holomorphic in $\text{Re} \ s > -\delta$, it is enough to show that

$$h(s) := g(s, x) - sz(s+1)x$$

is holomorphic in $\text{Re} \ s > -\delta$.

First, $h(s)$ is expressed in the following series in $\text{Re} \ s > 1$.

$$h(s) = \sum_{k=1}^{\infty} a_k^{-s} - \sum_{k=1}^{\infty} (a_k + x)^{-s} - s \sum_{k=1}^{\infty} a_k^{-s-1}x.$$ 

Suppose $|x| < a_{k_0}$. Then applying the Taylor expansion (negative binomial theorem)

$$(a_k + x)^{-s} = a_k^{-s} \sum_{j=0}^{\infty} \binom{s + j - 1}{j} \left( \frac{-x}{a_k} \right)^j$$

$$= a_k^{-s} + \lambda s a_k^{-s-1} + a_k^{-s} \sum_{j=2}^{\infty} \binom{s + j - 1}{j} \left( \frac{-x}{a_k} \right)^j, \quad k \geq k_0, \quad (7)$$
which converges absolutely, we see
\[ h(s) = -s \sum_{k=1}^{k_0-1} (a_k + x)^{-s} - s \sum_{k=k_0}^{\infty} a_k^{-s} \sum_{j=2}^{\infty} \frac{(s+1)(s+2) \cdots (s+j-1)}{j!} \left( \frac{-x}{a_1} \right)^j \left( \frac{a_1}{a_k} \right)^j. \]  
(8)

Since
\[
\sum_{k=k_0}^{\infty} a_k^{-s} \sum_{j=2}^{\infty} \frac{(s+1)(s+2) \cdots (s+j-1)}{j!} \left( \frac{-x}{a_1} \right)^j \left( \frac{a_1}{a_k} \right)^j
\]
\[
\leq \sum_{k=k_0}^{\infty} a_k^{-\text{Re} s} \left( \frac{a_1}{a_k} \right)^2 \sum_{j=2}^{\infty} \frac{(s+1)(s+2) \cdots (s+j-1)}{j!} \left( \frac{-x}{a_1} \right)^j
\]
\[
= \frac{a_1^2}{a_k} \sum_{k=k_0}^{\infty} a_k^{-\text{Re} s - 2} \sum_{j=2}^{\infty} \frac{(s+1)(s+2) \cdots (s+j-1)}{j!} \left( \frac{-x}{a_1} \right)^j
\]
is finite in \( \text{Re} s > -1 \) by Lemma 1, \( h(s) \) becomes holomorphic in \( \text{Re} s > -1 \).

Q.E.D.

**Definition 2** We define the zeta regularized product of \( \prod_{k=1}^{\infty} (1 + \frac{x}{a_k}) \) by
\[ z \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) := \exp \left( g'(0, x) \right). \]  
(9)

**Remark 2** If \( a = \{a_k\}_{k=1}^{\infty} \) is zeta regularizable, we have
\[ z \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) = z \prod_{k=1}^{\infty} \frac{(a_k + x)}{a_k} = \exp \left( z'(0) - z'(0, x) \right). \]

**2.2 Generalized Wallis formula**

**Proposition 3** (Weierstrass' infinite product formula, [4] Theorem 2, [6])
\[ z \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) = e^{qx} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) \exp \left( -\frac{x}{a_k} \right). \]

**Proof.** Noting \( \lim_{s \to 0} sz(s+1) = 1 \), we first calculate \( h'(0) \).
\[
h'(0) = g'(0, x) - \lim_{s \to 0} \frac{s z(s+1) - 1}{s} \lambda
\]
\[
= g'(0, x) - \lim_{s \to 0} \left[ z(s+1) - \frac{1}{s} \right] x
\]
\[
= g'(0, x) - qx \quad (\text{cf. (6)}). \]  
(10)

On the other hand, (8) implies \( h(0) = 0 \) and so that \( h'(0) = \lim_{s \to 0} h(s)/s \). Therefore
\[
h'(0) = -\sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \frac{(j-1)!}{j!} \left( \frac{-x}{a_k} \right)^j \left( \frac{a_1}{a_k} \right)^j
\]
\[
= \sum_{k=1}^{\infty} \left[ \log \left( 1 + \frac{x}{a_k} \right) - \frac{x}{a_k} \right].
\]
This and (10) imply that

$$g'(0, x) = qx + \sum_{k=1}^{\infty} \left[ \log \left( 1 + \frac{x}{a_k} \right) - \frac{x}{a_k} \right]. \quad (11)$$

Plugging this into the exponential function, we finally obtain

$$z^{-\infty} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) = e^{qx} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) e^{-\frac{x}{a_k}}.$$

Q.E.D.

**Theorem 1** (Generalized Wallis formula)

$$z^{-\infty} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) = \lim_{n \to \infty} n^{-x} \prod_{k=1}^{n} \left( 1 + \frac{x}{a_k} \right). \quad (12)$$

**Remark 3** For the special case where \( a_k = k, \, k = 1, 2, \ldots \), and \( x = -1/2 \), we have

$$z^{-\infty} \prod_{k=1}^{\infty} \left( 1 - \frac{1}{2k} \right) = \Gamma(1/2)^{-1} = \pi^{-1/2},$$

$$n^{1/2} \prod_{k=1}^{n} \left( 1 - \frac{1}{2k} \right) = n^{1/2} \left( \frac{2n}{n} \right)^{2n^2}.$$

So (12) implies now the classical Wallis formula.

**Proof of Theorem 1.** From (4) and Proposition 3, it follows that

$$z^{-\infty} \prod_{k=1}^{\infty} \left( 1 + \frac{\lambda}{a_k} \right) = \lim_{n \to \infty} \exp \left( \sum_{k=1}^{n} \left( a_k^{-1} + a_k^{-1} + \cdots + a_n^{-1} - \log n \right) x \right) \prod_{k=1}^{n} \left( 1 + \frac{x}{a_k} \right) e^{-\frac{x}{a_k}}$$

$$= \lim_{n \to \infty} n^{-x} \prod_{k=1}^{n} \left( 1 + \frac{x}{a_k} \right).$$

Q.E.D.

By definition, \( z^{-\infty} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{a_k} \right) \) is neither 0 nor infinite. Consequently, Proposition 3 and Theorem 1 have substantial meaning.

**Example 1** The square of the classical Wallis formula is in fact a zeta regularized product:

$$\pi^{-1} = z^{-\infty} \prod_{k=1}^{\infty} \left( 1 - \frac{1}{2k} \right)^2 = z^{-\infty} \prod_{k=1}^{\infty} \left( 1 - \frac{1}{a_k} \right),$$

where \( \frac{1}{a_k} = \frac{1}{k} - \frac{1}{4k^2} \) or

$$a_k = \frac{1}{k} - \frac{1}{4k^2} = k + \frac{1}{4} + \frac{1}{4(4k-1)}, \quad k = 1, 2, \ldots,$$

which satisfies Assumption 1. Then let us show that

$$q = \lim_{s \to 1} \left( z(s) - \frac{1}{s-1} \right) = \gamma - \frac{\pi^2}{24}. \quad (13)$$
Since
\[
\pi^{-1} = \lim_{n \to \infty} n \prod_{k=1}^{n} \left(1 - \frac{1}{2k}\right)^2 \exp \left(\frac{1}{k} - \frac{1}{4k^2}\right)
\]
we must have
\[
e^{-\varphi} = \lim_{n \to \infty} n \prod_{k=1}^{n} \exp \left(-\frac{1}{k} + \frac{1}{4k^2}\right),
\]
namely,
\[
\varphi = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{4k^2}\right) - \log n \right] = \gamma - \sum_{k=1}^{\infty} \frac{1}{4k^2} = \gamma - \frac{\pi^2}{24}.
\]

Remark 4 In case \(\sum_{k=1}^{\infty} a_k^{-1} < \infty\), \(z(s)\) becomes finite at \(s = 1\), so that its 'finite part' \(\varphi\) at \(s = 1\) is, of course, \(\sum_{k=1}^{\infty} a_k^{-1}\). Then it holds that \(z - \prod_{k=1}^{\infty} (1 + \frac{x}{a_k}) = \prod_{k=1}^{\infty} (1 + \frac{x}{a_k})\) and \(\lim_{n \to \infty} n \prod_{k=1}^{n} \exp \left(-\frac{1}{k} + \frac{1}{4k^2}\right)\).

Let us show it.

(i) For a finite sequence \(a = \{a_k\}_{k=1}^{N}\),
\[
z(s) := \sum_{k=1}^{N} a_k^{-s}, \quad z(s, x) := \sum_{k=1}^{N} (a_k + x)^{-s}, \quad 0 \leq \lambda < a_1,
\]
which are entire functions, it is easy to see that \(\exp(z'(0) - z'(0, x)) = \prod_{k=1}^{N} (1 + \frac{x}{a_k})\).

(ii) For an infinite sequence \(a = \{a_k\}_{k=1}^{\infty}\),
\[
z(s) := \sum_{k=1}^{\infty} a_k^{-s}, \quad z(s, x) := \sum_{k=1}^{\infty} (a_k + x)^{-s}, \quad 0 \leq \lambda < a_1,
\]
are finite at \(s = 1\), but we do not know whether they are analytically continued beyond \(\text{Re} \ s > 1\). Nevertheless their difference \(g(s, x) := z(s) - z(s, x)\) is analytically continued to a holomorphic function in \(\text{Re} \ s > -1\), which is shown in a similar way as Proposition 3. Indeed, by (7),
\[
g(s, x) = -\sum_{k=1}^{\infty} a_k^{-s} \sum_{j=1}^{\infty} \binom{s + j - 1}{j} \left(\frac{-x}{a_k}\right)^j,
\]
from which it follows that
\[
g'(0, x) = -\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{(j-1)!}{j!} \left(\frac{-x}{a_k}\right)^j = -\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{-x}{a_k}\right)^j = \sum_{k=1}^{\infty} \log \left(1 + \frac{x}{a_k}\right).
\]
Thus
\[
\exp(g'(0, x)) = \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right).
\]
2.3 Generalized digamma function

If $a = \{a_k\}_{k=1}^\infty$ satisfies Assumption 1, so does $\{a_k + x\}_{k=1}^\infty$ for each $x > 0$, and hence we can define

$$q(x) := \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{x + a_k} - \log n \right), \quad x > 0.$$  

Comparing with (2), we can say that $-q(x)$ is a generalized digamma function.

Suppose that unit electric charges are located at each point of $\{a_k\}_{k=1}^\infty$ on the real line $\mathbb{R}$. Then $q(x)$ can be regarded as the renormalized Coulomb potential at $-x$ caused by those electric charges. Indeed, we see

$$q'(x) = \frac{d}{dx} \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{x + a_k} - \log n \right) = \sum_{k=1}^{\infty} \frac{1}{(x + a_k)^2} = -z(2,x).$$

By (11), we have

$$q(x) = -\frac{d}{ds} \left( \zeta(s,x) - \zeta(s,x-1) \right) \bigg|_{s=0} + \sum_{k=1}^{\infty} \left[ \frac{1}{x + a_k} + \log \left( 1 - \frac{1}{x + a_k} \right) \right]. \quad (13)$$

Applying this formula to the sequence $\{a_k = k\}_{k=1}^\infty$, we have

$$-\psi(x+1) = -\log x + \sum_{k=1}^{\infty} \left[ \frac{1}{x + k} + \log \left( 1 - \frac{1}{x + k} \right) \right], \quad (x > 0), \quad (14)$$

because

$$\frac{d}{ds} \left( \zeta(s,x) - \zeta(s,x-1) \right) \bigg|_{s=0} = \frac{d}{ds} (-x^{-s}) \bigg|_{s=0} = \log x.$$

**Theorem 2** For any sequence $a = \{a_k\}_{k=1}^\infty$ satisfying Assumption 1 and $a_k k^{-1} = 1 + O(k^{-\delta'})$, $k \to \infty$, $\delta' > 0$, we have

$$q(x) = -\log x + O(x^{-1}), \quad x \to \infty.$$  

**Proof.** From (14) it follows that

$$-\psi(x+1) = -\log x + O(x^{-\min(1,\delta')}), \quad x \to \infty.$$  

On the other hand, for $x > 0$, we have

$$q(x) + \psi(x+1) = \sum_{k=1}^{\infty} \left( \frac{1}{x + k} - \frac{1}{x + a_k} \right)$$

$$= \sum_{k=1}^{\infty} \frac{a_k - k}{(x + k)(x + a_k)}$$

$$= \sum_{k=1}^{\infty} \frac{O(k^{1-\delta})}{(x + k)(x + a_k)} = O(x^{-\delta'}), \quad x \to \infty.$$
2.4 Generalized Gamma functions

The following lemma is easily derived from Theorem 1.

**Lemma 3** For each $n \in \mathbb{N}$,

$$z \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) = \prod_{k=1}^{n} \left(1 + \frac{x}{a_k}\right) z \prod_{k=n+1}^{\infty} \left(1 + \frac{x}{a_k}\right).$$

Now, recalling $z \prod_{k=1}^{\infty} (1 + \frac{x}{k}) = \Gamma(1 + x)^{-1}$, Lemma 3 implies

$$\Gamma(n + 1 + x) = \Gamma(1 + x) \prod_{k=1}^{n} (k + x) = \frac{\prod_{k=1}^{n} (k + x)}{z \prod_{k=n+1}^{\infty} \left(1 + \frac{x}{k}\right)}.$$

$$= \frac{n! \prod_{k=1}^{n} \left(1 + \frac{x}{k}\right)}{z \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right)} = \frac{n!}{z \prod_{k=n+1}^{\infty} \left(1 + \frac{x}{k}\right)}.$$

Therefore

$$\Gamma(x) = \frac{n!}{z \prod_{k=n+1}^{\infty} \left(1 + \frac{x - n - 1}{k}\right)}.$$

We consider an analogy of this.

**Definition 3** For each $n \in \mathbb{N}$, we define

$$G^{(n+1)}(x) := \frac{\prod_{k=1}^{n} a_k}{z \prod_{k=n+1}^{\infty} \left(1 + \frac{x - a_{n+1}}{a_k}\right)}.$$

Obviously, we have

$$G^{(n+1)}(a_{n+1}) = \prod_{k=1}^{n} a_k$$

$$z \prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right) = \frac{\prod_{k=1}^{n} (a_k + x)}{G^{(n+1)}(a_{n+1} + x)}, \quad n = 1, 2, \ldots$$

By (15), when $a_k = k$ for each $k \in \mathbb{N}$, $G^{(n+1)}(x) = \Gamma(x)$ holds for any $n \in \mathbb{N}$. In general, for $a = \{a_k\}_{k=1}^{\infty}$ satisfying the following assumption, the corresponding $G^{(n+1)}$ has a Gamma function-like property.
Assumption 2 There exists some $\alpha > 0$ such that $a_{k+1} - a_k = 1 + O(k^{-\alpha}), \ k \to \infty$.

Theorem 3 If $a = \{a_k\}_{k=1}^\infty$ satisfies Assumption 1 and Assumption 2, it holds for any $j \in \mathbb{N}$ that

\[ G^{(n+1)}(a_{n+1-j}) \sim \prod_{k=1}^{n-j} a_k, \ n \to \infty. \]

(19)

Here "~" indicates that the ratio of the both hand sides tends to 1 in the specified limit.

Proof. For $j < n$,

\[
G^{(n+1)}(a_{n+1-j}) = \prod_{k=1}^{n} \frac{a_k}{z^{-\prod_{k=n+1}^{k} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)}}
\]

\[
= \prod_{k=n+1-j}^{n} a_k^{-1} z^{-\prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)}.
\]

Therefore it is sufficient to show that

\[
\lim_{n \to \infty} \prod_{k=n+1-j}^{n} a_k^{-1} z^{-\prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)} = 1.
\]

(20)

By Proposition 3, we have

\[
z^{-\prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)} = \exp(-q_{n+1}(a_{n+1} - a_{n+1-j}))
\]

\[
\times \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \exp\left(\frac{a_{n+1} - a_{n+1-j}}{a_k}\right),
\]

where

\[ q_{n+1} := \lim_{N \to \infty} \left[ \sum_{k=n+1}^{N} a_k^{-1} - \log(N - n + 1) \right] \]

\[ = q - \sum_{k=1}^{n} a_k^{-1} = -\log n + o(1), \ n \to \infty. \]

Then Assumption 2 implies that

\[ \exp(-q_{n+1}(a_{n+1} - a_{n+1-j})) = n^{j-O(n^{-\alpha})}e^{o(1)(-j+O(n^{-\alpha}))} \sim n^j, \ n \to \infty. \]

The following is obvious.

\[ \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \exp\left(\frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \to 1, \ n \to \infty. \]
From these, it follows that
\[ z \prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right) \sim n^j, \quad n \to \infty. \]

And hence
\[
\lim_{n \to \infty} \prod_{k=n+1-j}^{n} a_k z^{-\prod_{k=n+1}^{\infty} \left(1 - \frac{a_{n+1} - a_{n+1-j}}{a_k}\right)} = \lim_{n \to \infty} \prod_{k=n+1-j}^{n} a_k = 1.
\]

Q.E.D.

If \( a = \{a_k\}_{k=1}^{\infty} \) satisfies Assumption 1 and Assumption 2, the expression (18) and Theorem 3 can be used for numerical evaluation of \( z^{-\prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right)} \) in some cases. The method is as follows: First, for a suitably large \( n \) and \( j_0 < n \), construct a Lagrange's polynomial \( h_a^{(n,j_0)}(x) \) of degree \((j_0 - 1)\) that interpolates the points \((x_j, y_j) = (a_{n+1-j}, \sum_{k=1}^{n-j} \log a_k)\), \( j = 0, 1, \ldots, j_0 - 1 \).

Substituting \( h_a^{(n,j_0)}(x) \) for \( \log G^{(n+1)}(x) \) in (18), we calculate
\[
c^{(n,n',j_0)}(x) := \frac{\prod_{k=1}^{n'} (a_k + x)}{\exp(h_a^{(n,j_0)}(a_{n'} + x))}
\]
as an approximated value of \( z^{-\prod_{k=1}^{\infty} \left(1 + \frac{x}{a_k}\right)} \). (In doing this, to prevent overflow or underflow, all calculations should be done by taking logarithm, i.e., we calculate
\[
\sum_{k=1}^{n'-1} \log(a_k + x) - h_a^{(n,j_0)}(a_{n'} + x)
\]
then plug the result into the exponential function.) Here \( n' \) is a suitable integer between \( n - j_0 \) and \( n \). Probably, it is better to pick up \( n' \) from the middle of the interval \([n+1-j_0, n]\).

**Example 2** Let us consider the square of the Wallis formula again. The sequence dealt in Example 1, i.e., \( a_k = k + \frac{1}{4} + \frac{1}{4(k-1)} \) satisfies Assumption 2 for \( \alpha = 2 \), so that we can apply the above method to get an approximated value of \( z^{-\prod_{k=1}^{\infty} \left(1 - \frac{1}{a_k}\right)} \).

For \( n = 30, 300, 3000 \), we constructed Lagrange polynomials \( h_a^{(n,5)}(x) \), and calculated \( c^{(n,n-2,5)}(1) \), which are listed in the table below. Since the true value is
\[
1/\pi = 1/\pi = 1/3.14159265\ldots,
\]
roughly speaking, the error decreases at the rate of \( O(n^{-2}) \).

For comparison, we also calculated \( u(n) := n \prod_{k=1}^{n} (1 - \frac{1}{a_k}) \) as approximated values due to the Wallis formula. This time, the error decreases at the rate of \( O(n^{-1}) \).
In this way, $c^{(n,n-2,5)}(1)$ is much better than $w(n)$. But this example may be a special case, and since we have not established a precise error estimate, we do not know if our method is valid for general cases.

### 3 Randomized special functions

By randomizing the objects in the previous sections, we can find a new type of limit theorems in probability theory.

#### 3.1 In the case of Poisson process

Let $\{\xi_i\}_{i=0}^{\infty}$ be a positive i.i.d. random variables whose common distribution is the exponential distribution with parameter 1, i.e.,

$$P(\xi_i \leq x) = \int_0^x e^{-t} dt = 1 - e^{-x}, \quad x \geq 0,$$

and set

$$X = \{X_k\}_{k=1}^{\infty} := \{\xi_1 + \cdots + \xi_k\}_{k=1}^{\infty}.$$

Then by virtue of the strong law of large numbers, the sequence $\{b + X_k\}_{k=1}^{\infty}, b > 0,$ satisfies Assumption 1 almost surely. Note that

$$\eta(t) := F_X(t) = \#\{k | X_k \leq t\}, \quad t \geq 0,$$

is a standard Poisson process.

#### 3.1.1 Randomization of the Wallis formula

First let us calculate the distribution of the following random variable.

$$W(b, \lambda) := z - \prod_{k=1}^{\infty} \left(1 + \frac{\lambda}{b + X_k}\right), \quad \lambda \geq -b, \; b > 0.$$

**Theorem 4** The $n$-th moment of $z(b, \lambda)$ is calculated as follows. 5

$$\mathbb{E}[W(b, \lambda)^n] = \begin{cases} b^{-\lambda}, & n = 1, \\ b^{-n\lambda} \exp\left(\sum_{r=2}^{n} \binom{n}{r} \frac{\lambda^r}{(r-1)b^{r-1}}\right), & n = 2, 3, \ldots \end{cases}$$

**Lemma 4** (Durrett[1], (5.1) Theorem, Chapt.3.) Under the conditional probability measure $P(\cdot | \eta(t) = N)$, $t > 0$, the distribution of $\{X_k\}_{k=1}^{N}$ coincides with that of the order statistics of $N$ independent uniformly distributed random variables in $[0, t]$.

---

4 $P$ stands for probability.

5 $\mathbb{E}$ stands for expectation.
Sketch of Proof of Theorem 4.

Step 1. Let \( \{X_{t,k}\}_{k=1}^{\infty} \) be independent uniformly distributed random variables in \([0, t]\).

First, we define a random variable

\[
W_{t,N,\lambda} := \prod_{k=1}^{N} \left( 1 + \frac{\lambda}{b + X_{t,k}} \right), \quad N \in \mathbb{N}.
\]

and calculate its moments.

\[
E \left[ (W_{t,N,\lambda})^n \right] = \prod_{k=1}^{N} E \left[ \left( 1 + \frac{\lambda}{b + X_{t,k}} \right)^n \right] = \left( \int_0^t \left( 1 + \frac{\lambda}{b + y} \right)^n \frac{dy}{t} \right)^N
\]

\[
= \left( \sum_{r=0}^{n} \frac{n!}{r!} \lambda^r \int_0^t \frac{dy}{(b+y)^r} \right)^N
\]

\[
= \left( 1 + \frac{n\lambda}{t} \left( \frac{r}{r-1} \frac{(b+y)^{r-1}}{r-1} \right) \right)^N
\]

\[
= \left( 1 + \frac{n\lambda}{t} (\log(b+t) - \log b) + \frac{1}{t} \sum_{r=2}^{n} \frac{n!}{(r-1)!} \lambda^r \frac{t}{r} \frac{1}{b^{r-1}} + O(t^{-2}) \right)^N
\]

\[
= \left( 1 + \frac{n\lambda \log(b+t)}{t} - \frac{n\lambda \log b + C}{t} + O(t^{-2}) \right)^N, \quad t \to \infty,
\]

where

\[
C := \sum_{r=2}^{n} \frac{n!}{(r-1)!} \frac{\lambda^r}{r b^{r-1}}.
\]

Step 2. Since \( P(\eta(t) = N) = t^Ne^{-t}/N! \), Lemma 4 implies that

\[
E \left[ \eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left( 1 + \frac{\lambda}{1+X_k} \right)^n \right] = \sum_{N=0}^{\infty} E \left[ N^{-n\lambda} \prod_{k=1}^{N} \left( 1 + \frac{\lambda}{1+X_k} \right)^n \mid \eta(t) = N \right] t^Ne^{-t} / N!
\]

\[
= \sum_{N=0}^{\infty} N^{-n\lambda} E \left[ (W_{t,N,\lambda})^n \right] t^Ne^{-t} / N!
\]

A change of variables \( u := N/t \) shows

\[
E \left[ \eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left( 1 + \frac{\lambda}{1+X_k} \right)^n \right] = \sum_{u \in \mathbb{N}} (tu)^{-n\lambda} E \left[ (W_{t,\eta(t),\lambda})^{tu} \right] (tu)^{-t} / (tu)!
\]

Since the distribution of \( \eta(t)/t \) is convergent to the Dirac measure \( \delta_1(du) \), we see

\[
E \left[ W(b, \lambda)^n \right] = E \left[ \lim_{t \to \infty} \eta(t)^{-n\lambda} \prod_{k=1}^{\eta(t)} \left( 1 + \frac{\lambda}{1+X_k} \right)^n \right]
\]
\[
\lim_{t \to \infty} \mathbb{E} \left[ \eta(t)^{-n \lambda} \prod_{k=1}^{\eta(t)} \left( 1 + \frac{\lambda}{1 + X_k} \right)^n \right] = \lim_{t \to \infty} (tu)^{-n \lambda} \mathbb{E} \left[ (W_{t,tu,\lambda})^n \right] \frac{t^{tu} e^{-t}}{(tu)!} u = \sum_{1} \frac{1}{t} N
\]
\[
= \lim_{t \to \infty} t^{-n \lambda} \mathbb{E} \left[ (W_{t,t,\lambda})^n \right] = b^{-n \lambda} e^{C}.
\]

The above calculation can be rigorously justified. Q.E.D.

In case \( b = 1 \), if \( |\lambda| \ll 1 \), then the \( n \)-th moments of \( W(1, \lambda) \) is approximately equal to \( \exp \left( \frac{n(n-1)}{2} \lambda^2 \right) \), which is nothing but the \( n \)-th moment of a log normal distribution, more precisely, the distribution of the random variable \( e^Y \) where \( Y \) is distributed as \( \mathcal{N}(\frac{-\lambda^2}{2}, \lambda^2) \). Hence when \( |\lambda| \ll 1 \), the distribution of \( W(1, \lambda) \) is close to that of \( e^Y \).  

### 3.1.2 Random digamma function

Proposition 3 implies that

\[
\log W(1, \lambda) = Q \lambda + \sum_{k=1}^{\infty} \left[ \log \left( 1 + \frac{\lambda}{1 + X_k} \right) - \frac{\lambda}{1 + X_k} \right],
\]

where

\[
Q = \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \frac{1}{1 + X_k} - \log N \right].
\]

This limit exists a.s.

Now, we have

\[
\mathbb{E}[Q] = 0, \quad \mathbb{E}[Q^2] = 1,
\]

which is shown in the following way. First, it is easy to see that

\[
\sum_{k=1}^{\infty} \left[ \log \left( 1 + \frac{\lambda}{1 + X_k} \right) - \frac{\lambda}{1 + X_k} \right] = O(\lambda^2), \quad \lambda \to 0.
\]

Hence we see

\[
\mathbb{E}[\log W(1, \lambda)] = -\mathbb{E}[Q] \lambda + O(\lambda^2), \quad \lambda \to 0.
\]

On the other hand, since the mean of \( \log W(1, \lambda) \) is approximately equal to \( -\lambda^2/2 \), we see \( \mathbb{E}[Q] = 0 \). And the 2-nd moment of \( \log W(1, \lambda) \) is approximately equal to \( \lambda^2 \), so we see \( \mathbb{E}[Q^2] = 1 \).

Suppose \( |\lambda| \ll 1 \). Since \( -Q \lambda \) is the main part of \( \log W(1, \lambda) \), and \( \log W(1, \lambda) \) is approximately distributed as \( \mathcal{N}(\frac{-\lambda^2}{2}, \lambda^2) \), one may well expect that \( Q \) is distributed as \( \mathcal{N}(0, 1) \). But although its distribution is close to \( \mathcal{N}(0, 1) \), it is not exactly distribute as \( \mathcal{N}(0, 1) \). In deed we have \( \mathbb{E}[Q^3] = 1/2 \).

Let us investigate a little bit more general case. Let

\[
Q(x) := \lim_{N \to \infty} \left[ \sum_{k=1}^{N} \frac{1}{x + X_k} - \log N \right], \quad x > 0.
\]

Comparing with (2), we can say that \( -Q(x) \) is a random digamma function.

---

6This may hold in any case where \( \{X_k\}_{k=1}^{\infty} \) is the partial sum of positive i.i.d. random variables with mean 1.
Theorem 5  The mean of $Q(x)$ is $E[Q(x)] = -\log x$, and the centered moments of $Q(x)$ are

$$E[(Q(x) + \log x)^n] = n! \sum_{p=1}^{\lfloor \sqrt{2n} \rfloor} \sum_{2 \leq n_1 < \cdots < n_p \leq n \atop k_1 n_1 + \cdots + k_p n_p = n} \frac{1}{x^{n-(k_1+\cdots+k_p)}} \times \prod_{j=1}^{p} \frac{1}{k_j!(n_j)!(n_j-1)^{k_j}}, \quad n = 2, 3, \ldots (21)$$

More concretely, the centered moments of order $1, 2, \ldots, 6$ are

$$0, \quad \frac{1}{x}, \quad \frac{1}{2x^2}, \quad \frac{3}{x^3} + \frac{1}{3x^3}, \quad \frac{5}{x^3} + \frac{1}{4x^4}, \quad \frac{15}{x^3} + \frac{15}{2x^4} + \frac{1}{5x^5}. $$

Sketch of Proof. As in the previous section, let $\{X_{t,k}\}_{k=1}^{N}$ be i.i.d. uniform random variables in $[0, t]$. Define

$$Y_{t,N} := \sum_{k=1}^{N} \left( \frac{1}{x+X_{t,k}} - c(t) \right), \quad c(t) = \frac{1}{t}(\log(x+t) - \log x). $$

Then, when $t \to \infty$, we have

$$E \left[ \left( \frac{1}{x+X_{t,k}} - c(t) \right)^n \right] = \begin{cases} 0 & (n = 1) \\ \frac{1}{x^{n-1}(n-1)t} + O(t^{-2+\epsilon}) & (n \geq 2) \end{cases} \quad (22)$$

Here $\epsilon > 0$ can be arbitrarily small. Indeed, if we write the L.H.S. of the above expression by integrals,

$$= \int_0^t \left( \frac{1}{x+y} - c(t) \right)^n \frac{dy}{t}$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-c(t))^{n-r} \frac{1}{t} \int_0^t (x+y)^{-r} dy$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-c(t))^{n-r} \times \left\{ \begin{array}{ll} \frac{1}{c(t)} \frac{1}{r!} \frac{1}{t} (x^{-r+1} - (x+t)^{-r+1}) & (r = 0) \\ \frac{1}{c(t)} \frac{1}{(r-1)t} \frac{1}{r^{r-1}} (x^{r-1} - 1) & (r = 1) \\ \frac{1}{c(t)} \frac{1}{(r-1)t} \frac{1}{r^{r-1}} (x^{r-1} - 1) & (r \geq 2) \end{array} \right. $$

Here we have $c(t) = O(t^{-1+\epsilon}), \, t \to \infty$, and hence we obtain (22).

Under these preparations, we see

$$E \left[ Y_{t,N}^n \right] $$

$$= \sum_{n_1+n_2+\cdots+n_N=n} \prod_{k=1}^{N} \left( \frac{1}{x+X_{t,k}} - c(t) \right)^{n_k} $$

$$= \sum_{n_1+n_2+\cdots+n_N=n} \prod_{k=1}^{N} \left( \frac{1}{x+X_{t,k}} - c(t) \right)^{n_k} $$
\[
\sum_{p=1}^{n} \frac{N!}{(N-p)!} \prod_{j=1}^{p} E \left( \frac{1}{x + X_{t,1} - c(t)} \right)^{n_{j}^{k_{j}}} \times \frac{1}{k_{1}! k_{2}! \cdots k_{p}!} \frac{n!}{(n_{1}!)^{k_{1}} (n_{2}!)^{k_{2}} \cdots (n_{p}!)^{k_{p}}} \cdot \prod_{j=1}^{p} \frac{1}{(n_{j} - 1)^{k_{j}}}.
\]

From this, it follows similarly to the previous theorem that

\[
\mathbb{E}[(Q(x) + \log x)^n] = \mathbb{E} \left[ \lim_{t \to \infty} \left( \sum_{k=1}^{\eta(t)} \left( \frac{1}{x + X_{k}} - c(t) \right) \right)^n \right]
\]

\[
= \lim_{t \to \infty} \mathbb{E} \left[ \left( \sum_{k=1}^{\eta(t)} \left( \frac{1}{x + X_{k}} - c(t) \right) \right)^n \right]
\]

\[
= \lim_{t \to \infty} \sum_{n=0}^{\infty} \mathbb{E} \left[ Y_{t,N}^{n} \right] \frac{tN e^{-t}}{N!}
\]

\[
= \lim_{t \to \infty} \sum_{u \in t^{N}} \mathbb{E} \left[ Y_{t,u}^{n} \right] \frac{tu e^{-t}}{(tu)!} \left( = \lim_{t \to \infty} \mathbb{E} \left[ Y_{t,u}^{n} \right] \right)
\]

\[
= \sum_{p=1}^{n} \sum_{2 \leq n_{1} \cdots n_{p} \leq n} \frac{1}{x^{n-(k_{1} + k_{2} + \cdots + k_{p})}} \times \frac{1}{k_{1}! k_{2}! \cdots k_{p}!} \frac{n!}{(n_{1}!)^{k_{1}} (n_{2}!)^{k_{2}} \cdots (n_{p}!)^{k_{p}}} \prod_{j=1}^{p} \frac{1}{(n_{j} - 1)^{k_{j}}}.
\]

Now it is easy to get (21). Q.E.D.

The following theorem is an easy consequence of the law of iterated logarithm and Theorem 2.

**Theorem 6** For any \( \epsilon > 0 \), \( Q(x) = -\log x + O(x^{-(1/2)+\epsilon}) \), \( x \to \infty \), a.s.
Regarding this theorem as a law of large numbers, the following corresponds to the central limit theorem.

**Theorem 7**  The distribution of $\sqrt{x}(Q(x) + \log x)$ converges to $\mathcal{N}(0, 1)$ as $x \to \infty$.

**Proof.** The $n$-th moment of $\sqrt{x}(Q(x) + \log x)$ is

$$E\left[ (\sqrt{x}(Q(x) + \log x))^n \right] = n! \sum_{p=1}^{\lfloor \sqrt{2n} \rfloor} \sum_{2 \leq n_1 < \cdots < n_p, k_1 n_1 + \cdots + k_p n_p = n} \frac{1}{x^{n/2-(k_1+\cdots+k_p)}} \prod_{j=1}^{p} \frac{1}{k_j!(n_j!)^{k_j}(n_j-1)^{k_j}}.$$  

In the sum $\sum_{p=1}^{\lfloor \sqrt{2n} \rfloor}$ of the R.H.S., the sum for $p$ such that $n/2 > k_1 + \cdots + k_p$ converge to 0 as $x \to \infty$. Therefore as $x \to \infty$, what survive are the terms for $p$ such that $n/2 = k_1 + \cdots + k_p$ (from this $n$ must be even), that is, for $p = 1$, $k_1 = n/2$, $n_1 = 2$. Hence we see that

$$\lim_{x \to \infty} E\left[ (\sqrt{x}(Q(x) + \log x))^n \right] = \frac{n!}{2^{n/2}(n/2)!}, \quad (n: \text{even}).$$

This is nothing but the $n$-th moment of $\mathcal{N}(0, 1)$.

Q.E.D.

Suppose that unit electric charges are located at each random point of $\{X_k\}_{k=1}^\infty$, the renormalized Coulomb potential $Q(x)$ at the location $-x$ is distributed approximately as $\mathcal{N}(-\log x, 1/x)$ if $x$ is large.

**Another proof of Theorem 7.** Since the distribution function $F_{\{x+X_k\}}(t)$ corresponding to the sequence $\{x+X_k\}_{k=1}^\infty$ is exactly $\eta(t-x)$, (5) implies that

$$Q(x) = -\log x + 1 + \int_x^\infty (\eta(t-x) - t) \frac{dt}{t^2} = -\log x + \int_0^\infty (\eta(t) - t) \frac{dt}{(t+x)^2}.$$  

Since $E[\eta(t) - t] = 0$, we readily see $E[Q(x)] = -\log x$. From the following expression

$$\sqrt{x} (Q(x) + \log x) = \int_0^\infty (\eta(t) - t) \frac{\sqrt{x}}{(t+x)^2} dt,$$  

let us derive Theorem 7 by using the Lindeberg-Feller theorem ([2] Chapt.2 (4.5) Theorem).

First, note that $\tilde{\eta}(t) := \eta(t) - t$ is a martingale with mean 0. The Fubini theorem (or integration by parts formula) implies

$$\int_S^T \tilde{\eta}(t) \frac{\sqrt{x}}{(t+x)^2} dt = \int_S^T \left( \tilde{\eta}(S) + \int_S^t d\tilde{\eta}(s) \right) \frac{\sqrt{x}}{(t+x)^2} dt$$

$$= \int_S^T \left( \int_S^T \frac{\sqrt{x}}{(t+x)^2} dt \right) d\tilde{\eta}(s) + \tilde{\eta}(S) \int_S^T \frac{\sqrt{x}}{(t+x)^2} dt$$

$$= \int_S^T \left( \frac{\sqrt{x}}{s+x} - \frac{\sqrt{x}}{T+x} \right) d\tilde{\eta}(s) + \left( \frac{\sqrt{x}}{S+x} - \frac{\sqrt{x}}{T+x} \right) \tilde{\eta}(S)$$

$$= \int_S^T \frac{\sqrt{x}}{s+x} d\tilde{\eta}(s) - \sqrt{x} \frac{\tilde{\eta}(T) + \sqrt{x}}{S+x}.$$
Letting $S \rightarrow 0$, $T \rightarrow \infty$, we have the following expression. 

$$\sqrt{x}(Q(x) + \log x) = \int_0^\infty \tilde{\eta}(t) \frac{\sqrt{x}}{(t + x)^2} dt = \int_0^\infty \frac{\sqrt{x}}{t + x} d\tilde{\eta}(t), \text{ a.s.}$$

Now put

$$\begin{align*}
U_{n,m} &:= \int_{(m-1)^2}^{m^2} \frac{\sqrt{n}}{t + n} d\tilde{\eta}(t), \quad m = 1, 2, \ldots, n - 1, \\
U_{n,n} &:= \int_{(n-1)^2}^\infty \frac{\sqrt{n}}{t + n} d\tilde{\eta}(t).
\end{align*}$$

Let us the triangular array $\{U_{n,m}\}_{1 \leq m \leq n} \leq n$ satisfies the Lindeberg - Feller's conditions.

**Step 1.** Since $\{\eta(t)\}_{t \geq 0}$ is an independent increment process, $\{U_{n,m}\}_{m=1}^n$ is independent sequence for each $n$.

**Step 2.** It holds (without letting $n \rightarrow \infty$) that

$$\sum_{m=1}^{n} \mathbb{E}[U_{n,m}^2] = \sum_{m=1}^{n-1} \int_{(m-1)^2}^{m^2} \left( \frac{\sqrt{n}}{t + n} \right)^2 dt + \int_{(n-1)^2}^\infty \left( \frac{\sqrt{n}}{t + n} \right)^2 dt = \int_0^\infty \frac{n}{(t + n)^2} dt = 1.$$ 

**Step 3.** Now, to prove the theorem, it is sufficient to show the last Lindeberg - Feller's condition: for any $\epsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \sum_{m=1}^{n} \mathbb{E}[U_{n,m}^2; |U_{n,m}| > \epsilon] = 0.$$  

(24)

**Lemma 5** If a random variable $U$ has the 4-th moment, then

$$\mathbb{E}\left[U^2; |U| > \epsilon\right] \leq \epsilon^{-2} \mathbb{E}[U^4].$$

**Proof.**

$$\mathbb{E}\left[U^2; |U| > \epsilon\right] \leq \mathbb{E}\left[U^2 \cdot \frac{U^2}{\epsilon^2}; |U| > \epsilon\right] \leq \frac{1}{\epsilon^2} \mathbb{E}[U^4].$$

Q.E.D.

Since

$$\mathbb{E}[U_{n,n}^2] = \int_{(n-1)^2}^\infty \left( \frac{\sqrt{n}}{t + n} \right)^2 dt = \frac{n}{(n-1)^2 + n} \rightarrow 0, \quad n \rightarrow \infty,$$

it is sufficient to prove

$$\lim_{n \rightarrow \infty} \sum_{m=1}^{n-1} \mathbb{E}[U_{n,m}^4] = 0$$

(25)

to show (24) by Lemma 5.

Let us estimate each of $\mathbb{E}[U_{n,m}^4]$. Putting

$$U_m(t) := \int_{(m-1)^2}^t \frac{\sqrt{n}}{s + n} d\tilde{\eta}(s), \quad m \leq t,$$
and applying the Itô formula, we have

$$
\begin{align*}
\mathbb{E} \left[ U_{n,m}^4 \right] &= \mathbb{E} \left[ \int_{(m-1)^2}^{m^2} \left( \left( U_{m}(t) + \frac{\sqrt{n}}{t+n} \right)^4 - U_{m}(t)^4 - 4U_{m}(t)^3 \frac{\sqrt{n}}{t+n} \right) \, dt \right] \\
&= \int_{(m-1)^2}^{m^2} \left( 6 \mathbb{E} \left[ U_{m}(t)^2 \right] \left( \frac{\sqrt{n}}{t+n} \right)^2 + \left( \frac{\sqrt{n}}{t+n} \right)^4 \right) \, dt \\
&= n^2 \int_{(m-1)^2}^{m^2} \left( 6 \cdot \frac{t - (m-1)^2}{((m-1)^2 + n)(t+n)^3} + \frac{1}{(t+n)^4} \right) \, dt \\
&< n^2 \int_{(m-1)^2}^{m^2} \left( 6 \cdot \frac{2m-1}{((m-1)^2 + n)^4} \right) \, dt \\
&= 6n^2 \cdot \frac{2m-1}{((m-1)^2 + n)^4} + n^2 \cdot \frac{2m-1}{((m-1)^2 + n)^4} \\
&= 6n^2 \cdot \frac{2m-1}{((m-1)^2 + n)^4}.
\end{align*}
$$

From this, we derive

$$
\sum_{m=1}^{n-1} \mathbb{E} \left[ U_{n,m}^4 \right] < n^2 \sum_{m=1}^{n-1} \frac{6(2m-1)^2 + 2m-1}{((m-1)^2 + n)^4} = O(n^{-1/2}), \quad n \to \infty, ^7
$$

thus (25) holds. Now the proof of Theorem 7 is complete. Q.E.D.

3.2 In the case of random walk

We next consider the case where \( \{\xi_i\}_{i=1}^{\infty} \) is a Bernoulli sequence with \( P(\xi_i = 0) = P(\xi_i = 2) = 1/2 \), and \( X_n := \sum_{i=1}^{n} \xi_i \). Again by the strong law of large numbers, \( \{x + X_k\}_{k=1}^{\infty} \), \( x > 0 \), satisfies Assumption 1 almost surely.

3.2.1 Random digamma function

Defining \( G_k \) by

$$
G_k := \# \{ n \in \mathbb{N} ; X_n = 2k \}, \quad k = 1, 2, \ldots.
$$

\( \{G_k\}_{k=1}^{\infty} \) is an i.i.d. sequence with a geometric distribution \( P(G_k = n) = 2^{-n}, \ n \in \mathbb{N} \), and we have

$$
\sum_{k=1}^{G_1 + \cdots + G_N} \frac{1}{x + X_k} = \sum_{k=0}^{N} \frac{G_k}{x + 2k}.
$$

Let us first look at the law of large numbers. Since \( \mathbb{E}[G_k] = 2 \), for sufficiently large \( N \),

$$
\log(G_1 + G_2 + \cdots + G_N) = \log \left( \frac{1}{N} \sum_{k=1}^{N} G_k \right) + \log N \approx \log 2 + \log N.
$$

^7For this estimate, use \( n^2 \int_{0}^{\infty} x^2/(x^2 + n)^4 \, dx = \pi/(32\sqrt{n}) \).
Therefore, with probability 1,

\[ Q(x) = \lim_{N \to \infty} \left[ \sum_{k=0}^{N} \frac{G_k}{x+2k} - (\log 2 + \log N) \right] \]

converges. Theorem 2 implies that with probability 1,

\[ Q(x) = -\log x + O(x^{-1}), \quad x \to \infty. \]

The mean of \( Q(x) \) is computed as follows.

\[
\mathbb{E}[Q(x)] = \lim_{N \to \infty} \mathbb{E} \left[ \sum_{k=0}^{N} \frac{G_k}{x+2k} - (\log 2 + \log N) \right] = \lim_{N \to \infty} \left[ \sum_{k=0}^{N} \frac{2}{x+2k} - (\log 2 + \log N) \right] = \lim_{N \to \infty} \left[ \sum_{k=0}^{N} \frac{1}{(x/2)+k} - (\log 2 + \log N) \right] = -\psi((x/2)+1) - \log 2 = -\log x + \sum_{k=0}^{\infty} \left[ \frac{1}{(x/2)+k} + \log \left( 1 - \frac{1}{(x/2)+k} \right) \right].
\]

Next, let us look at the central limit theorem. We put

\[ Q_N(x) := \sum_{k=0}^{N} \frac{G_k}{x+2k} - (\log 2 + \log N), \]

and calculate its characteristic function (Fourier transform). Noting that

\[ \mathbb{E}[\exp(itG_k)] = \frac{\frac{1}{2}e^{it}}{1 - \frac{1}{2}e^{it}}, \]

we have

\[
\mathbb{E}[e^{itQ_N(x)}] = \prod_{k=0}^{N} \mathbb{E} \left[ \exp \left( it \cdot \frac{G_k}{x+2k} \right) \right] \exp (-it(\log 2 + \log N)) = \prod_{k=0}^{N} \left[ \frac{1}{2} \exp \left( it \cdot \frac{1}{x+2k} \right) \right] \exp (-it(\log 2 + \log N)) = \prod_{k=0}^{N} \left[ \frac{1}{2} \exp \left( -it \cdot \frac{1}{x+2k} \right) \right] \exp \left[ -it \left( -\sum_{k=0}^{N} \frac{2}{x+2k} + \log 2 + \log N \right) \right].
\]

Thus

\[
\mathbb{E}[e^{itQ(x)}] = \prod_{k=0}^{\infty} \left[ \frac{\exp \left( -it \cdot \frac{1}{x+2k} \right)}{2 - \exp \left( it \cdot \frac{1}{x+2k} \right)} \right] \exp \left[ -it \left( -\psi((x/2)+1) + \log 2 \right) \right].
\]
Let us take the limit \( x \to \infty \) of the infinite product

\[
\prod_{k=0}^{\infty} \frac{\exp(-it \cdot \frac{1}{x+2k})}{2 - \exp(it \cdot \frac{1}{x+2k})} = \prod_{k=0}^{\infty} \frac{1}{2\exp(it \cdot \frac{1}{x+2k}) - \exp(it \cdot \frac{2}{x+2k})}.
\]

Developing the denominator, we see

\[
2\exp(it \cdot \frac{1}{x+2k}) - \exp(it \cdot \frac{2}{x+2k}) = 2\left(1 + \frac{it}{x + 2k} - \frac{1}{2} \frac{t^2}{(x + 2k)^2} + \cdots\right) - \left(1 + \frac{2it}{x + 2k} - \frac{1}{2} \frac{4t^2}{(x + 2k)^2} + \cdots\right) = 1 + \frac{t^2}{(x + 2k)^2} + \cdots,
\]

and hence

\[
\prod_{k=0}^{\infty} \frac{\exp(-it \cdot \frac{1}{x+2k})}{2 - \exp(it \cdot \frac{1}{x+2k})} = \prod_{k=0}^{\infty} \left(1 + \frac{t^2}{(x + 2k)^2} + \cdots\right)^{-1} \sim \prod_{k=0}^{\infty} \exp\left(-\frac{t^2}{2} \frac{1}{\left(\frac{x}{2} + k\right)^2}\right), \quad x \to \infty.
\]

From the above, when \( x \gg 1 \), the distribution of \( Q(x) \) is close to the normal distribution with mean \(-\psi((x/2) + 1) + \log 2\) and variance \( \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{((x/2) + k)^2} \). Since we have

\[
-\psi((x/2) + 1) + \log 2 = -\log x + O(x^{-1}),
\]

\[
\frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{((x/2) + k)^2} = x^{-1} + O(x^{-2}),
\]

as \( x \to \infty \), in the long run, we proved the following convergence in distribution.

\[
\sqrt{x}(Q(x) + \log x) \to \mathcal{N}(1, 0), \quad x \to \infty.
\]

That is, the assertion of Theorem 7 holds in this case, too.

4 Further discussions

4.1 Extension of Theorem 6 and Theorem 7

Recently, Theorem 6 and Theorem 7 have been much more generalized by S. Takanobu.

**Theorem 8** ([5]) Let \( \{\xi_i\}_{i=1}^{\infty} \) be an i.i.d. sequence with \( \xi_i > 0 \), \( E[\xi_i] = 1 \), and \( E[\xi_i^\beta] < \infty \) for some \( \beta > 1 \). Then we have \( Q(x) = -\log x + O(x^{-1}), \quad x \to \infty, \ a.s. \).

**Theorem 9** ([5]) Let \( \{\xi_i\}_{i=1}^{\infty} \) be an i.i.d. sequence with \( \xi_i > 0 \), \( E[\xi_i] = 1 \), and \( v := V[\xi_i^2] < \infty \). Then, for \( X = \{X_k\}_{k=1}^{\infty} = \{\xi_1 + \cdots + \xi_k\}_{k=1}^{\infty} \), it holds that the distribution of \( \sqrt{x}(Q_X(x) + \log x) \) converges to \( \mathcal{N}(0, v) \) as \( x \to \infty \).
For i.i.d. sequences $\{\xi_i\}_{i=1}^\infty$ with $\xi_i > 0$, $E[\xi_i] = 1$, but $V[\xi_i^2] = \infty$, we have following limit theorem.

**Theorem 10** ([5]) (i) If $[0, \infty) \ni s \mapsto E[\xi_i^2; \xi_i \leq s] \in [0, \infty)$ is slowly varying at $\infty$, there exists a positive sequence $\{B_n\}_{n=1}^\infty$ such that

$$\frac{x}{B_{\lfloor x\rfloor}}(Q_X(x) + \log x) \rightarrow N(0, 1), \quad x \rightarrow \infty,$$

in distribution.

(ii) If there exist a $\beta \in (1, 2)$ and an $L(\cdot)$ which is slowly varying at $\infty$ such that

$$P(\xi_1 > x) \sim L(x)x^{-\beta}, \quad x \rightarrow \infty,$$

then there exists $\{B_n\}_{n=1}^\infty$ such that

$$\lim_{x \rightarrow \infty} E \left[ \exp \left( \sqrt{-1} t(\beta - 1)^{1/\beta} \frac{x}{B_{\lfloor x\rfloor}}(Q_X(x) + \log x) \right) \right] = \exp \left( \beta \int_0^\infty \left( e^{\sqrt{-1} \alpha y} - 1 - \sqrt{-1} \alpha y \right) \frac{dy}{y^{\beta+1}} \right).$$

These results with proofs will be written in a paper in near future.

### 4.2 The case of two dimensional random array of electric charges

We mentioned about the electro-static interpretation of random digamma function in § 3.1.2. In this context, a natural question arises: Suppose that unit electrical charges are located at random in an unbounded domain of $\mathbb{R}^2$. Then, can we define a renormalized Coulomb potential as a random variable?

**Example 3** Suppose that the distribution of the unit electrical charges are described by a Poisson random measure on the out side of centered circle $B(O, x)^c$ with the Lebesgue measure as the intensity. Then the Coulomb potential at $O$ will be expressed as

$$\int_x^\infty \frac{dN(\pi t^2)}{t} = \sqrt{\pi} \int_x^\infty \frac{dN(t)}{\sqrt{t}}$$

by a standard Poisson process $N(t)$, which is of course divergent. The renormalized potential would be

$$\sqrt{\pi} \int_{x^2}^\infty \frac{d\tilde{N}(t)}{\sqrt{t}}, \quad \tilde{N}(t) := N(t) - t,$$

but it is not well-defined because

$$E \left[ \left( \int_{x^2}^\infty \frac{d\tilde{N}(t)}{\sqrt{t}} \right)^2 \right] = \int_{x^2}^\infty \frac{dt}{t} = \infty.$$

To look at the situation closely, let us observe the following deterministic case.: The sequence $a = \{\sqrt{k}\}_{k=1}^\infty$ is zeta regularizable, because the corresponding zeta function is $z(s) = \zeta(s/2)$. Hence by Theorem 2 in [4] and Theorem 1.8 in [3], we have

$$z \prod_{k=1}^\infty \left( 1 + \frac{x}{\sqrt{k}} \right) = \exp \left( \zeta \left( \frac{1}{2} \right) x - \gamma \frac{t^2}{2} \right) \prod_{k=1}^\infty \left( 1 + \frac{x}{\sqrt{k}} \right) \exp \left( -\frac{x}{\sqrt{k}} + \frac{x^2}{2k} \right),$$
where
\[
\zeta\left(\frac{1}{2}\right) = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n} \right] = -1.46035\ldots
\]

For \( X = \{X_k\}_{k=1}^{\infty} := \{\xi_1 + \cdots + \xi_k\}_{k=1}^{\infty} \), partial sums of i.i.d. random variables, Assumption 1 is satisfied with \( \delta < 1/2 \), and then the corresponding zeta function
\[
Z(s) = \sum_{k=1}^{\infty} X_k^{-s}
\]
will become meromorphic in \( \text{Re} \ s > 1/2 \), but \( Z(1/2) \) may not be defined. This fact has something to do with the non-existence of the limit
\[
\int_{x^2}^{\infty} \frac{dN(t)}{\sqrt{t}} = \lim_{y \to \infty} \left[ \sum_{x^2 \leq X_k \leq y} \frac{1}{\sqrt{X_k}} - 2\sqrt{y} + 2x \right].
\]

References


