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Variance of randomized values of Riemann's zeta function on the critical line

Tomoyuki SHIRAI (Kyushu University) *

1 The Riemann zeta function and the Lindelöf hypothesis

The Riemann zeta function is defined as an absolutely convergent series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^{s}}\quad(\Re s>1)$$

in the right half plane $\Re s>1$ and it can be extended meromorphically to the whole complex plane with only a simple pole at $s=1$ with residue 1. It satisfies the functional equation as

$$\zeta(s)=2\Gamma(1-s)\sin\frac{\pi s}{2}(2\pi)^{s-1}\zeta(1-s)\quad(s\in\mathbf{C}). \tag{1}$$

So the values for $\Re s\leq 0$ is given by those for $\Re s\geq 1$. It is easy to see that

$$\zeta(\sigma)^{-1}\leq|\zeta(\sigma+it)|\leq\zeta(\sigma) \tag{2}$$

for $\sigma>1$ since $\zeta(s)^{-1}=\sum_{n=1}^{\infty}\mu(n)n^{-s}$ for $\mathrm{Re}\, s>1$, where $\mu(n)\in\{0,\pm 1\}$ is the Möbius function. Here we adopt the notation of the complex variable $s=\sigma+it$. This inequality (2) implies that there is no zero in the right half plane $\Re s>1$. For $\Re s<0$, $\Gamma(1-s)$ is non-zero, and $\sin\frac{\pi s}{2}$ has zeros of order 1 at $s=-2n, n\leq 0$ and $\zeta(1-s)$ has only a pole of order 1 at $s=0$. Therefore, $\zeta(s)$ has zeros only at even negative integers in $\Re s<0$. Hence other zeros are located inside the so-called critical strip $0\leq\Re s\leq 1$. The Riemann hypothesis says that

(RH). All non-real zeros lie on the critical line $\Re s=1/2$.

As everyone knows, it still remains open.

*This note is based on the talk on Oct. 16, 2007 at IIAS in Kyoto

The following figures are drawn by using Mathematica 5.2.

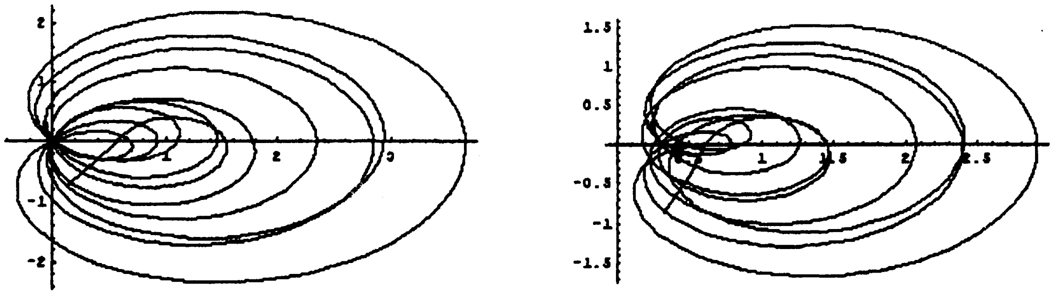

Figure 1: $\zeta(\frac{1}{2}+it)$ and $\zeta(\frac{3}{4}+it)$ $\quad (0 \leq t \leq 50)$.

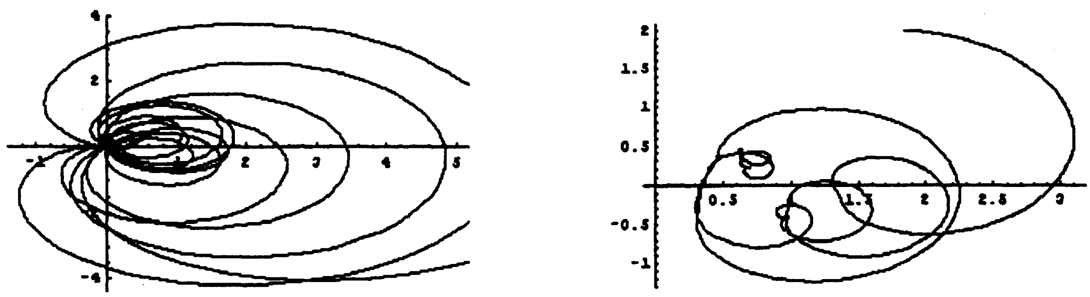

Figure 2: $\zeta(\frac{1}{2}+it)$ and $\zeta(\frac{3}{4}+it)$ $\quad (100000 \leq t \leq 100010)$.

Another conjecture closely related to (RH) is the so-called Lindelöf hypothesis which is about how fast $\zeta(\sigma+it)$ grows as the imaginary part t goes to ∞. We define the order $\mu(\sigma)$ as the least upper bound of the real numbers c such that $|\zeta(\sigma+it)||t|^{-c}$ is bounded as $t \to \infty$, i.e.,

$$\mu(\sigma) = \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log t}$$

For $\sigma > 1$, it is obvious that $\mu(\sigma) = 0$ because of (2). For $\sigma < 0$, by the functional equation (1) and the asymptotics of the Γ-function

$$|\Gamma(s)| = (2\pi)^{1/2}(|t|+2)^{\sigma-1/2}e^{-\pi|t|/2}(1+O(|t|+2)^{-1}))$$

uniformly for arbitrary strip $\sigma_1 \leq \sigma \leq \sigma_2$, we get

$$\zeta(\sigma+it) \asymp (|t|+2)^{-\sigma+1/2} \quad (\sigma < 0).$$

Hence $\mu(\sigma) = -\sigma + 1/2$ if $\sigma < 0$. The remaining problem is to see what occurs in between $0 \leq \sigma \leq 1$. It is known that $\mu(\sigma)$ is non-negative, continuous and non-increasing, and by the Phragmén-Lindelöf convexity principle, one can see that the order $\mu(\sigma)$ is convex. So we can conclude at least $\mu(\sigma) \le (1-\sigma)/2$ if $0 \le \sigma \le 1$. In particular, on the critical line $\sigma = 1/2$, we have

$$\zeta(1/2+it) \ll |t|^{1/4}.$$

Hardy-Littlewood showed that $\mu(\sigma) \le 1/6 = 0.16666$, after that many mathematicians improved again and again. So far, Huxley [4] has obtained the best result $\mu(\sigma) \le 32/205 \fallingdotseq 0.156098$. The ultimate goal of this problem is considered as the following.

(LH). The Lindelöf hypothesis says that $\mu(1/2) = 0$. In other words,

$$\zeta(1/2+it) \ll |t|^{\epsilon} \quad \forall\epsilon > 0.$$

Remark. The Riemann hypothesis (RH) implies the Lindelöf hypothesis (LH). Indeed, it is known that if (RH) is true, then

$$\zeta(1/2+it) = O\left(\exp\left(c\frac{\log t}{\log\log t}\right)\right)$$

for some c, and it is stronger than (LH).

(LH) says that the Riemann-zeta function grows very slowly, on the other hand, it is shown that for $1/2 < \sigma \le 1$, the set $\{\zeta(\sigma+it), t \in \mathbf{R}\}$ is dense in $\mathbf{C}$, and it is believed that this is also the case for $\sigma = 1/2$.

2 Asymptotic behavior of the variance of randomized values of Riemann's zeta function

The behavior of the mean-value, for example,

$$\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^{2k} dt \quad (k = 1, 2, \dots)$$

is easier to analyze than that of the order of $|\zeta(\sigma+it)|$ itself. The integral above is considered as the $2k$-th moment of the random variable $\zeta(\sigma+iU)$ with uniform random variable U on $[0, T]$, and $\zeta(\sigma+it)$ is considered as a (trivial) stochastic process $\{\zeta(\sigma+iS_t), t \ge 0\}$ with $\{S_t = t, t \ge 0\}$. It might be natural to study a random variable $\zeta(\sigma+iX)$ or a stochastic process $\{\zeta(\sigma+iS_t), t \ge 0\}$.

Let $\xi_1, \dots, \xi_n$ are i.i.d. Cauchy random variables whose probability law is given by

$$\frac{P(\xi_1 \in dx)}{dx} = \frac{1}{\pi(1+x^2)}$$

and its characteristic function is $e^{-|\lambda|}$. The Cauchy random walk is defined as the sum of i.i.d. Cauchy random variables, $S_n = \xi_1 + \cdots + \xi_n$, $n = 1, 2, \dots$. Lifshits and Weber [5] discussed the limiting behavior of randomized values of the Riemann zeta function on the critical line

$$\zeta_n = \zeta(\frac{1}{2} + iS_n)$$

sampled by the Cauchy random walk. The mean $E\zeta_n$ goes to 1 as $n \to \infty$ (see (5) below). For the second moment, they obtained the following results.

Theorem 1 ([5]). As $n \to \infty$, $\mathrm{var}(\zeta_n) = \log n + A + o(1)$. The constant A is explicitly given as an integral.

Also, they show an almost sure convergence result for the sum of randomized values.

Theorem 2 ([5]). For any real $b > 2$,

$$\sum_{k=1}^{n} \zeta(\frac{1}{2} + iS_k) = n + o(n^{1/2}(\log n)^b) \quad \text{a.s.}$$

as $n \to \infty$.

Since $\zeta(1/2 + it)$ lives beyond the domain of absolutely convergence, we cannot directly deal with the power series in the definition of the Riemann zeta function. The main idea of the proof of these results is to consider the random variable

$$\begin{aligned} Z_n(x) &= \sum_{k \le x} k^{-\frac{1}{2}-iS_n} - \int_0^x u^{-\frac{1}{2}-iS_n} du \\ &= \sum_{k \le x} k^{-\frac{1}{2}} e^{i(-\log k)S_n} - \int_0^x u^{-\frac{1}{2}} e^{i(-\log u)S_n} du \end{aligned} \tag{3}$$

which approximates $\zeta(1/2+iS_n)$ by using the approximate functional equation due to Hardy-Littlewood:

Proposition 1. Let $x > 1$.

$$\zeta(s) = \sum_{k \le x} k^{-s} - \int_0^x u^{-s} ds + O(x^{-\sigma})$$

uniformly on $\sigma_0 \le \sigma < 1$ and $|t| \le \frac{2\pi x}{C}$ with $C > 1$.

Proof. See Theorem 4.11 in [10] and recall the identity

$$\frac{x^{1-s}}{1-s} = \int_0^x u^{-s} ds$$

which is valid for $0 < \Re s < 1$. □

When S_n is the Cauchy random walk, all quantity such as the mean and the variance for $Z_n(x)$ can be computed explicitly (for example, see (5) below) and that is one reason why they take up the Cauchy random walk.

We would like to see what happens if we replace the Cauchy random walk by other stochastic processes. The natural candidate for this problem is the Lévy processes. In this note, we discuss the same problem as studied in [5] for a special class of Lévy processes, the symmetric α-stable process with $1 \le \alpha \le 2$ which includes the Cauchy random walk as a special case of $\alpha = 1$. Our argument goes essentially parallel to [5] with a little modification.

Here we recall the Lévy processes (cf. [1, 9]). The Lévy process $\{S_t, t \ge 0\}$ is a stochastic process which has stationary independent increments. The outstanding feature coming from stationary independent increments is that there exists a function $\Psi(\lambda)$ such that the characteristic function is given by the formula

$$Ee^{i\lambda S_t} = e^{-t\Psi(\lambda)} \tag{4}$$

The function $\Psi(\lambda)$ is called the characteristic exponent and has the Lévy-Khintchin representation:

$$\Psi(\lambda) = i\lambda a + \frac{c\lambda^2}{2} + \int_{\mathbf{R}} (1 - e^{i\lambda x} + i\lambda x I_{|x|\le 1})\Pi(dx),$$

where $a \in \mathbf{R}$, $c \ge 0$ and $\Pi(dx)$ is the so-called Lévy measure satisfying $\Pi(\{0\}) = 0$ and $\int_{\mathbf{R}}(1 \wedge x^2)\Pi(dx) < \infty$. The symmetric α-stable process is the special case where

$$\Psi(\lambda) = |\lambda|^\alpha \quad (0 < \alpha \le 2), \quad \Pi(dx) = |x|^{-\alpha-1}dx \quad (0 < \alpha < 2).$$

The case $\alpha = 1$ corresponds to the Cauchy process and the case $\alpha = 2$ the Brownian motion.

Remark. The symmetric α-stable process is recurrent when $1 \le \alpha \le 2$ and transient when $0 < \alpha < 1$.

By (3) and (4), it is easy to see that

$$EZ_t(x) = \sum_{k \le x} k^{-\frac{1}{2}} e^{-t\Psi(-\log k)} - \int_0^x u^{-\frac{1}{2}} e^{-t\Psi(-\log u)} du$$

and, in particular, when $\alpha = 1$, i.e., $\Psi(\lambda) = |\lambda|$, we see that

$$\begin{aligned} \lim_{x \to \infty} EZ_t(x) &= \zeta(t + 1/2) - \frac{2t}{t^2 - 1/4} \\ &\to 1 \end{aligned} \tag{5}$$

as $t \to \infty$. Remark that by the property that $\zeta(\bar{s}) = \overline{\zeta(s)}$, the mean $E\zeta(\sigma + iX)$ is real when X is a symmetric random variable in the sense that X and $-X$ have the same law. For a wide class of Lévy processes even in the non-symmetric case, $E\zeta(\sigma + iS_t)$ goes to 1 as $t \to \infty$.

How well does the random variable $Z_t(x)$ approximate $\zeta(\frac{1}{2} + iS_t)$?

Lemma 1. Suppose that X is a random variable with density $p(x)$ which satisfies $p(x) \ll |x|^{-\gamma}$ with $\gamma > 1$. Then, $\zeta(1/2+iX)$ is an L^2 random variable. Furthermore, there exists $M > 0$ such that $p(x)$ is differentiable for $|x| > M$ and satisfies $p(x) + \int_x^\infty |p'(u)|du \ll |x|^{-\gamma}$ with $\gamma > 3/2$, then $Z_X(x)$ converges to $\zeta(1/2+iX)$ in L^2 as $x \to \infty$, where $Z_X(x)$ is defined by (3) using X in place of S_n.

If we believe (LH), the integrability of $\zeta(1/2+iX)$ becomes much stronger. See Problem 3 below. We do not know whether (LH) is true, so instead, we use a mean value theorem

$$\int_0^T |\zeta(1/2+it)|^2 dt \sim T \log T \quad (T \to \infty)$$

for the proof of this lemma.

From now on, we consider the symmetric α-process with $1 \leq \alpha \leq 2$. Let $p_t(x)$ be the transition density of the α-stable process which is given by

$$p_t(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{izx - t|z|^\alpha} dz = \frac{1}{\pi} \Re \int_0^\infty e^{izx - tz^\alpha} dz.$$

It is clear from the above that the transition density has the scaling property

$$p_t(x) = t^{-1/\alpha} p_1(t^{-1/\alpha} x).$$

As in [12], by using the contour $\{z \in \mathbf{C}; \mathrm{Im}(izx - z^\alpha) = 0\}$, Cauchy's theorem gives us that

$$p_1(x) = \frac{\alpha}{\pi(\alpha-1)} x^{\frac{1}{\alpha-1}} \int_0^{\pi/2} \varphi(\theta) \exp\left(-x^{\frac{\alpha}{\alpha-1}} \varphi(\theta)\right) d\theta$$

and also

$$p_1'(x) = \frac{-1}{\pi(\alpha-1)} x^{\frac{2}{\alpha-1}} \int_0^{\pi/2} r(\theta)^2 \left(1 - \frac{\alpha \sin(\alpha-2)\theta}{\sin \alpha\theta}\right) \exp\left(-x^{\frac{\alpha}{\alpha-1}} \varphi(\theta)\right) d\theta$$

where

$$r(\theta) = \left(\frac{\cos\theta}{\sin\alpha\theta}\right)^{\frac{1}{\alpha-1}}, \quad \varphi(\theta) = r(\theta)^\alpha \frac{\cos(\alpha-1)\theta}{\cos\theta}.$$

The main contribution comes from near $\theta = \pi/2$ as $x \to \infty$, we see that

$$p_1(x) + \int_x^\infty |p_1'(u)| du \leq C|x|^{-\alpha-1}.$$

Hence, Lemma 1 shows that $Z_t(x)$ approximates $\zeta(1/2+iS_t)$ in L^2 and it is enough to consider $Z_t(x)$ in order to compute the variance of $\zeta(1/2+iS_t)$.

Remark. For the case $0 < \alpha < 1$, there are similar integral representations for the transition density by duality around $\alpha = 1$.

Proposition 2. Let S_t be the symmetric α-stable process with $1 \leq \alpha \leq 2$ and $\zeta_n = \zeta(1/2 + iS_n)$. Then, as $n \to \infty$
(i) $E\zeta_n = 1 + O(n^{-1/\alpha})$.
(ii) $\mathrm{var}(\zeta_n) = O(\log n)$.
(iii) $\mathrm{cov}(\zeta_n, \zeta_m) = O(m^{-1/\alpha}) \vee O(C_\alpha^{n-m})$ whenever $n > m$ for some $C_\alpha > 0$.

From this proposition, it is easy to see that

$$\mathrm{var}(\sum_{k=i}^{j} \zeta_k) = \begin{cases} O((j-i)j^{1-1/\alpha}) & 1 \leq \alpha \leq 2 \\ O((j-i)\log j) & \alpha = 1. \end{cases}$$

and we obtain the following almost sure convergence.

Theorem 3. Let S_t be the symmetric α-stable process with $1 \leq \alpha \leq 2$. Then,

$$\sum_{k=1}^{n} \zeta(1/2 + iS_n) = n + o(n^{1-\frac{1}{2\alpha}}(\log n)^b) \quad a.s.$$

for any $b > 3/2$ if $1 < \alpha \leq 2$; for any $b > 2$ if $\alpha = 1$.

Although the random variables $\zeta_n - 1, n = 1, 2, \ldots$ in our case are not orthogonal but weakly dependent, an extension of Rademacher-Menchoff type almost sure convergence theorem can be applied. Here we use a simplified version of a result obtained in [11].

Proposition 3 ([11]). Let $\xi_1, \xi_2, \ldots$ be random variables and $\Phi : (0, \infty) \to (0, \infty)$ a concave nondecreasing function. Suppose

$$E\left|\sum_{k=i}^{j} \xi_k\right|^2 \leq \Phi(j)(j-i)$$

for any $1 \leq i < j$. Then,

$$\frac{\sum_{k=1}^{n} \xi_k}{(n\Phi(n))^{1/2} \log^{\tau}(1+n)} \to 0 \quad a.s.$$

with $\tau > 3/2$.

3 Some related problems

Here we list some problems related to this topic.

Problem 1. What happens if we consider Dirichlet's L-functions in place of the Riemann zeta function?

Problem 2. Lifshits and Weber gave the exact asymptotics of the variance in Theorem 1, while here we just show $O(\log n)$ for it. Give the exact asymptotics for the α-stable case.

Problem 3. Heuristically, (LH) implies that

$$E|\zeta(1/2+iX)|^p \ll E|X|^{p\epsilon} < \infty$$

if $\epsilon > 0$ is arbitrary small. Does it hold that $\zeta(\frac{1}{2}+iX) \in \bigcap_{p>0} L^p$ for a random variable $X \in \bigcup_{p>0} L^p$?

Problem 4. Here we only dealt with the case where S_t is the α-stable process with $1 \leq \alpha \leq 2$. As mentioned in the remark, the symmetric α-stable process is transient or recurrent according to $0 < \alpha < 1$ or $1 \leq \alpha < 2$. Since the Lindelöf hypothesis corresponds to the deterministic (transient) process $S_t = t$, the transient case should be more appropriate to study. What happens for the α-stable process with $0 < \alpha < 1$? Moreover, what happens for α-stable subordinator(increasing Lévy process) with $0 < \alpha < 1$ or, furthermore, for general Lévy processes?

Problem 5. Related to Problem 4, we consider the Brownian motion S_t with constant drift whose characteristic exponent is $\Psi(\lambda) = \epsilon\lambda^2/2 + ia\lambda$ ($\epsilon > 0, a \in \mathbf{R}$). In this case, we can compute directly the moments of $\zeta(1/2+iS_t)$ by using another integral representation of the Riemann zeta function

$$\zeta(s) = s\int_0^\infty Q(x)x^{-(s+1)}dx \quad (0 < \Re s < 1)$$

where $Q(x) = [x] - x$. If $E|S_t| < \infty$,

$$E\zeta(\sigma+iS_t) = \int_{-\infty}^\infty Q(e^{-u})e^{\sigma u}(\sigma - t\Psi'(u))e^{-t\Psi(u)}du$$

and if S_t is the Brownian motion with constant drift

$$\begin{aligned}&E[|\zeta(\sigma+iX_t)|^2]\\&= 2\int_0^\infty Q(x)x^{-(1+2\sigma)}dx\int_0^\infty Q(xe^{-u})e^{\sigma u}\Re\left(\{\sigma^2 + t\Psi''(u) - (t\Psi'(u))^2\}e^{-t\Psi(u)}\right)du.\end{aligned}$$

In particular, when $\Psi(\lambda) = \frac{\epsilon\lambda^2}{2} + ia\lambda$, setting $b = \epsilon^{-1}a$ and $T = t\epsilon$, we obtain

$$E[|\zeta(\sigma + iX_t)|^2] = 2\int_0^\infty Q(x)x^{-(1+2\sigma)}dx \times \int_0^\infty Q(xe^{-u})e^{\sigma u}f(u;T,b)e^{-T\frac{u^2}{2}}du$$

where

$$f(u;T,b) = (\sigma^2 + T - T^2(u^2 - b^2))\cos(buT) - 2T^2ub\sin(buT).$$

Compute the asymptotic behavior of the second moment as $T \to \infty$, especially when b is non-zero.

Problem 6. It is well-known that (LH) is equivalent to one of the following conditions

1. $E[|\zeta(1/2 + iU_T)|^{2k}] = O(T^\epsilon), \quad k = 1, 2, \ldots$
2. $E[|\zeta(\sigma + iU_T)|^{2k}] = O(T^\epsilon), \quad \sigma > \frac{1}{2}, k = 1, 2, \ldots$
3. $\lim_{T\to\infty} E[|\zeta(\sigma + iU_T)|^{2k}] = \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}}, \quad \sigma > \frac{1}{2}, k = 1, 2, \ldots$

where U_T is a random variable uniformly distributed on $[0, T]$ and $d_k(n)$ denotes the number of ways of representing integer n as a product of k factors. What kind of family of random variables has the same property as $\{U_T, T \gg 1\}$ does.

Problem 7. It is known that $\frac{\log\zeta(1/2+iU_T)}{\sqrt{\log\log T}} \to N_{\mathbb{C}}(0,1)$ as $T \to \infty$ where U_T is distributed uniformly on $[0, T]$ (cf. [3]). Discuss limit theorems or the large deviations for the process $\{\zeta(1/2 + iS_t), t \geq 0\}$ or $\{\log\zeta(1/2 + iS_t), t \geq 0\}$.

Problem 8. The following normalized zeta function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s).$$

is sometimes used. This is an entire function which satisfies the following simple functional equation:

$$\xi(s) = \xi(1 - s).$$

One of the important feature of this version is that it is *real-valued* on the critical line. This function is related to probability theory in this way: let Y be a random variable defined by

$$Y = \sqrt{\frac{2}{\pi}}(\max_{0\leq s\leq 1} b_s - \min_{0\leq s\leq 1} b_s)$$

where $\{b_s, 0 \leq s \leq 1\}$ is the standard Brownian bridge pinned at $(t, x) = (0, 0)$ and $(t, x) = (1, 0)$. So Y is the range of the Brownian bridge. The interesting connection between this random variable and the Riemann zeta function is given by the following relation

$$EY^s = 2\xi(s) \quad (s \in \mathbf{C})$$

Another related random variable is defined by

$$S = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\gamma_n}{n^2}$$

where $\{\gamma_n, n = 1, 2, \dots\}$ are i.i.d. random variables such that the density $P(\gamma_1 \in dx)/dx = xe^{-x}$. It is known that

$$Y \stackrel{d}{=} \left(\frac{\pi}{2} S\right)^{1/2}.$$

See [2] for this topic. Are there any approaches to the growth problem by using this probabilistic interpretation?

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