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Universality for Dirichlet L-functions and Lerch zeta-functions

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Abstract

In this article, we will give a on the survey theory of universality for Dirichlet L-functions and Lerch zeta-functions (Sections 1 to 4), then state main results talked in the conference "Number theory and probability theory" (Section 5) and problems on the universality for zeta-functions (Section 6).

Contents

This article treats only a small part of the theory. If you are interested in the theory of universality, see [6] and [20].

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1 Introduction

In this section we define the Riemann zeta-function, Dirichlet L-functions, and Lerch zeta-functions. We mainly treat these three types of functions because we do not assume the background knowledge of number theory. Next we explain some well-known properties of these functions, for example, analytic continuation and functional equations. We discuss the number of non-trivial of zeros of the Riemann zeta function, the Riemann hypothesis, zero-free region of $\zeta(s)$, and the relation with the prime number theorem.

1.1 Definitions

Definition 1.1. *The Riemann zeta function is a function of a complex variable $s = \sigma + it$, for $\sigma > 1$ given by*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \Bigl(1 - \frac{1}{p^s}\Bigr)^{-1}, \tag{1.1}$$

where the letter p is a prime number, and the product of $\prod_p$ is taken over all primes.

The Dirichlet series and the Euler product of $\zeta(s)$ converges absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane.

By partial summation, we have

$$\zeta(s) = \sum_{n \le N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + s \int_N^{\infty} \frac{[x] - x}{x^{s+1}} dx,$$

here and in the sequel $[x]$ denotes the maximal integer less than or equal to x. The above formula gives the analytic continuation for $\zeta(s)$ to the half-plane $\sigma > 0$ with a simple pole at $s = 1$ of residue 1. Riemann gave the functional equation

$$\pi^{-s/2}\Gamma\Bigl(\frac{s}{2}\Bigr)\zeta(s) = \pi^{-(1-s)/2}\Gamma\Bigl(\frac{1-s}{2}\Bigr)\zeta(1-s), \tag{1.2}$$

where $\Gamma(s)$ denotes Euler's Gamma-function. We can continue $\zeta(s)$ analytically to the whole complex plane except for $s = 1$.

Next we define Dirichlet characters and Dirichlet L-functions. Let q be a positive integer. A Dirichlet character χ mod q is a non-vanishing group homomorphism from the group $(\mathbb{Z}/q\mathbb{Z})^*$ of prime residue classes modulo q to $\mathbb{C}$. The character, which is identically one, is called principal, and denoted by χ_0. By setting $\chi(n) = \chi(a)$ for $n \equiv a \mod q$, we can extend the character to a completely multiplicative arithmetic function on $\mathbb{Z}$.

Definition 1.2. *For $\sigma > 1$, the Dirichlet L-function $L(s, \chi)$ attached to a character χ mod q is given by*

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \Bigl(1 - \frac{\chi(p)}{p^s}\Bigr)^{-1}. \tag{1.3}$$

The Riemann zeta function $\zeta(s)$ may be regarded as the Dirichlet L-function to the principal character χ_0 mod 1. It is possible that for values of n coprime with q the character $\chi(n)$ may have a period less than q. If so, we say that χ is imprimitive, and otherwise primitive. Every non-principal imprimitive character is induced by a primitive character. Two characters are non-equivalent if they are not induced by the same character. Characters to a common modulus are pairwise non-equivalent.

It is well-known that if χ is a non-principal Dirichlet character, $L(s,\chi)$ converges for $\sigma > 0$ according to Abel's partial summation. We can show that $L(s,\chi)$ is continued analytically to $\mathbb{C}$, similarly to the case of the Riemann zeta-function, and regular at $s = 1$ if and only if χ is non-principal by partial summation. Furthermore, Dirichlet L-functions to primitive characters satisfy a functional equation of the Riemann-type.

Finally, we define the Lerch zeta-function.

Definition 1.3. *The Lerch zeta-function $L(\lambda, \alpha, s)$, for $0 < \lambda \le 1$, $0 < \alpha \le 1$ and $\Re(s) > 1$, is defined by*

$$L(\lambda, \alpha, s) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+\alpha)^s}. \tag{1.4}$$

When $\lambda = 1$, the Lerch-zeta function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function $\zeta(s,\alpha)$. If $\lambda \ne 1$, the function $L(\lambda, \alpha, s)$ is analytically continuable to an entire function. But the function $\zeta(s,\alpha)$ is analytically continuable to a meromorphic function, which has a simple pole at $s = 1$. We can see that $L(\lambda, \alpha, s)$ converges for $\sigma > 0$ according to Abel's partial summation when $\lambda \ne 1$. The Lerch zeta-function also has the functional equation. It should be noted that the Dirichlet L-function is written by a linear combination of Hurwitz zeta-functions,

$$L(s,\chi) = \sum_{r=1}^{q} \sum_{n=0}^{\infty} \frac{\chi(r+nq)}{(r+nq)^s} = \sum_{r=1}^{q} \chi(r) \sum_{n=0}^{\infty} \frac{1}{(r+nq)^s} = q^{-s} \sum_{r=1}^{q} \chi(r) \zeta(s, r/q).$$

1.2 Zeros of the Riemann zeta-function

In view of the Euler product (1.1), it is seen easily that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$. It follows from the functional equation (1.2) and basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\sigma < 0$ exactly at the so-called trivial zeros $s = -2n$, $n \in \mathbb{N}$. All other zeros of $\zeta(s)$ are said to be non-trivial, and we denote them by $\rho = \beta + i\gamma$. Obviously, they have to lie inside the strip $0 \le \sigma \le 1$. The functional equation (1.2) and the identity $\zeta(\overline{s}) = \overline{\zeta(s)}$ shows some symmetries of $\zeta(s)$. Especially, the non-trivial zeros of $\zeta(s)$ are distributed symmetrically with respect to the real axis and to the vertical line $\sigma = 1/2$.

In 1859, Riemann conjectured that the number $N(T)$ of non-trivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma \le T$ (counted with multiplicity) satisfies an asymptotic formula. This was proved by von Mangoldt in 1895 who found more precisely

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

Riemann worked the function $t \mapsto \zeta(1/2+it)$ and wrote that very likely all roots T are real, i.e., all non-trivial zeros lie on the so-called critical line $\sigma = 1/2$. This is the famous, yet unproved Riemann hypothesis which we rewrite equivalently as

Riemann hypothesis. $\zeta(s) \neq 0$ for $\sigma > 1/2$.

A classical density theorem due to Bohr and Landau that states the most of the zeros lie close to the critical line. Denote by $N(\sigma, T)$ the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ for which $\beta > \sigma$ and $0 < \gamma \leq T$ (counted with multiplicity) Bohr and Landau proved that for any fixed $1/2 < \sigma < 1$

$$N(\sigma, T) = O(T^{4\sigma(1-\sigma)+\varepsilon}), \tag{1.5}$$

here and in the sequel ε stands for an arbitrarily small positive constant, not necessarily the same at each appearance. Hence almost all zeros of the Riemann zeta-function are clustered around the critical line.

Next we introduce information on the distribution of the non-trivial zeros. In 1896, de la Vallée-Poussin showed that

$$\zeta(s) \neq 0, \qquad \sigma \geq 1 - c\big(\log(|t|+2)\big)^{-1},$$

where c is some positive constant. The largest known zero-free region for $\zeta(s)$ was found by Vinogradov and Korobov (in 1958, independently) who proved

$$\zeta(s) \neq 0, \qquad \sigma \geq 1 - c\big(\log(|t|+2)\big)^{-1/3}\big(\log\log(|t|+3)\big)^{-2/3}.$$

Finally we present relations between the Riemann zeta-function and the distribution of prime numbers. Gauss conjectured in 1791 for the number $\pi(x)$ of primes $p \leq x$ the asymptotic formula

$$\pi(x) \sim \mathrm{li}(x), \qquad \mathrm{li}(x) := \int_2^x \frac{du}{\log u}.$$

By using the zero-free region proved by Vinogradov and Korobov, we obtain the prime number theorem with the strongest known reminder term,

$$\pi(x) = \mathrm{li}(x) + O\big(x\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})\big).$$

On the other hand, in 1900 von Koch showed that for fixed $1/2 \leq \theta < 1$,

$$\pi(x) - \mathrm{li}(x) = O(x^{\theta+\varepsilon}) \quad\Longleftrightarrow\quad \zeta(s) \neq 0 \text{ for } \sigma > \theta.$$

Hence we can see that studying the zeros of $\zeta(s)$ is important and difficult (since no one can improve the zero-free region in about 50 years). In the next section, we will see an answer of the question "Why is determining the zeros of $\zeta(s)$ so difficult ?" by universality.

2 Universality and self-similarity

Firstly we briefly introduce the history of universality, which means any non-vanishing analytic function can be uniformly approximated by shifts on the Riemann zeta-function. Next, we sketch the proof of the universality theorem. Finally, we present the notion of almost periodicity and self-similarlity. These conceptions are in some sense equivalent to the (generalized) Riemann hypothesis.

2.1 Universality

The distribution of the values of the Riemann zeta function $\zeta(\sigma + it)$ for fixed σ and variable $t > 0$ was investigated by H. Bohr. In 1914, he showed the following denseness theorem, as a joint work with Courant.

Theorem 2.1 (see [11, Theorem 1]). *For any fixed σ satisfying $1/2 < \sigma < 1$, the set $\{\zeta(\sigma + it) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}$.*

This theorem should be compared with the following inequality,

$$0 < |\zeta(s)| \leq \zeta(\sigma), \qquad \sigma > 1.$$

This theorem of Bohr was the first remarkable denseness result for the Riemann zeta function and it was generelized by S. M. Voronin in 1972. He proved that if $s_1, s_2, \ldots, s_m$ are distinct points lying in the strip $1/2 < \sigma < 1$, and $h > 0$ is an arbitrary fixed number then the sequence

$$\big(\zeta(s_1 + inh), \zeta(s_2 + inh), \ldots, \zeta(s_m + inh)\big) \qquad n \in \mathbb{N}$$

is dense in $\mathbb{C}^m$. He also obtained that the sequence

$$\big(\zeta(s_0 + inh), \zeta'(s_0 + inh), \ldots, \zeta^{(m-1)}(s_0 + inh)\big) \qquad n \in \mathbb{N}$$

is dense in $\mathbb{C}^m$ for any fixed s_0 such that $1/2 < \Re(s_0) \leq 1$.

The question on differential properties of the Riemann zeta function was raised by D. Hilbert in 1900 during the International Congress of Mathematicians. He noted that an algebraic-differential independence of $\zeta(s)$ can be proved by the algebraic-differential independence of the gamma function $\Gamma(s)$ and the functional equation of $\zeta(s)$.

As a generalization of this mention of Hilbert, we obtain the following theorem by using the above theorem of Voronin.

Theorem 2.2 (see [8, Theorem 6.6.3]). *Let F_k, $k = 0, 1, \ldots, n$, be continuous functions, and let*

$$\sum_{k=0}^{n} s^k F_k\big(\zeta(s), \zeta'(s), \ldots, \zeta^{(j-1)}(s)\big) = 0$$

be valid identically for $s \in \mathbb{C}$. Then $F_k \equiv 0$, for $k = 0, 1, \ldots, n$.

A natural next step is to study the situation on infinite dimensional spaces, namely on function spaces. Concerning this problem, in 1975, S. M. Vornin [22] showed the next theorem, which is now called the universality. We prepare some notation for universality. By meas$\{A\}$ we denote the Lebesgue measure of the set A, and, for $T > 0$, we use the notation

$$\nu_T^\tau\{\ldots\} := T^{-1}\text{meas}\{\tau \in [0,T] : \ldots\}$$

where in place of dots some condition satisfied by τ is to be written.

Theorem 2.3 (see [8, Theorem 6.5.1] or [20, Theorem 1.7]). *Let $0 < r < 1/4$ and suppose that $g(s)$ is a non-vanishing continuous function on the disk $|s| \leq r$ which is analytic in the interior. Then for any $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \nu_T^\tau\left\{\max_{|s|\leq r}\left|\zeta(s+3/4+i\tau) - f(s)\right| < \varepsilon\right\} > 0. \tag{2.1}$$

This theorem means that any non-vanishing analytic function can be uniformly approximated by certain purely imaginary shifts of the Riemann zeta-function $\zeta(s)$. Moreover the set of approximating shifts has positive lower density.

Reich [19] and Bagchi [1] improved Voronin's universality theorem significantly in replacing the disk by an arbitrary compact subset in the critical strip $D := \{s \in \mathbb{C} : 1/2 < \Re(s) < 1\}$ with connected complement, and by giving a lucid proof in the language of probability theory. The strongest version of Voronin's theorem has the form;

Theorem 2.4 (see [8, Theorem 6.5.2] or [20, Theorem 1.9]). *Let K be a compact subset of the strip D with connected complement, and $f(s)$ be a non-vanishing function analytic in the interior of K and continuous on K. Then for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T^\tau\left\{\sup_{s\in K}\left|\zeta(s+i\tau) - f(s)\right| < \varepsilon\right\} > 0. \tag{2.2}$$

It should be noted that the restriction on $f(s)$ can not be removed. This is closely related to the Riemann hypothesis and self-similarlity (see Theorem 2.9).

2.2 Sketch of the proofs of universality theorems

We sketch the proof of Theorem 2.4. We will prove the universality theorem for $L(s,\chi)$ instead of $\zeta(s)$. The proof of the universality theorem is divided into two parts, a limit theorem and a denseness lemma. Firstly, we show the limit theorem for Dirichlet L-functions.

We quote definitions and theorems from [6] and [20]. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta.

Let $\mathfrak{B}(S)$ stand for the class of Borel sets of the space S. Define on $(H(D), \mathfrak{B}(H(D)))$ the probability measures

$$P_{DL}^{T}(A) := \nu_T^{\tau}\{L(s+i\tau, \chi) \in A\}, \qquad A \in \mathfrak{B}(H(D)).$$

What we need is a limit theorem in the sense of weak convergence of the probability measure for P_{DL}^{T} as $T \to \infty$, with an explicit form of the limit measure. Denote by γ the unit circle on $\mathbb{C}$, and let $\Omega := \prod_p \gamma(p)$, where $\gamma(p) = \gamma$ for all primes p. With the product topology and pointwise multiplication the infinite dimensional torus Ω is a compact topological Abelian group.

Denoting by m_H the probability Haar measure on $(\Omega, \mathfrak{B}(\Omega))$, we obtain a probability space $(\Omega, \mathfrak{B}(\Omega), m_H)$. We define the $H(D)$-valued random element $L(s, \chi \,|\, \omega)$ by

$$L(s, \chi \,|\, \omega) := \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}, \qquad s \in D, \quad \omega \in \Omega. \tag{2.3}$$

Let P_{DL} stand for the distribution of $L(s, \chi \,|\, \omega)$, i.e.

$$P_{DL}(A) := m_H(\omega \in \Omega : L(s, \chi \,|\, \omega) \in A), \qquad A \in \mathfrak{B}(H(D)).$$

Proposition 2.5 (see [6, Theorem 5.1.8]). *The probability measure P_{DL}^{T} converges weakly to P_{DL} as $T \to \infty$.*

This is called the "limit theorem". The key of the proof is the uniqueness of decomposition of integers into the product of prime numbers.

Next we consider the support of the measure P. Let $H^m(D) := H(D) \times \cdots \times H(D)$. We recall that the minimal closed set $S_P \subseteq H^m(D)$ such that $P(S_P) = 1$ is called the support of P. The set S_P consists of all $\underline{f} \in H^m(D)$ such that for every neighborhood V of $\underline{f}$ the inequality $P(V) > 0$ is satisfied. The support of the distribution of the random element X is called the support of X and is denoted by S_X.

Lemma 2.6 (see [20, Lemma 12.7]). *Let $\{X_n\}$ be a sequence of independent $H^m(D)$-valued random elements, and suppose that the series $\sum_{n=1}^{\infty} X_n$ converges almost surely. Then the support of the sum of this series is the closure of the set of all $\underline{f} \in H^m(D)$ which may be written as a convergent series $\underline{f} := \sum_{n=1}^{\infty} \underline{f}_n$, $\underline{f}_n \in S_{X_n}$.*

We quote well-known results for the weak convergence of probability measures. Suppose P_n and P are probability measures on $(S, \mathfrak{B}(S))$ for some metric space S.

Lemma 2.7. *P_n converges weakly to P as $n \to \infty$ if and only if $\liminf_{n\to\infty} P_n(G) \geq P(G)$ for all open sets $G \in \mathfrak{B}(S)$.*

The next lemma are commonly used for proving universality theorems.

Lemma 2.8 (see [6, Theorem 6.3.10]). *Let $\{f_n\}$ be a sequence in $H(D)$ which satisfies :*
(a) If μ is a complex measure on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact support contained in D such that $\sum_{n=0}^{\infty} |\int_{\mathbb{C}} f_n d\mu| < \infty$, then $\int_{\mathbb{C}} s^r d\mu(s) = 0$, for all $r \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
(b) The series $\sum_{n=0}^{\infty} f_n$ converges in $H(D)$;
(c) For any compact set $\mathcal{K} \subset D$, $\sum_{n=0}^{\infty} \sup_{s\in\mathcal{K}} |f_n(s)|^2 < \infty$.
Then the set of all convergent series $\sum_{n=0}^{\infty} b_n f_n$ with $|b_n| = 1$ is dense in $H(D)$.

Now we show the outline of the proof of Theorem 2.4 (see [6, Section6] and [20, Section 5] for the details). We define $T(D) := \{x \in H(D) : x(s) \neq 0 \text{ for all } s \in D \text{ or } x \equiv 0\}$. By using Lemmas 2.6 and 2.8, we see that the support of $\log L(s, \chi \mid \omega)$ is $H(D)$. Hence the support of $L(s, \chi \mid \omega)$ contains $T(D)$. Now suppose $f(s) \in T(D)$. Denote by Φ the set of functions $\phi \in H(D)$ such that $\sup_{s \in K} |\phi(s) - f(s)| < \varepsilon$. By Proposition 2.5, Lemma 2.7 and the fact that Φ is open, we have

$$\liminf_{T\to\infty} \nu_T^T\Big\{\sup_{s\in K}\big|L(s+i\tau,\chi) - f(s)\big| < \varepsilon\Big\} = \liminf_{T\to\infty} P_{DL}^T(\Phi) \geq P_{DL}(\Phi) > 0. \tag{2.4}$$

This (outline of) proof the theorem is the proof when function $f(s)$ have a non-vanishing analytic continuation to D. Note that here the restriction on K to have connected complement is not necessary. When $f(s)$ is as in Theorem 2.4, we apply a complex analogue of Weierstrass' approximation theorem, that is the theorem of Mergelyan on the approximation of analytic functions by polynomials (see [20, Theorem 5.15]).

2.3 Self-similarity

Firstly, we define the almost periodicity.

An analytic function $f(s)$, defined on some vertical strip $a < \sigma < b$, is called almost periodic in the sense of Bohr (uniformly almost periodic) if, for any positive $\varepsilon > 0$, and any α, β with $a < \alpha < \beta < b$, there exists a length $l := l(f, \alpha, \beta, \varepsilon) > 0$ such that every interval (τ_1, τ_2) of length l contains an almost period of f relatively to ε in the closed strip $\alpha \leq \sigma \leq \beta$, i.e., there exists a number $d \in (\tau_1, \tau_2)$ such that

$$\big|f(\sigma + id + i\tau) - f(\sigma + i\tau)\big| < \varepsilon, \qquad \alpha \leq \sigma \leq \beta, \quad \tau \in \mathbb{R}.$$

Bohr [4] proved that every Dirichlet series is almost periodic in the sense of Bohr in its half-plane of absolute convergence. Moreover, Bohr showed if χ is non-principal, then the Riemann hypothesis for Dirichlet L-function $L(s, \chi)$ is equivalent to the almost periodicity in the sense of Bohr of $L(s, \chi)$ in $\sigma > 1/2$ (see also [20, Section 8.2]).

The condition on the character looks artificial but it is necessary. The Dirichlet L-function $L(s, \chi)$ with a non-principal character χ converges throughout the critical strip, but the Riemann zeta-function does not.

More than 50 years later from Bohr's paper [4], Bagchi in his Ph. D. Thesis [1], proved that the Riemann hypothesis is true if and only if the Riemann zeta function can be approximated by itself in the sense of universality.

In fact, his result asserts that the Riemann hypothesis is true if and only if, for any compact subset K in the strip D with connected complement and for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T^T\Big\{\sup_{s\in K}\big|\zeta(s+i\tau) - \zeta(s)\big| < \varepsilon\Big\} > 0.$$

In Bagchi [2, Theorem 3.7], it is shown that the above statement is also hold for $L(s, \chi)$ instead of $\zeta(s)$. We call this property "*self-similarity*" (strong recurrence). We extend Bagchi's result slightly to

Theorem 2.9 (see [20, Theorem 8.3]). *Let $\theta \geq 1/2$. Then $\zeta(s)$ is non-vanishing in the half-plane $\sigma > \theta$ if and only if, any $\varepsilon > 0$ with $\theta < \Re(z) < 1$, and for any $0 < r < \min\{\Re(z-\theta), 1-\Re(z)\}$,*

$$\liminf_{T\to\infty} \nu_T^\tau \Big\{ \sup_{|s-z|\leq r} \big|\zeta(s+i\tau) - \zeta(s)\big| < \varepsilon \Big\} > 0. \tag{2.5}$$

We sketch the proof (see [20, Theorem 8.3] for the details). If the Riemann hypothesis is true we can apply universality Theorem 2.3, which implies the self-similarty. The idea for the proof of the other implication is that if there is at least one zero to the right of the line $\sigma = \theta$, then the self-similarty (and Rouché's theorem) implies the existence of too many zeros with regard to the classic density theorem written by (1.5).

Note that self-similarity implies almost periodicity in the sense of Bohr. By modifying the proof of Theorem 3.3, written in the next section with $m = 1$, we can see that the Lerch zeta function $L(\lambda, \alpha, s)$ has the universality property (see also [8, Theorem 6.1.1]). Applying the universality Theorem 3.3 with $f(s) = L(\lambda, \alpha, s)$, we obtain the self-similarity for the Lerch zeta-function, which also implies the almost periodicity in the sense of Bohr. We remark that $L(\lambda, \alpha, s)$ has infinitely many zeros in the critical strip despite of self-similarlity for $L(\lambda, \alpha, s)$ when α is transcendental.

3 Joint universality

In this section we introduce joint universality. Firstly we show the joint universality for numerator part of Dirichlet L-functions and Lerch zeta-functions. Next we show the joint universality for denominator part of zeta-functions. The former type of joint universality was proved by Laurinčikas and Matsumoto [10], recently. Finally we introduce the joint universality for denominator part between the Riemann zeta-function and the Hurwitz zeta-function.

3.1 Joint universality for numerators

As a generalization of Theorem 2.4, Voronin also proved the next theorem, that means a collection of Dirichlet L-functions of non-equivalent characters uniformly approximate simultaneously non-vanishing analytic functions. In slightly different form this was also established by S. M. Gonek [5] and B. Bagchi [1], independently (all of these papers are unpublished doctoral theses).

Theorem 3.1 (see [20, Theorem 1.10]). *For $1 \leq l \leq m$, let $\chi_1 \mod q_1, \ldots, \chi_m \mod q_m$ be pairwise non-equivalent Dirichlet characters, K_l be a compact subset of the strip D with connected complement, and $f_l(s)$ be a non-vanishing function analytic in the interior of K_l and continuous on K_l for each $1 \leq l \leq m$. Then for every $\varepsilon > 0$*

$$\liminf_{T\to\infty} \nu_T^\tau \Big\{ \sup_{1\leq l\leq m} \sup_{s\in K_l} \big|L(s+i\tau, \chi_l) - f_l(s)\big| < \varepsilon \Big\} > 0.$$

We call this type of results joint universality for numerators. By using this theorem, we obtain the following theorem, the joint functional independence.

Proposition 3.2 (see [8, Theorem 6.6.3]). *Let F_k, $k = 0, 1, \ldots, n$, be continuous functions, and let*

$$\sum_{k=0}^{n} s^k F_k \left(L(s, \chi_1), \ldots, L^{(j-1)}(s, \chi_1), \ldots, L(s, \chi_m), \ldots, L^{(j-1)}(s, \chi_m) \right) = 0$$

be valid identically for $s \in \mathbb{C}$. Then $F_k \equiv 0$, for $k = 0, 1, \ldots, n$.

Next we consider the joint universality for numerators of Lerch zeta-functions. The following theorem is essentially included in Laurinčikas and Matsumoto [9].

Theorem 3.3 (see [9, Theorem 1]). *Let α be a transcendental number, $b_l, q_l \in \mathbb{N}$, q_l are distinct, $\lambda_l = b_l/q_l$, $(b_l, q_l) = 1$ and $b_l < q_l$. Let K_l be a compact subset of the strip D with connected complement and $f_l(s)$ be functions analytic in the interior of K_l and continuous on K_l for $1 \le l \le m$. Then for every $\varepsilon > 0$, it holds that*

$$\liminf_{T \to \infty} \nu_T^{\tau} \Big\{ \sup_{1 \le l \le m} \sup_{s \in K_l} \big| L(\lambda_l, \alpha, s + i\tau) - f_l(s) \big| < \varepsilon \Big\} > 0. \tag{3.1}$$

Remark 3.4. *The statement of [9, Theorem 1] is the joint universality of the Lerch zeta-functions $\{L(\lambda_l, \alpha_l, s)\}_{1 \le l \le m}$ where $\alpha_1, \ldots, \alpha_m$ are transcendental numbers. However some additional assumption is necessary to verify their proof. When $\alpha_1 = \cdots = \alpha_m$, their proof is valid as it is, which gives the above Theorem 3.3. On the other hand, in [10] they mentioned that their proof is also valid when $\alpha_1, \ldots, \alpha_m$ are algebraically independent over $\mathbb{Q}$ (see Thorem 3.5).*

Laurinčikas and Matsumoto [10] made the assumptions of λ_l weaker. Let $\lambda_1, \ldots, \lambda_m$ be arbitrary rational numbers with denominators $q_1, \ldots, q_m$, respectively. Denote by $k = [q_1, \ldots q_m]$ the least common multiple, and define

$$A := \begin{pmatrix} \exp(2\pi i \lambda_1) & \exp(2\pi i \lambda_2) & \cdots & \exp(2\pi i \lambda_m) \\ \exp(4\pi i \lambda_1) & \exp(4\pi i \lambda_2) & \cdots & \exp(4\pi i \lambda_m) \\ \vdots & \vdots & \ddots & \vdots \\ \exp(2k\pi i \lambda_1) & \exp(2k\pi i \lambda_2) & \cdots & \exp(2k\pi i \lambda_m) \end{pmatrix}.$$

Laurinčikas and Matsumoto [10] showed that if we have rank $(A) = r$ instead of the assumptions for λ_l in Theorem 3.3, then we have the joint universality for numerators of the Lerch zeta-functions $L(\lambda_l, \alpha, s)$. It should be noted that we can obtain the functional independence for the Lerch zeta-functions $L(\lambda_l, \alpha, s)$ by using Theorem 3.3.

Moreover Nagoshi [14] showed the joint universality for numerators of the Lerch zeta-functions $L(\lambda_l, \alpha, s)$ when $\lambda_1, \ldots, \lambda_m$ are algebraic real numbers such that $1, \lambda_1, \ldots, \lambda_m$

are linearly independent over $\mathbb{Q}$. And the author [15] showed the joint universality of the Lerch zeta-functions $L(\lambda_l, \alpha, s)$, where $\lambda_l := \lambda + l/m$, $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ or $\lambda = j/k \in \mathbb{Q} \setminus \mathbb{Z}$, and k, m are relatively prime.

In this subsection, we treat joint universality for Lerch zeta-functions in the case that the parameter α is common. In the next subsection, we consider the case when $\alpha_1, \ldots, \alpha_m$ are algebraically independent over $\mathbb{Q}$.

3.2 Joint universality for denominators

The first joint universality for denominators of Lerch zeta-functions is proved by Laurinčikas and Matsumoto [10].

Theorem 3.5 (see [10, Theorem 2]). *Suppose that $\alpha_1, \ldots, \alpha_m$ are algebraically independent over $\mathbb{Q}$ and that* rank $(A) = r$. *Let K_l be a compact subset of the strip D with connected compliment, and $f_l(s)$ be functions analytic in the interior of K_l and continuous on K_l for each $1 \le l \le m$. Then for every $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \nu_T^{\tau}\Big\{ \sup_{1\le l\le m} \sup_{s\in K_l} \big|L(\lambda_l, \alpha_l, s+i\tau) - f_l(s)\big| < \varepsilon \Big\} > 0. \tag{3.2}$$

Afterwards the author [16] removed the conditions for λ_l in Theorem 3.5.

Theorem 3.6 (see [16, Theorem 1.2]). *Suppose that $0 < \alpha_l < 1$ are algebraically independent numbers and $0 < \lambda_l \le 1$ for $1 \le l \le m$. Let K_l be a compact subset of the strip D with connected compliment, and $f_l(s)$ be functions analytic in the interior of K_l and continuous on K_l for each $1 \le l \le m$. Then for every $\varepsilon > 0$, it holds that*

$$\liminf_{T\to\infty} \nu_T^{\tau}\Big\{ \sup_{1\le l\le m} \sup_{s\in K_l} \big|L(\lambda_l, \alpha_l, s+i\tau) - f_l(s)\big| < \varepsilon \Big\} > 0. \tag{3.3}$$

Hence we can say that the assumption rank $(A) = r$ is important for the joint universality for numerator of Lerch zeta-functions. On the other hand, the condition α_ls are algebraically independent is essential for the joint universality for denominators of Lerch zeta-functions.

Moreover Mishou [13] showed the joint universality between the Riemann zeta-function and the Hurwitz zeta-function.

Theorem 3.7 (see [13, Theorem 2]). *Suppose $0 < \alpha < 1$ is a transcendental number. Let K_1 and K_2 be compact subsets of the strip D with connected complements. Assume that $f_j(s)$ is continuous on K_j and analytic in the interior of K_j for each $j = 1, 2$. In addition we suppose that $f_1(s)$ does not vanish on K_1. Then for all positive ε, we have*

$$\liminf_{T\to\infty} \nu_T^{\tau}\Big\{ \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon,\ \sup_{s\in K_2} |\zeta(s+i\tau, \alpha) - f_2(s)| < \varepsilon \Big\} \tag{3.4}$$

On the other hand, the author [15] showed the following statement. Let α be a transcendental number. Suppose that K is a compact subset of the strip D with connected complement and $f(s)$ is a function analytic in the interior of K and continuous on K. Then for every $\varepsilon > 0$ it holds that

$$\liminf_{T\to\infty} \nu_T^\tau \Big\{ \sup_{0\le j\le m-1} \sup_{s\in K} \big| m^{it} L(\lambda, \alpha + k/m, s+i\tau) - f(s)\big| < \varepsilon \Big\} > 0$$

where $k = 0, 1, \dots m-1$. The above inequality means that Lerch zeta-functions $L(\lambda, \alpha + k/m, s)$ can approximate only one function $f(s)$. Obviously, $\alpha + k/m$ are algebraically dependent over $\mathbb{Q}$.

4 Main results

In this section, we state the main results presented in the conference "Number theory and probability theory" (see also [18]). We show the joint universality for $\{L(s+idd_l\tau, \chi)\}_{l=1}^m$, where $1 = d_1, d_2, \dots, d_m$ are algebraic real numbers and linearly independent over $\mathbb{Q}$ and $d \in \mathbb{R} \setminus \{0\}$. From this property, we obtain that $\{L(s+idd_l\tau, \chi)\}_{l=j,k}$, where d_j and d_k are two of the above, has a kind of generalized self-similarity. Moreover, as a kind of generalization of the above theorems, we show the joint universality and the generalized self-similarity for $\{L(s+i\delta_l\tau, \chi)\}_{l=1}^2$, where $\delta_1 = 1$, for almost all δ_2.

4.1 Statement of main results

We may regard that the following theorem is a kind of joint universality for denominators.

Theorem 4.1. *Let $1 = d_1, d_2, \dots, d_m$ be algebraic real numbers and linearly independent over $\mathbb{Q}$ and $d \in \mathbb{R} \setminus \{0\}$. Suppose K_l is a compact subset of the strip D with connected complement, and $f_l(s)$ is a non-vanishing function analytic in the interior of K_l and continuous on K_l for each $1 \le l \le m$. Then for every $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \nu_T^\tau \Big\{ \sup_{1\le l\le m} \sup_{s\in K_l} \big| L(s+idd_l\tau, \chi) - f_l(s)\big| < \varepsilon \Big\} > 0. \tag{4.1}$$

The assumption for $1 = d_1, d_2, \dots, d_m$ in Theorem 4.1 is essential (see the proof of Theorem 4.1 and Remark 4.5).

By Putting $K := K_j = K_k$ and $1 \equiv f_j(s) \equiv f_k(s)$ in Theorem 4.1, and using

$$\big|L(s+idd_j\tau, \chi) - L(s+idd_k\tau, \chi)\big| \le \big|L(s+idd_j\tau, \chi) - 1\big| + \big|L(s+idd_k\tau, \chi) - 1\big|,$$

we also obtain the next theorem, which may be called "*generalized self-similarity*".

Theorem 4.2. *Let $d_1, d_2, \dots, d_m$ and d be as Theorem 4.1. Then for any compact subset K of the strip D with connected complement, and for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T^\tau \Big\{ \sup_{s\in K} \big| L(s+idd_j\tau, \chi) - L(s+idd_k\tau, \chi)\big| < \varepsilon \Big\} > 0. \tag{4.2}$$

We will also show the following theorems, which are generalizations of Theorems 4.1 and 4.2.

Theorem 4.3. *Let $\delta_1 = 1$, $f_1(s), f_2(s)$ and K_1, K_2 be as Theorem 4.1. Then for almost all $\delta_2 \in \mathbb{R}$ and every $\varepsilon > 0$, it holds that*

$$\liminf_{T\to\infty} \nu_T^\tau \Big\{ \sup_{1\le l\le 2} \sup_{s\in K_l} \big| L(s+i\delta_l\tau,\chi) - f_l(s) \big| < \varepsilon \Big\} > 0. \tag{4.3}$$

Theorem 4.4. *For almost all $\delta_2 \in \mathbb{R}$ and for any compact subset K of the strip D with connected complement, and for any $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \nu_T^\tau \Big\{ \sup_{s\in K} \big| L(s+i\tau,\chi) - L(s+i\delta_2\tau,\chi) \big| < \varepsilon \Big\} > 0. \tag{4.4}$$

If we could take $\delta_2 = 0$ in (4.4), we could obtain self-similarity, which is equivalent to the (generalized) Riemann hypothesis (see Theorem 2.9)!

Remark 4.5. *We have examples for which (4.1) is not true when d_1, d_2 are linearly dependent over $\mathbb{Q}$. For instance, the case $d_2 = -1$ is proved as follows. Let $K_1 = K_2$ be a one point set on the real axis in D. In this case, any τ satisfying $|L(\sigma+i\tau,\chi)+i| < \varepsilon$ must fulfill $|L(\sigma-i\tau,\chi)-i| < \varepsilon$ for any real Dirichlet character.*

It should be noted that $1, dd_1, dd_2$ are not always linearly independent over $\mathbb{Q}$ when $1, d_1, d_2$ are linearly dependent over $\mathbb{Q}$. For instance $1, d\sqrt{2}$ and $d\sqrt{3}$ are linearly dependent over $\mathbb{Q}$ when $d^{-1} = \sqrt{2}+\sqrt{3}$.

4.2 Sketch of the proofs of main theorems

We sketch the proofs of main theorems. See [18] for the details. Firstly, we quote Baker's theorem, which is a well-known result in transcendental number theory.

Lemma 4.6 (see [3, Theorem 2.4]). *The numbers $\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n}$ are transcendental for any algebraic numbers $\alpha_1, \dots \alpha_n$, other than 0 or 1, and any algebraic numbers $\beta_1, \dots, \beta_n$ with $1, \beta_1, \dots, \beta_n$ linearly independent over the rationals.*

By using this lemma, we obtain the following proposition, which is a key for the proof of Theorem 4.1.

Proposition 4.7. *Let p_n be the n-th prime number and $1 = d_1, d_2, \dots, d_m$ be algebraic real numbers which are linearly independent over $\mathbb{Q}$. Then $\{\log p_n^{d_l}\}_{n\in\mathbb{N}}^{1\le l\le m}$ is linearly independent over $\mathbb{Q}$.*

Now let $H^m(D) := H(D) \times \cdots \times H(D)$, where $H(D)$ is defined by the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Define on $(H^m(D), \mathfrak{B}(H^m(D)))$ the probability measure

$$\underline{P}_{DL}^T(A) := \nu_T^\tau\{(L(s+id_1\tau,\chi),\ldots,L(s+id_m\tau,\chi)) \in A\}, \qquad A \in \mathfrak{B}(H^m(D)).$$

Denoting by $\underline{m}_H$ the probability Haar measure on $(\Omega^m, \mathfrak{B}(\Omega^m))$, where $\Omega^m := \Omega \times \cdots \times \Omega$, we obtain a probability space $(\Omega^m, \mathfrak{B}(\Omega^m), \underline{m}_H)$. Let $\omega_l(p)$ be the projection of $\omega_l \in \Omega$ to the coordinate space $\gamma(p)$, and define on the probability space $(\Omega^m, \mathfrak{B}(\Omega^m), \underline{m}_H)$ the $H^m(D)$-valued random element $\underline{L}(s,\chi\,|\,\underline{\omega}) := \big(L(s,\chi\,|\,\omega_1),\ldots,L(s,\chi\,|\,\omega_m)\big)$, where

$$L(s,\chi\,|\,\omega_l) := \prod_p \left(1 - \frac{\chi(p)\omega_l(p)}{p^s}\right)^{-1}, \qquad s \in D, \quad 1 \le l \le m. \tag{4.5}$$

Let $\underline{P}_{DL}$ stand for the distribution of the random element $\underline{L}(s,\chi\,|\,\underline{\omega})$, i.e.

$$\underline{P}_{DL}(A) := \underline{m}_H(\underline{\omega} \in \Omega^m : \underline{L}(s,\chi\,|\,\underline{\omega}) \in A), \qquad A \in \mathfrak{B}(H^m(D)).$$

Proposition 4.8. *The probability measure $\underline{P}_{DL}^T$ converges weakly to $\underline{P}_{DL}^T$ as $T \to \infty$.*

The key for the proof of Theorem 3.5 (see also [10, Theorem 1]) or Theorem 3.7 (see also [13, Theorem 1]) is the linear independence over $\mathbb{Q}$ of $\{\log(n+\alpha_l)\}_{n\in\mathbb{N}_0}^{1\le l\le m}$ or $\{\log p_n\}_{n\in\mathbb{N}} \cup \{\log(n+\alpha)\}_{n\in\mathbb{N}_0}$, where $\alpha \in \mathbb{R}\setminus\overline{\mathbb{Q}}$, and $\alpha_1,\ldots,\alpha_m$ are algebraically independent over $\mathbb{Q}$. In the proof of Proposition 4.8, we use the fact that $\{\log p_n^{d_l}\}_{n\in\mathbb{N}}^{1\le l\le m}$ is linearly independent over $\mathbb{Q}$, proved by Proposition 4.7.

The next lemma when $m = 1$ coincides with Lemma 2.8

Lemma 4.9. *Let $\{\underline{f}_n\}$ be a sequence in $H^m(D)$ which satisfies :*
(a) If μ_l are complex measures on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact support contained with D such that $\sum_{n=1}^\infty |\int_{\mathbb{C}} f_{ln} d\mu_l| < \infty$, then $\int_{\mathbb{C}} s^r d\mu_l(s) = 0$, for all $1 \le l \le m$ and $r \in \mathbb{N}_0$;
(b) The series $\sum_{n=1}^\infty \underline{f}_n$ converges in $H^m(D)$;
(c) For any compact set $\mathcal{K}_l \subset D$, $\sum_{n=1}^\infty \sup_{1\le l\le m} \sup_{s\in\mathcal{K}_l} |f_{ln}(s)|^2 < \infty$.
Then the set of all convergent series $\sum_{n=1}^\infty (a_{1n}x_{1n},\ldots,a_{mn}x_{mn})$ with $|a_{ln}| = 1$, is dense in $H^m(D)$.

Hence by using Proposition 4.8 and Lemma 4.9, and modifying the proof of Theorem 2.4, we obtain Theorem 4.1. For proving Theorem 4.3, we use Lemma 4.9 and the following lemma instead of Proposition 4.8.

Lemma 4.10. *For almost all $\delta_2 \in \mathbb{R}$, $\{\log p_n\} \cup \{\log p_n^{\delta_2}\}$ is linearly independent over $\mathbb{Q}$.*

5 Problems

In this section, we give some problems on universality. We can also consider other problems, for example, universality for multiple zeta-functions and Selberg zeta-functions. But here we only treat problems on universality for Dirichlet L-functions and Lerch zeta-functions.

5.1 The non-existence of universality

Firstly, we consider the non-existence of (single) universality. Let $\eta = x+iy$, $x, y \geq 0$. We define the Hurwitz-Lerch zeta-funtion $L(\eta, \alpha, s)$ as a generalization of Lerch zeta-functions. When $y > 0$, $L(\eta, a, s)$ converges absolutely in the critical strip D. Hence $L(\eta, \alpha, s)$ with $y > 0$ does not have universality because $L(\eta, \alpha, s)$ obviously can not approximate the constant $2\sup_{s\in D} L(\eta, a, s) < \infty$. This is a trivial example of the non-existence of (single) universality. Hence the next question is important.

Problem 1. Find non-trivial examples for the non-existence of universality.

Now we have only trivial examples of non-existence of "single" universality. But for "joint" universality, we have a bit complicated example.

Consider the joint universality between $\zeta(s)$ and $\zeta^2(s)$. Obviously, $\zeta(s)$ and $\zeta^2(s)$ are unbounded in D. If there exist τ such that $\sup_{s\in K} |\zeta(s+i\tau) - 2| < 1$ for a compact set K, the τ must satisfy $\sup_{s\in K} |\zeta^2(s+i\tau) - 4| < 1$. Therefore $\zeta(s)$ and $\zeta^2(s)$ can not approximate simultaneously the constants 2 and 10. By generalizing the proof of this fact, we obtain the next example.

Example 5.1 (see [16, Proposition 6.2]). *Let α be a positive number and λ be a real number. If we put $\lambda_n = \lambda + n/m$, $\alpha_n = m\alpha$ for $0 \leq n \leq m-1$, and $\lambda_m = m\lambda$, $\alpha_m = \alpha + j/m$, $(0 \leq j \leq m-1)$, then there exists an $\varepsilon > 0$ and analytic functions $f_l(s)$ on K_l, for which there does not exist τ satisfying*

$$\sup_{0\leq l\leq m} \sup_{s\in K_l} \big| L(\lambda_l, \alpha_l, s+i\tau) - f_l(s) \big| \leq \varepsilon.$$

Needless to say, all $L(\lambda_l, \alpha_l, s+i\tau)$ in the above example are not absolute convergent in the critical strip D. This example is proved by using the following functional relation :

$$L\Big(m\lambda, a + \frac{j}{m}, s\Big) = m^{s-1} e^{-2\pi i\lambda j} \sum_{n=0}^{m-1} \omega_m^{-jn} L\Big(\lambda + \frac{n}{m}, ma, s\Big),$$

where ω_m^j by $\omega_m^j := \exp(2\pi i j/m)$, $j, m \in \mathbb{N}$, $0 \leq j \leq m-1$. Recall Proposition 3.2, which means that joint universality implies joint functional independence. Thus we can say functional relations deduce a kind of non-existence of universality. Therefore we can see that joint universality is essentially more difficult than single universality because of its connection with functional relations.

In [16, Section 6], there is another type of non-existence of universality. This is caused by the fact that "distance" of the zeta-functions is close. This non-existence is also a phenomenon which only occurs in the case of "joint" universality.

5.2 Value approximation and universality

In this subsection, we consider the following property, which is weaker than the joint universality, and stronger than the joint denseness.

Definition 5.2 (Joint value approximation, see [17]). *The joint value approximation (of positive density) for $\zeta(s)$ is the following property: Let σ_0 be a fixed number in the range $1/2 < \sigma < 1$ and $C_l \in \mathbb{C}$ for $1 \le l \le m$. Then for every $\varepsilon > 0$*

$$\liminf_{T\to\infty} \nu_T^{\tau}\Big\{ \sup_{1\le l\le m} \big|\zeta(\sigma_0 + i\tau) - C_l\big| < \varepsilon \Big\} > 0.$$

We can interpret the joint value approximation as the joint universality in the complex plane. We can also consider the joint value approximation as a kind of universality in the case that the compact subset K is a one point set. These view points are very important.

We can show the next proposition. Note that $C_1, C_2 \in \mathbb{C}$ do not need the assumption $C_1, C_2 \neq 0$ since the closure of $\{\mathbb{C} \setminus \{0\}\}^2$ is $\mathbb{C}^2$.

Proposition 5.3. *Suppose σ_0 is a fixed number in the range $1/2 < \sigma_0 < 1$, $d_1 = 0$, and $0 \neq d_2 \in \mathbb{R}$. Then for any $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \nu_T^{\tau}\Big\{ \sup_{1\le l\le 2} \big|L(\sigma_0 + id_l + i\tau, \chi) - C_l\big| < \varepsilon \Big\} > 0. \tag{5.1}$$

By this proposition, we can obtain the following corollaries.

Corollary 5.4. *Suppose σ_0 is a fixed number in the range $1/2 < \sigma_0 < 1$ and $0 \neq d \in \mathbb{R}$, Then for any $\varepsilon > 0$, it holds that*

$$\liminf_{T\to\infty} \nu_T^{\tau}\left\{ \big|L(\sigma_0 + id + i\tau, \chi) - L(\sigma_0 + i\tau, \chi)\big| < \varepsilon \right\} > 0. \tag{5.2}$$

Corollary 5.5. *Let σ_0 and σ_1 be fixed numbers in the range $(1/2, 1)$. Then for any $\varepsilon > 0$*

$$\liminf_{T\to\infty} \nu_T^{\tau}\left\{ \big|L(\sigma_0 + i\tau, \chi) - L(\sigma_1, \chi)\big| < \varepsilon \right\} > 0. \tag{5.3}$$

Remark 5.6. *Recall that both the almost periodicity in the sense of Bohr and the self-similarity for L-functions are equivalent to the (generalized) Riemann hypothesis. The difference between Corollary 5.4 and the almost periodicity in the sense of Bohr is the difference between positive density and uniformity. The difference between Corollary 5.5 with $\sigma_0 = \sigma_1$ and the self-similarity is the difference between the complex plane and a function space (difference caused by the fact the compact set K is a one point set or not). If we could fill one of these differences, we could prove the Riemann hypothesis. Needless to say, to fill such a difference is very difficult.*

Moreover, we give an example which satisfies joint value approximation (see Proposition 5.3) but does not satisfy joint universality.

Example 5.7. *Suppose* $d_1 = 0$, $0 \neq d_2 := d \in \mathbb{R}$. *There exists an* $\varepsilon > 0$ *and* $(f_1(s), f_2(s)) \in H^2(D)$ *satisfying*

$$\limsup_{T\to\infty} \nu_T^{\tau}\Big\{ \sup_{1\le l\le 2} \sup_{s\in K_l} \big|L(s+id_l+i\tau,\chi) - f_l(s)\big| \le \varepsilon \Big\} = 0. \tag{5.4}$$

Hence we can say that the difference between the complex plane and a function space is very big (see Remark 5.6). Thus we finish this article with the following problem.

Problem 2. Determine exactly the difference between the phenomenon on the complex plane and that function spaces for universality.

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