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Characteristic polynomial averages of a random matrix from compact symmetric spaces [1]

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Abstract

We calculate the average of products of characteristic polynomials of random matrices associated with classical compact symmetric spaces. These averages are expressed in terms of a Jack polynomial or a Heckman and Opdam's Jacobi polynomial.

1 Introduction

One can consider the following general problem: Let S be a set of $n \times n$ matrices and let dM be a probability measure on S. Then we would like to calculate the average

$$\left\langle \prod_{i=1}^{m} \det(I + x_i M) \right\rangle_S := \int_S \prod_{i=1}^{m} \det(I + x_i M) dM, \qquad x_1, \dots, x_m \in \mathbb{C}, \tag{1.1}$$

where $I = I_n$ is the $n \times n$ identity matrix. The consideration of this problem is motivated by its connection with Riemann zeta functions and L-functions, as developed by Keating and Snaith [KS1, KS2], which we will briefly review in §2.

In the present note, we consider the following compact symmetric spaces:

$$U(n)/O(n),\ U(2n)/Sp(2n),\ U(n+m)/(U(n) \times U(m)),\ O(n+m)/(O(n) \times O(m)),$$
$$Sp(2n)/U(n),\ Sp(2n+2m)/(Sp(2n) \times Sp(2m)),\ SO(2n)/U(n).$$

Let G/K be a compact symmetric space given above. Then G is a classical group and K is a closed subgroup of G. The space G/K can be realized as a subset S of G: $S \cong G/K$. For example, if $G/K = U(n)/O(n)$, then we can take S as the set of all symmetric (unitary) matrices in $U(n)$. We consider the probability measure dM on S given by the Haar measure on G, in which case the pair (S, dM) is a probability space over matrices. We call this space the random matrix ensemble associated with the symmetric space G/K.

We also treat the classical groups $U(n)$, $SO(n)$, and $Sp(2n)$. For these cases we let S be the group itself while dM is the normalized Haar measure. Note that these Lie groups G can be identified with the symmetric space $(G \times G)/G$.

Cartan's classifications for classical groups and compact symmetric spaces are given in the following List 1.

[1]Workshop "Number Theory and Probability Theory", 15-16 October, 2007 at RIMS and IIAS.

List 1.

space (G/K or G)	type	parameter(s)
$U(n)$	A	$\beta = 2$
$U(n)/O(n)$	A I	$\beta = 1$
$U(2n)/Sp(2n)$	A II	$\beta = 4$
$SO(2n+1)$	B	$(k_1, k_2, k_3) = (1, 0, 1)$
$Sp(2n)$	C	$(k_1, k_2, k_3) = (0, 1, 1)$
$SO(2n)$	D	$(k_1, k_2, k_3) = (0, 0, 1)$
$U(2n+r)/(U(n+r) \times U(n))$	A III - r	$(k_1, k_2, k_3) = (r, \frac{1}{2}, 1)$
$O(2n+r)/(O(n+r) \times O(n))$	BD I - r	$(k_1, k_2, k_3) = (\frac{r}{2}, 0, \frac{1}{2})$
$Sp(2n)/U(n)$	C I	$(k_1, k_2, k_3) = (0, \frac{1}{2}, \frac{1}{2})$
$Sp(4n+2r)/(Sp(2n+2r) \times Sp(2n))$	C II - r	$(k_1, k_2, k_3) = (2r, \frac{3}{2}, 2)$
$SO(4n+2)/U(2n+1)$	D III - odd	$(k_1, k_2, k_3) = (2, \frac{1}{2}, 2)$
$SO(4n)/U(2n)$	D III - even	$(k_1, k_2, k_3) = (0, \frac{1}{2}, 2)$

We will calculate the characteristic polynomial average (1.1) on $S \cong G/K$ (or $S = G$). The characteristic polynomial of a matrix M depends only on its eigenvalues, and so we require a density function for these eigenvalues. As described by [Du] for example, we know from classical representation theory that these density functions are given as follows:

For type A, A I, and A II, the probability density function (pdf) for eigenvalues $z_1, z_2, \ldots, z_n$ of a matrix M in $S \cong G/K$ is proportional to

$$\Delta^{\text{Jack}}(z_1, \ldots, z_n; 2/\beta) = \prod_{1 \le i < j \le n} |z_i - z_j|^\beta, \tag{1.2}$$

where β is $1, 2, 4$ as given in List 1. Similarly, for type B, C, D, A III, BD I, C I, C II, and D III, the corresponding pdf is proportional to

$$\begin{aligned} &\Delta^{\text{HO}}(z_1, \ldots, z_n; k_1, k_2, k_3) \\ &= \prod_{1 \le i < j \le n} |1 - z_i z_j|^{2k_3} |1 - z_i z_j^{-1}|^{2k_3} \cdot \prod_{1 \le j \le n} |1 - z_j|^{2k_1} |1 - z_j^2|^{2k_2}, \end{aligned} \tag{1.3}$$

where the k_i's are given in List 1 for each case.

Our goal in this note is to express the characteristic polynomial averages as a Jack polynomial for type A, A I, A II spaces, or as a Heckman and Opdam's Jacobi polynomial for symmetric spaces of other types. These results will be given in §5. In §2, we review the Keating-Snaith conjecture. In order to describe our main results in §5 we recall Jack polynomials and Heckman and Opdam's Jacobi polynomials in §3 and §4, respectively. In §6, we discuss some related works.

2 Keating-Snaith conjecture

In this section, we recall the motivation for the calculation of characteristic polynomial averages. Specifically, we review the Keating-Snaith conjecture for the Riemann zeta function stated in [KS1], see also [KS2, CFKRS1, CFKRS2].

2.1 Unitary groups

Let $U(n)$ be the unitary group:

$$U(n) = \{M \in GL(n, \mathbb{C}) \mid UU^* = I\}.$$

Let dM be the normalized Haar measure for $U(n)$. By definition, the measure dM satisfies the invariance

$$d(M_1 M M_2) = dM, \qquad M_1, M_2 \in U(n),$$

and $\int_{U(n)} dM = 1$. By employing Selberg's integral evaluation, Keating and Snaith [KS1] calculated the moment of the characteristic polynomial as

$$\langle |\det(I + \xi M)|^{2m} \rangle_{U(n)} = \prod_{j=0}^{n-1} \frac{j!\,(j+2m)!}{\{(j+m)!\}^2}, \qquad |\xi| = 1. \tag{2.1}$$

Note that this value does not depend on ξ. Bump and Gamburd [BG] (see also §3 and §5) gave a simple proof of expression (2.1) by using Schur polynomials. Furthermore, in the limit as the matrix size n goes to the infinity, we have

$$\langle |\det(I + \xi M)|^{2m} \rangle_{U(n)} \sim f_{\text{unitary}}(m) \cdot n^{m^2}$$

for m fixed, with

$$f_{\text{unitary}}(m) = \prod_{j=0}^{m-1} \frac{j!}{(j+m)!}. \tag{2.2}$$

2.2 Riemann zeta functions

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Re(s) > 1.$$

It has an Euler product expression

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

where p runs over all prime numbers. The zeta function $\zeta(s)$ can be extended to the whole complex plane $\mathbb{C}$ as a meromorphic function that has only one simple pole, at

$s = 1$. In addition, $\zeta(s)$ has the following *functional equation* with respect to the *critical line* $\Re(s) = 1/2$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{s\pi}{2}\right)\Gamma(s)\zeta(s) \qquad \text{for any } s \in \mathbb{C},$$

where $\Gamma(s)$ is the gamma function. We have $\zeta(-2n) = 0$ for $n = 1, 2, \ldots$. These zeros are called trivial zeros. It is known that other zeros, called nontrivial zeros, are in the *critical strip* $0 < \Re(s) < 1$.

The well-known *Riemann Hypothesis* claims that nontrivial zeros belong to the critical line $\Re(s) = 1/2$. We are interested in the behavior of $\zeta(s)$ on the critical line.

The following statement has been conjectured concerning the moment of $\zeta(s)$ on the critical line:

Conjecture 2.1 (Keating and Snaith [KS1]). *For each positive integer m, the limit*

$$\lim_{T\to\infty} \frac{1}{(\log T)^{m^2}} \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2m} \frac{dt}{T}$$

exists and equals

$$a(m) f_{\text{unitary}}(m),$$

where $a(m)$ is defined by an Euler product

$$a(m) = \prod_{p:\text{prime}} \left[(1-p^{-1})^{m^2} \sum_{k=0}^{\infty} \binom{m+k-1}{k}^2 p^{-k}\right]$$

and $f_{\text{unitary}}(m)$ is given by relation (2.2).

We remark that in [KS1] Keating and Snaith present this conjecture without a restriction that m be an integer. The value $a(m)$ is called the arithmetic part, while $f_{\text{unitary}}(m)$ is called the random matrix part.

The arithmetic part $a(m)$ arises as follows (see e.g. [BH, Appendix]). The mth power of $\zeta(s)$ is written as

$$\zeta(s)^m = \sum_{n=1}^{\infty} \frac{d_m(n)}{n^s} = \prod_p (1 + d_m(p)p^{-s} + d_m(p^2)p^{-2s} + \cdots), \qquad \Re(s) > 1,$$

where

$$d_m(n) = \sum_{n_1 n_2 \cdots n_m = n} 1.$$

In particular, we have $d_m(p^k) = \binom{m+k-1}{k}$. Consider $\sum_{n=1}^{\infty} \frac{d_m(n)^2}{n^s}$, which is the "diagonal" term of $|\zeta(s)|^{2m} = |\sum_{n=1}^{\infty} \frac{d_m(n)}{n^s}|^2$. Then we see that

$$\sum_{n=1}^{\infty} \frac{d_m(n)^2}{n^s} = \zeta(s)^{m^2} g_m(s),$$

where

$$g_m(s) = \prod_p \left[(1-p^{-s})^{m^2} \sum_{k=0}^{\infty} \frac{d_m(p^k)^2}{p^{ks}} \right].$$

The function $g_m(s)$ is analytic at $s=1$, and the value $a(m)$ is equal to $g_m(1)$.

Conjecture 2.1 has only been proved for the cases $m=1$ and 2, with

$$a(1)f_{\text{unitary}}(1) = 1 \times 1 = 1 \quad \text{and} \quad a(2)f_{\text{unitary}}(2) = \frac{1}{\zeta(2)} \times \frac{1}{2!3!} = \frac{6}{\pi^2} \times \frac{1}{12} = \frac{1}{2\pi^2},$$

see e.g. [T].

2.3 Generalized conjecture

For two nonnegative integers K and L, we let $\Xi_{L,K}$ be the set of the $\binom{L+K}{L}$ permutations $\sigma \in \mathfrak{S}_{L+K}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(L)$ and $\sigma(L+1) < \sigma(L+2) < \cdots < \sigma(L+K)$.

With this notation the following conjecture, which is a generalization of Conjecture 2.1, has been made:

Conjecture 2.2 ([CFKRS1, CFKRS2]).

$$\begin{aligned} &\int_0^T \prod_{l=1}^m \zeta\left(\frac{1}{2}+\alpha_l+it\right) \cdot \prod_{k=1}^m \zeta\left(\frac{1}{2}-\alpha_{m+k}-it\right) dt \\ &= \int_0^T W_m(t;\alpha_1,\dots,\alpha_m;\alpha_{m+1},\dots,\alpha_{2m}) \left(1+O(t^{-\frac{1}{2}+\epsilon})\right) dt, \end{aligned}$$

where

$$\begin{aligned} &W_m(t;\alpha_1,\dots,\alpha_m;\alpha_{m+1},\dots,\alpha_{2m}) \\ &= e^{\frac{1}{2}\log\frac{t}{2\pi}(-\alpha_1-\cdots-\alpha_m+\alpha_{m+1}+\cdots+\alpha_{2m})} \\ &\quad \sum_{\sigma \in \Xi_{m,m}} e^{\frac{1}{2}\log\frac{t}{2\pi}(-\alpha_{\sigma(1)}-\cdots-\alpha_{\sigma(m)}+\alpha_{\sigma(m+1)}+\cdots+\alpha_{\sigma(2m)})} \\ &\qquad \times A_m(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(2m)}) \prod_{1\le l,k\le m} \zeta(1+\alpha_{\sigma(l)}-\alpha_{\sigma(m+k)}). \end{aligned}$$

Here $A_m(u_1,\dots,u_m)$ is

$$\begin{aligned} A_m(u_1,\dots,u_m) = \prod_p \Bigg[&\prod_{k=1}^m \prod_{l=1}^m (1-p^{-1-u_l-u_{m+k}}) \\ &\times \int_0^1 \prod_{k=1}^m (1-e^{2\pi i\theta}p^{-\frac{1}{2}-u_k})^{-1}(1-e^{-2\pi i\theta}p^{-\frac{1}{2}+u_{m+k}})^{-1} d\theta \Bigg]. \end{aligned}$$

For the unitary group, we have the following statement.

Theorem 2.3 ([CFKRS1, BG]). *For two nonnegative integers L and K,*

$$\left\langle \prod_{l=1}^{L} \det(I + x_l^{-1} M^{-1}) \cdot \prod_{k=1}^{K} \det(I + x_{L+k} M) \right\rangle_{U(n)}$$

$$=(x_1 \cdots x_L)^{-n} s_{(n^L)}(x_1, \dots, x_{L+K}) \tag{2.3}$$

$$= \sum_{\sigma \in \Xi_{L,K}} \frac{\prod_{k=1}^{K} (x_{\sigma(L+k)}^{-1} x_{L+k})^n}{\prod_{l=1}^{L} \prod_{k=1}^{K} (1 - x_{\sigma(l)}^{-1} x_{\sigma(L+k)})}, \tag{2.4}$$

where $s_\lambda(x_1, \dots, x_N)$ is the Schur polynomial (whose definition will be given in the next section).

If we consider $x_k = e^{-i\alpha_k}$ in equation (2.4), we obtain

$$e^{n(-\alpha_{L+1} - \cdots - \alpha_{L+K})} \sum_{\sigma \in \Xi_{L,K}} e^{n(\alpha_{\sigma(L+1)} + \cdots + \alpha_{\sigma(L+K)})} \prod_{l=1}^{L} \prod_{k=1}^{K} (1 - e^{\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)}})^{-1}.$$

Compare this with the function W_m in Conjecture 2.2.

In order to prove expression (2.3), Conrey et al. [CFKRS1] employ the Selberg integral evaluation, while Bump and Gamburd [BG] employ symmetric polynomial theory. The expression (2.4) follows from (2.3) by the determinantal expression of the Schur polynomial and its Laplace expansion.

There are similar relations between other classical groups and arithmetic L-functions, see [KS2] and its generalizations [CFKRS1, CFKRS2]. This is our motivation for the calculation of characteristic polynomial averages.

Our purpose in this note is to obtain analogues of equation (2.3) for some random matrices.

3 Jack polynomials

In order to calculate characteristic polynomial averages associated with symmetric spaces of type A, A I, and A II, we will employ the Jack polynomials reviewed in this section.

3.1 Partitions

We employ the standard notation used in [Mac, Chapter I-1].

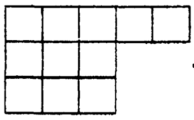
A partition is a weakly decreasing sequence of nonnegative integers with finitely many nonzero entries:

$$\lambda = (\lambda_1, \lambda_2, \dots).$$

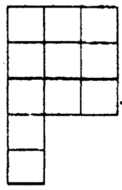
We put

$$\ell(\lambda) = \#\{j \geq 1 \mid \lambda_j > 0\} \qquad \text{and} \qquad |\lambda| = \sum_{j\geq 1} \lambda_j,$$

and call $\ell(\lambda)$ the length and $|\lambda|$ the weight. We identify a partition with the associated Young diagram $\{(i,j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i\}$. For example, the Young diagram of $\lambda = (5,3,3)$ is given by

.

In particular, when all nonzero λ_j are equal, the Young diagram is rectangular and we say that such a partition λ is rectangular-shaped. For a partition λ, the conjugate partition λ' is determined by the transpose of the Young diagram λ on the diagonal line. For example, for $\lambda = (5,3,3)$, we have $\lambda' = (3,3,3,1,1)$ with the diagram

.

It is sometimes convenient to write a partition in the form $\lambda = (1^{m_1}2^{m_2}\cdots)$, where $m_i = m_i(\lambda)$ is the multiplicity of i in λ given by $m_i = \lambda'_i - \lambda'_{i+1}$. In particular, a rectangular-shaped partition is written in the form

$$(n^k) = (\underbrace{n, n, \ldots, n}_{k}).$$

For two partitions λ and μ, we write $\lambda \subset \mu$ if the diagram of μ covers the diagram of λ, that is, if $\lambda_i \leq \mu_i$ for all i. In particular, the notation $\lambda \subset (m^n)$ means that λ satisfies $\lambda_1 \leq m$ and $\lambda'_1 = \ell(\lambda) \leq n$.

3.2 Definition of Jack polynomials

In this section we recall relevant details of Jack polynomials; the reader may refer to [Mac, Chapter VI] for further details.

Let $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ and let dz be the normalized Haar measure on $\mathbb{T}$. By definition, we have

$$\int_{\mathbb{T}} f(z)dz = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta$$

for a continuous function f on $\mathbb{T}$, where $d\theta$ is the Lebesgue measure on the interval $[0, 2\pi]$.

Fix a positive real number α. Define a function on $\mathbb{T}^n$ by

$$\Delta^{\mathrm{Jack}}(z;\alpha) = \prod_{1\le i<j\le n} |z_i - z_j|^{2/\alpha}, \qquad z = (z_1,\dots,z_n) \in \mathbb{T}^n \tag{3.1}$$

(cf. equation (1.2)). The function $\Delta^{\mathrm{Jack}}(z,\alpha)$ is the probability density function (pdf) for eigenvalues of random matrices associated with $U(n), U(n)/O(n)$, or $U(2n)/Sp(2n)$.

Denote by $\mathbb{C}[x_1,\dots,x_n]^{\mathfrak{S}_n}$ the space of symmetric polynomials in n variables, and define an inner product on $\mathbb{C}[x_1,\dots,x_n]^{\mathfrak{S}_n}$ by

$$\langle \phi, \psi\rangle_{\Delta^{\mathrm{Jack}}} = \frac{1}{n!}\int_{\mathbb{T}^n} \phi(z)\overline{\psi(z)}\Delta^{\mathrm{Jack}}(z;\alpha)dz, \tag{3.2}$$

where $dz = dz_1\cdots dz_n$.

For a partition λ of length $\ell(\lambda)\le n$, put

$$m_\lambda^{\mathrm{A}}(x_1,\dots,x_n) = \sum_{\nu=(\nu_1,\dots,\nu_n)\in\mathfrak{S}_n\lambda} x_1^{\nu_1}\cdots x_n^{\nu_n}, \tag{3.3}$$

where the sum runs over the $\mathfrak{S}_n$-orbit $\mathfrak{S}_n\lambda = \{(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(n)}) \mid \sigma\in\mathfrak{S}_n\}$. Here the suffix 'A' means that $\mathfrak{S}_n$ is the Weyl group of type A [2]. The set $\{m_\lambda^{\mathrm{A}} \mid \lambda \text{ are partitions with } \ell(\lambda)\le n\}$ is a basis of $\mathbb{C}[x_1,\dots,x_n]^{\mathfrak{S}_n}$.

Jack polynomials

$$\{P_\lambda^{\mathrm{Jack}}(x_1,\dots,x_n;\alpha) \mid \lambda \text{ are partitions with } \ell(\lambda)\le n\}$$

are uniquely determined by the polynomials in $\mathbb{Q}(\alpha)[x_1,\dots,x_n]^{\mathfrak{S}_n}$ satisfying the following conditions:

- $P_\lambda^{\mathrm{Jack}} = m_\lambda^{\mathrm{A}} + \sum_{\mu:\mu<_{\mathrm{A}}\lambda} u_{\lambda\mu}^{(\alpha)} m_\mu^{\mathrm{A}}, \qquad u_{\lambda\mu}^{(\alpha)}\in\mathbb{Q}(\alpha).$
- $\langle P_\lambda^{\mathrm{Jack}}, P_\mu^{\mathrm{Jack}}\rangle_{\Delta^{\mathrm{Jack}}} = 0 \qquad \text{if } \lambda\ne\mu.$

Here "$<_{\mathrm{A}}$" denotes the dominance order for root systems of type A:

$$\mu \le_{\mathrm{A}} \lambda \quad \overset{\mathrm{def}}{\Leftrightarrow} \quad |\lambda| = |\mu| \quad \text{and} \quad \mu_1+\mu_2+\cdots+\mu_i \le \lambda_1+\lambda_2+\cdots+\lambda_i \quad \text{for all } i\ge 1.$$

Note that for the empty partition (0) it holds that $P_{(0)}^{\mathrm{Jack}} = 1$.

It is well known that the Jack polynomial at $\alpha = 1$ agrees with a Schur polynomial:

$$P_\lambda^{\mathrm{Jack}}(x_1,\dots,x_n;1) = s_\lambda(x_1,\dots,x_n) := \frac{\det(x_j^{\lambda_i+n-i})_{1\le i,j\le n}}{\det(x_j^{n-i})_{1\le i,j\le n}}.$$

[2]In Macdonald's Book [Mac], polynomials m_λ^{A} are written as m_λ for conciseness. In the present study, however, we must distinguish the Laurent polynomial m_λ^{BC} given in the next section.

The Schur polynomials are irreducible characters of $U(n)$ associated with the highest weight $(\lambda_1, \dots, \lambda_n)$, further details may be found in any standard text on representation theory of classical groups. Moreover, when $\alpha = 2$ and $\alpha = 1/2$, the Jack polynomial is (a constant times) a spherical function associated with the symmetric space $U(n)/O(n)$ and $U(2n)/Sp(2n)$ respectively, see [Mac, Chapter VII]. However, while the Schur polynomials may be expressed as a quotient of determinants, such an expression is not known for Jack polynomials, even for $\alpha = 2, 1/2$.

In this note, we use the following properties of Jack polynomials:

Lemma 3.1. *Jack polynomials satisfy the following properties.*

- *([Mac, Chapter VI (4.17)]) If $\ell(\lambda) = n$, then*

$$P_\lambda^{\text{Jack}}(x_1, \dots, x_n; \alpha) = x_1 x_2 \cdots x_n P_\mu^{\text{Jack}}(x_1, \dots, x_n; \alpha) \tag{3.4}$$

  with $\mu = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_n - 1)$.

- *([Mac, Chapter VI (5.4)]) Dual Cauchy identity:*

$$\sum_{\lambda \subset (m^n)} P_{\lambda'}^{\text{Jack}}(x_1, \dots, x_m; 1/\alpha) P_\lambda^{\text{Jack}}(y_1, \dots, y_n; \alpha) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + x_i y_j). \tag{3.5}$$

- *([Mac, Chapter VI (10.38)]) For a positive real number α, a positive integer n, and a partition λ with $\ell(\lambda) \leq n$, we have*

$$\begin{aligned} &\langle P_\lambda^{\text{Jack}}(\cdot; \alpha), P_\lambda^{\text{Jack}}(\cdot; \alpha) \rangle_{\Delta^{\text{Jack}}} \\ &= \prod_{1 \leq i < j \leq n} \frac{\Gamma(\lambda_i - \lambda_j + (j - i + 1)/\alpha) \Gamma(\lambda_i - \lambda_j + 1 + (j - i - 1)/\alpha)}{\Gamma(\lambda_i - \lambda_j + (j - i)/\alpha) \Gamma(\lambda_i - \lambda_j + 1 + (j - i)/\alpha)}, \end{aligned}$$

  where Γ is the gamma function. In particular, for any nonnegative integer L,

$$\langle P_{(L^n)}^{\text{Jack}}(\cdot; \alpha), P_{(L^n)}^{\text{Jack}}(\cdot; \alpha) \rangle_{\Delta^{\text{Jack}}} = \langle 1, 1 \rangle_{\Delta^{\text{Jack}}}. \tag{3.6}$$

- *([Mac, Chapter VI (10.20)]) Principal specialization: for any partition λ of length $\ell(\lambda) \leq n$,*

$$P_\lambda^{\text{Jack}}(\underbrace{1, 1, \dots, 1}_{n}; \alpha) = \prod_{(i,j) \in \lambda} \frac{n + \alpha(j-1) - (i-1)}{\alpha(\lambda_i - j) + (\lambda'_j - i) + 1}, \tag{3.7}$$

  where (i, j) run over all boxes in the Young diagram λ, i.e., $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$.

Remark 3.1. If you read Macdonald's book [Mac], you should attend the fact that the Jack polynomial is a degenerate case of a two-parameter symmetric polynomial $P_\lambda(x_1, \dots, x_n; q, t)$. Specifically, Jack polynomials are obtained by setting $q = t^\alpha$ and examining the limit as $t \to 1$. The polynomial $P_\lambda(x_1, \dots, x_n; q, t)$ is called the Macdonald polynomial. Note that $P_\lambda^{\text{Jack}}(x_1, \dots, x_n; \alpha)$ is written as $P_\lambda^{(\alpha)}(x_1, \dots, x_n)$ in Macdonald's book.

3.3 Characteristic polynomial averages for type A

For a real number $\beta > 0$ and a function ϕ on $\mathbb{T}^n$, we define the value $\langle \phi \rangle_{n,\beta}$ by

$$\langle \phi \rangle_{n,\beta} = \langle \phi(\boldsymbol{z}) \rangle_{n,\beta} = \frac{\int_{\mathbb{T}^n} \phi(\boldsymbol{z}) \Delta^{\mathrm{Jack}}(\boldsymbol{z}; 2/\beta) d\boldsymbol{z}}{\int_{\mathbb{T}^n} \Delta^{\mathrm{Jack}}(\boldsymbol{z}; 2/\beta) d\boldsymbol{z}}, \tag{3.8}$$

where the denominator is equal to $\langle 1, 1 \rangle_{\Delta^{\mathrm{Jack}}}$ with parameter $\alpha = 2/\beta$. We do not need an explicit expression of the denominator.

The value defined by equation (3.8) is reduced from the average of a function on random matrices associated with $U(n)$, $U(n)/O(n)$, or $U(2n)/Sp(2n)$ at $\beta = 2, 1$, or 4 respectively.

We consider a polynomial on $\mathbb{T}^n$ defined by

$$\Psi^{\mathrm{A}}(\boldsymbol{z}; x) = \prod_{j=1}^{n} (1 + x z_j), \qquad \boldsymbol{z} \in \mathbb{T}^n, \ x \in \mathbb{C}.$$

This corresponds to the characteristic polynomial of a (unitary) matrix with eigenvalues $z_1, \dots, z_n$. The following theorem gives an average of the product of characteristic polynomials.

Theorem 3.2. *Let L and K be nonnegative integers and let $x_1, x_2, \dots, x_{L+K}$ be complex numbers. Then we have*

$$\left\langle \prod_{l=1}^{L} \Psi^{\mathrm{A}}(\boldsymbol{z}^{-1}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi^{\mathrm{A}}(\boldsymbol{z}; x_{L+k}) \right\rangle_{n,\beta} = (x_1 \cdots x_L)^{-n} P^{\mathrm{Jack}}_{(n^L)}(x_1, \dots, x_{L+K}; \beta/2).$$

Here $\boldsymbol{z}^{-1} = (z_1^{-1}, \dots, z_n^{-1})$.

Proof. By the dual Cauchy identity (3.5) we have

$$\begin{aligned} \prod_{l=1}^{L} \Psi^{\mathrm{A}}(\boldsymbol{z}^{-1}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi^{\mathrm{A}}(\boldsymbol{z}; x_{L+k}) &= \prod_{l=1}^{L} x_l^{-n} \cdot (z_1 \cdots z_n)^{-L} \cdot \prod_{k=1}^{L+K} \prod_{j=1}^{n} (1 + x_k z_j) \\ &= \prod_{l=1}^{L} x_l^{-n} \cdot (z_1 \cdots z_n)^{-L} \sum_{\lambda} P^{\mathrm{Jack}}_{\lambda'}(x_1, \dots, x_{L+K}; \beta/2) P^{\mathrm{Jack}}_{\lambda}(z_1, \dots, z_n; 2/\beta). \end{aligned}$$

Since $P^{\mathrm{Jack}}_{(L^n)}(z_1,\ldots,z_n;2/\beta)=(z_1\cdots z_n)^L$ by (3.4), we have

$$\begin{aligned}
&\left\langle \prod_{l=1}^{L}\Psi^{\mathrm{A}}(\boldsymbol{z}^{-1};x_l^{-1})\cdot\prod_{k=1}^{K}\Psi^{\mathrm{A}}(\boldsymbol{z};x_{L+k})\right\rangle_{n,\beta}\\
&=\prod_{l=1}^{L}x_l^{-n}\cdot\sum_{\lambda}P^{\mathrm{Jack}}_{\lambda'}(x_1,\ldots,x_{L+K};\beta/2)\langle P^{\mathrm{Jack}}_{\lambda}(\boldsymbol{z};2/\beta)\overline{P^{\mathrm{Jack}}_{(L^n)}(\boldsymbol{z};2/\beta)}\rangle_{n,\beta}\\
&=\prod_{l=1}^{L}x_l^{-n}\cdot\sum_{\lambda}P^{\mathrm{Jack}}_{\lambda'}(x_1,\ldots,x_{L+K};\beta/2)\frac{\langle P^{\mathrm{Jack}}_{\lambda}(\cdot;2/\beta),P^{\mathrm{Jack}}_{(L^n)}(\cdot;2/\beta)\rangle_{\Delta^{\mathrm{Jack}}}}{\langle 1,1\rangle_{\Delta^{\mathrm{Jack}}}}.
\end{aligned}$$

If we make use of the orthogonality property of Jack polynomials and relation (3.6), then the above expression can be re-expressed as

$$\prod_{l=1}^{L}x_l^{-n}\cdot P^{\mathrm{Jack}}_{(n^L)}(x_1,\ldots,x_{L+K};\beta/2).$$

□

Corollary 3.3. *For $\xi\in\mathbb{T}$ we have*

$$\left\langle|\Psi^{\mathrm{A}}(\boldsymbol{z};\xi)|^{2m}\right\rangle_{n,\beta}=\prod_{i=0}^{m-1}\frac{\Gamma(\frac{2}{\beta}(i+1))\Gamma(n+\frac{2}{\beta}(m+i+1))}{\Gamma(\frac{2}{\beta}(m+i+1))\Gamma(n+\frac{2}{\beta}(i+1))}.$$

Moreover,

$$\lim_{n\to\infty}\frac{1}{n^{2m^2/\beta}}\left\langle|\Psi^{\mathrm{A}}(\boldsymbol{z};\xi)|^{2m}\right\rangle_{n,\beta}=\prod_{i=0}^{m-1}\frac{\Gamma(\frac{2}{\beta}(i+1))}{\Gamma(\frac{2}{\beta}(m+i+1))}.$$

Proof. In Theorem 3.2, let $L=K=m$ and $x_1=\cdots=x_{2m}=\xi$. Since Jack polynomials are homogeneous, we have

$$\left\langle|\Psi^{\mathrm{A}}(\boldsymbol{z};\xi)|^{2m}\right\rangle_{n,\beta}=\xi^{-nm}P^{\mathrm{Jack}}_{(n^m)}(\xi,\ldots,\xi;\beta/2)=P^{\mathrm{Jack}}_{(n^m)}(1^{2m};\beta/2).$$

The first claim follows from relation (3.7) and a straightforward calculation. The second claim is obtained from the first claim together with the asymptotics

$$\lim_{n\to\infty}\frac{\Gamma(n+a)}{\Gamma(n)n^a}=1\qquad\text{for } a \text{ fixed.}$$

□

4 Heckman and Opdam's Jacobi polynomials

In this section, we review multivariate Jacobi polynomials due to Heckman and Opdam (see e.g. [Di]). Note that the structure of this review follows that conducted for Jack polynomials in the preceding section.

4.1 Definition of multivariate Jacobi polynomials

Fix three real numbers k_1, k_2, and k_3 such that

$$k_1 + k_2 > -1/2, \qquad k_2 > -1/2, \qquad k_3 \geq 0.$$

Define a function on $\mathbb{T}^n$ by

$$\Delta^{\mathrm{HO}}(\boldsymbol{z}; k_1, k_2, k_3) = \prod_{1 \leq i < j \leq n} |1 - z_i z_j^{-1}|^{2k_3} |1 - z_i z_j|^{2k_3} \cdot \prod_{1 \leq j \leq n} |1 - z_j|^{2k_1} |1 - z_j^2|^{2k_2} \tag{4.1}$$

(cf. equation (1.3)). For special parameters (k_1, k_2, k_3) given in List 1, the function $\Delta^{\mathrm{HO}}(\boldsymbol{z}, k_1, k_2, k_3)$ is the pdf for eigenvalues of random matrices for type B, C, D, A III, BD I, and so on.

Denote by $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ the algebra of Laurent polynomials in n variables. Let W be the wreath product $\mathbb{Z}_2 \wr \mathfrak{S}_n = \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$, which is the Weyl group of type BC. The group W acts naturally on $\mathbb{Z}^n$ and $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ respectively. Denote by $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W$ the subalgebra of all W-invariants. Define an inner product on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W$ by

$$\langle \phi, \psi \rangle_{\Delta^{\mathrm{HO}}} = \frac{1}{2^n n!} \int_{\mathbb{T}^n} \phi(\boldsymbol{z}) \overline{\psi(\boldsymbol{z})} \Delta^{\mathrm{HO}}(\boldsymbol{z}; k_1, k_2, k_3) d\boldsymbol{z}. \tag{4.2}$$

For a partition λ of length $\ell(\lambda) \leq n$, put

$$m_\lambda^{\mathrm{BC}}(x_1, \ldots, x_n) = \sum_{\nu = (\nu_1, \ldots, \nu_n) \in W\lambda} x_1^{\nu_1} \cdots x_n^{\nu_n}, \tag{4.3}$$

where the sum runs over the W-orbit of λ in $\mathbb{Z}^n$. The set $\{m_\lambda^{\mathrm{BC}} \mid \lambda \text{ are partitions with } \ell(\lambda) \leq n\}$ is a basis of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W$.

The Heckman and Opdam's Jacobi polynomials

$$\{P_\lambda^{\mathrm{HO}}(x_1, \ldots, x_n; k_1, k_2, k_3) \mid \lambda \text{ are partitions with } \ell(\lambda) \leq n\}$$

are uniquely determined by the Laurent polynomials in $\mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W$ satisfying the following conditions:

- $P_\lambda^{\mathrm{HO}} = m_\lambda^{\mathrm{BC}} + \sum_{\mu : \mu <_{\mathrm{BC}} \lambda} u_{\lambda\mu} m_\mu^{\mathrm{BC}}, \qquad u_{\lambda\mu} \in \mathbb{R}.$
- $\langle P_\lambda^{\mathrm{HO}}, P_\mu^{\mathrm{HO}} \rangle_{\Delta^{\mathrm{HO}}} = 0 \qquad \text{if } \lambda \neq \mu.$

Here "$<_{\mathrm{BC}}$" denotes the dominance order for root systems of type BC:

$$\mu \leq_{\mathrm{BC}} \lambda \quad \overset{\mathrm{def}}{\Leftrightarrow} \quad \mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i \quad \text{for all } i \geq 1.$$

It is known that the Jacobi polynomials agree with the irreducible character of $SO(2n+1), Sp(2n)$, and $O(2n)$ at $(k_1, k_2, k_3) = (1,0,1), (0,1,1)$ and $(0,0,1)$ respectively. Hence in these cases, P_λ^{HO} can be expressed as a quotient of determinants (see e.g. [BG]); however, such expressions are not known for other cases.

Lemma 4.1. *Jacobi polynomials satisfy the following properties:*

- *([Mi]) Dual Cauchy identity:*

$$\begin{aligned}(4.4)\qquad &\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + x_i^{-1} - y_j - y_j^{-1}) \\ &= \sum_{\lambda \subset (m^n)} (-1)^{|\tilde{\lambda}|} P_\lambda^{\mathrm{HO}}(x_1, \ldots, x_n; k_1, k_2, k_3) P_{\tilde{\lambda}}^{\mathrm{HO}}(y_1, \ldots, y_m; \tilde{k}_1, \tilde{k}_2, \tilde{k}_3),\end{aligned}$$

where $\tilde{\lambda} = (n - \lambda'_m, n - \lambda'_{m-1}, \ldots, n - \lambda'_1)$ and

$$(4.5)\qquad \tilde{k}_1 = k_1/k_3, \qquad \tilde{k}_2 = (k_2+1)/k_3 - 1, \qquad \tilde{k}_3 = 1/k_3.$$

- *([Di]) For a partition λ of length $\leq m$,*

$$\begin{aligned}(4.6)\quad P_\lambda^{\mathrm{HO}}(\underbrace{1, \ldots, 1}_{m}; k_1, k_2, k_3) =& 2^{2|\lambda|} \prod_{1 \leq i < j \leq m} \frac{(\rho_i + \rho_j + k_3)_{\lambda_i + \lambda_j} (\rho_i - \rho_j + k_3)_{\lambda_i - \lambda_j}}{(\rho_i + \rho_j)_{\lambda_i + \lambda_j} (\rho_i - \rho_j)_{\lambda_i - \lambda_j}} \\ &\times \prod_{j=1}^{m} \frac{(\frac{k_1}{2} + k_2 + \rho_j)_{\lambda_j} (\frac{k_1+1}{2} + \rho_j)_{\lambda_j}}{(2\rho_j)_{2\lambda_j}}\end{aligned}$$

with $\rho_j = (m-j)k_3 + \frac{k_1}{2} + k_2$. Here $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

4.2 Characteristic polynomial averages for type BC

For a function ϕ on $\mathbb{T}^n$, we define the value $\langle \phi \rangle_n^{k_1,k_2,k_3}$ by

$$(4.7)\qquad \langle \phi \rangle_n^{k_1,k_2,k_3} = \frac{\int_{\mathbb{T}^n} \phi(\boldsymbol{z}) \Delta^{\mathrm{HO}}(\boldsymbol{z}; k_1, k_2, k_3) d\boldsymbol{z}}{\int_{\mathbb{T}^n} \Delta^{\mathrm{HO}}(\boldsymbol{z}; k_1, k_2, k_3) d\boldsymbol{z}}.$$

The value defined by expression (4.7) is reduced from the average of a function on random matrices associated with symmetric spaces with BC type root system (recall List 1).

We consider a polynomial on $\mathbb{T}^n$ defined by

$$\Psi^{\mathrm{BC}}(\boldsymbol{z};x)=\prod_{j=1}^{n}(1+xz_j)(1+xz_j^{-1}),\qquad \boldsymbol{z}\in\mathbb{T}^n,\ x\in\mathbb{C},$$

which corresponds to the characteristic polynomial of a (unitary) matrix with eigenvalues $z_1, z_1^{-1}, \dots, z_n, z_n^{-1}$.

Theorem 4.2. *The following relation holds:*

$$\left\langle \prod_{j=1}^{m}\Psi^{\mathrm{BC}}(\boldsymbol{z};x_j)\right\rangle_n^{k_1,k_2,k_3}=(x_1\cdots x_m)^n P^{\mathrm{HO}}_{(n^m)}(x_1,\dots,x_m;\tilde{k}_1,\tilde{k}_2,\tilde{k}_3), \tag{4.8}$$

where parameters $\tilde{k}_i$ are defined by relations (4.5).

Proof. We see that

$$\Psi^{\mathrm{BC}}(\boldsymbol{z};x_1)\Psi^{\mathrm{BC}}(\boldsymbol{z};x_2)\cdots\Psi^{\mathrm{BC}}(\boldsymbol{z};x_m)=(x_1\cdots x_m)^n\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+x_i^{-1}+z_j+z_j^{-1}).$$

Using expression (4.4) we have

$$\begin{aligned}&\left\langle\Psi^{\mathrm{BC}}(\boldsymbol{z};x_1)\Psi^{\mathrm{BC}}(\boldsymbol{z};x_2)\cdots\Psi^{\mathrm{BC}}(\boldsymbol{z};x_m)\right\rangle_n^{k_1,k_2,k_3}\\ =&(x_1\cdots x_m)^n\sum_{\lambda\subset(m^n)}P^{\mathrm{HO}}_{\tilde{\lambda}}(x_1,\dots,x_m;\tilde{k}_1,\tilde{k}_2,\tilde{k}_3)\langle P^{\mathrm{HO}}_{\lambda}(\boldsymbol{z};k_1,k_2,k_3)\rangle_n^{k_1,k_2,k_3}.\end{aligned}$$

By the orthogonality relation for Jacobi polynomials, we have

$$\langle P^{\mathrm{HO}}_{\lambda}(\boldsymbol{z};k_1,k_2,k_3)\rangle_n^{k_1,k_2,k_3}=\begin{cases}1, & \text{if } \lambda=(0),\\ 0, & \text{otherwise,}\end{cases}$$

and we thus obtain the theorem. □

Corollary 4.3. *Let*

$$\mathcal{F}(m;k_1,k_2,k_3)=\prod_{j=0}^{m-1}\frac{\sqrt{\pi}}{2^{k_1+2k_2+jk_3-1}\Gamma(k_1+k_2+\frac{1}{2}+jk_3)}.$$

The m-th moment of $\Psi^{\mathrm{BC}}(\boldsymbol{z};1)$ is given by

$$\left\langle\Psi^{\mathrm{BC}}(\boldsymbol{z};1)^m\right\rangle_n^{k_1,k_2,k_3}=\mathcal{F}(m;\tilde{k}_1,\tilde{k}_2,\tilde{k}_3)\cdot\prod_{j=0}^{m-1}\frac{\Gamma(n+\tilde{k}_1+2\tilde{k}_2+j\tilde{k}_3)\Gamma(n+\tilde{k}_1+\tilde{k}_2+\frac{1}{2}+j\tilde{k}_3)}{\Gamma(n+\frac{\tilde{k}_1}{2}+\tilde{k}_2+\frac{j\tilde{k}_3}{2})\Gamma(n+\frac{\tilde{k}_1}{2}+\tilde{k}_2+\frac{1+j\tilde{k}_3}{2})}.$$

Moreover,

$$\lim_{n\to\infty}\frac{\left\langle\Psi^{\mathrm{BC}}(\boldsymbol{z};1)^m\right\rangle_n^{k_1,k_2,k_3}}{n^{m(\tilde{k}_1+\tilde{k}_2)+\frac{1}{2}m(m-1)\tilde{k}_3}}=\mathcal{F}(m;\tilde{k}_1,\tilde{k}_2,\tilde{k}_3).$$

Proof. The proof follows from Theorem 4.2 and expression (4.6). □

5 Random characteristic polynomial averages

We consider random matrix ensembles associated with classical groups and compact symmetric spaces (see [Du]). Our goal is to express the average of characteristic polynomials on each ensemble as a Jack polynomial or as a Jacobi polynomial. Note that while our results for classical groups have previously been presented in [CFKRS1, BG], the results for symmetric spaces have not, to our knowledge, appeared in any previous studies.

5.1 $U(n)$ – type A

Consider the unitary group $U(n)$ with the normalized Haar measure. (Recall §2.1.) This space has a simple root system of type A. The corresponding pdf for eigenvalues $z_1, \dots, z_n$ of $M \in U(n)$ is proportional to

$$\Delta^{\mathrm{Jack}}(\boldsymbol{z}; 1) = \prod_{1 \le i < j \le n} |z_i - z_j|^2.$$

This random matrix ensemble is well known, and is called the circular unitary ensemble (CUE).

For complex numbers $x_1, \dots, x_{L+K}$, it follows from Theorem 3.2 that ([CFKRS1] and [BG, Proposition 4])

$$\begin{aligned}
&\left\langle \prod_{i=1}^{L} \det(I + x_i^{-1} M^{-1}) \cdot \prod_{i=1}^{K} \det(I + x_{L+i} M) \right\rangle_{U(n)} \\
&= \left\langle \prod_{i=1}^{L} \Psi^{\mathrm{A}}(\boldsymbol{z}^{-1}; x_i^{-1}) \cdot \prod_{i=1}^{K} \Psi^{\mathrm{A}}(\boldsymbol{z}; x_{L+i}) \right\rangle_{n,2} = \prod_{i=1}^{L} x_i^{-n} \cdot s_{(n^L)}(x_1, \dots, x_{L+K}).
\end{aligned}$$

In addition, from Corollary 3.3 we obtain ([KS1, BG])

$$\langle |\det(I + \xi M)|^{2m} \rangle_{U(n)} = \prod_{j=0}^{m-1} \frac{j!(n+j+m)!}{(j+m)!(n+j)!} \overset{n\to\infty}{\sim} \prod_{j=0}^{m-1} \frac{j!}{(j+m)!} \cdot n^{m^2}$$

for any $\xi \in \mathbb{T}$.

5.2 $U(n)/O(n)$ – type A I

Let $S^{\mathrm{A\,I}}(n)$ be the set of all symmetric matrices in $U(n)$:

$$S^{\mathrm{A\,I}}(n) := \{M \in U(n) \mid M \text{ is symmetric}\}.$$

Then $S^{\mathrm{A\,I}}(n)$ is the ensemble associated with the symmetric space $U(n)/O(n)$:

$$S^{\mathrm{A\,I}}(n) \cong U(n)/O(n).$$

The corresponding pdf for eigenvalues $z_1, \dots, z_n$ of $M \in S^{\mathrm{A\,I}}(n)$ is proportional to $\Delta^{\mathrm{Jack}}(\boldsymbol{z}; 2) = \prod_{1 \le i < j \le n} |z_i - z_j|$. This random matrix ensemble is called the circular orthogonal ensemble (COE). We have

$$\left\langle \prod_{i=1}^{L} \det(I + x_i^{-1} M^{-1}) \cdot \prod_{i=1}^{K} \det(I + x_{L+i} M) \right\rangle_{S^{\mathrm{A\,I}}(n)}$$
$$= \left\langle \prod_{i=1}^{L} \Psi^{\mathrm{A}}(\boldsymbol{z}^{-1}; x_i^{-1}) \cdot \prod_{i=1}^{K} \Psi^{\mathrm{A}}(\boldsymbol{z}; x_{L+i}) \right\rangle_{n,1} = \prod_{i=1}^{L} x_i^{-n} \cdot P^{\mathrm{Jack}}_{(n^L)}(x_1, \dots, x_{L+K}; 1/2).$$

For $\xi \in \mathbb{T}$, we obtain ([KS1])

$$\langle |\det(I + \xi M)|^{2m} \rangle_{S^{\mathrm{A\,I}}(n)} = \prod_{j=0}^{m-1} \frac{(2j+1)!(n+2m+2j+1)!}{(2m+2j+1)!(n+2j+1)!} \sim \prod_{j=0}^{m-1} \frac{(2j+1)!}{(2m+2j+1)!} \cdot n^{2m^2}.$$

5.3 $U(2n)/Sp(2n)$ – type A II

Let

$$S^{\mathrm{A\,II}}(n) := \{M \in U(2n) \mid M = J M^{\mathrm{T}} J^{\mathrm{T}}\},$$

where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ and M^{T} stands for the transposed matrix of M. Then $S^{\mathrm{A\,II}}(n) \cong U(2n)/Sp(2n)$. This random matrix ensemble is called the circular symplectic ensemble (CSE). The eigenvalues of $M \in S^{\mathrm{A\,II}}(n)$ are of the form $z_1, z_1, z_2, z_2, \dots, z_n, z_n$, and so the characteristic polynomial is given as

$$\det(I + xM) = \prod_{j=1}^{n} (1 + x z_j)^2 = \Psi^{\mathrm{A}}(\boldsymbol{z}; x)^2.$$

The corresponding pdf for $z_1, \dots, z_n$ is proportional to $\Delta^{\mathrm{Jack}}(\boldsymbol{z}; 1/2) = \prod_{1 \le i < j \le n} |z_i - z_j|^4$. We have

$$\left\langle \prod_{i=1}^{L} \det(I + x_i^{-1} M^{-1})^{1/2} \cdot \prod_{i=1}^{K} \det(I + x_{L+i} M)^{1/2} \right\rangle_{S^{\mathrm{A\,II}}(n)}$$
$$= \left\langle \prod_{i=1}^{L} \Psi^{\mathrm{A}}(\boldsymbol{z}^{-1}; x_i^{-1}) \cdot \prod_{i=1}^{K} \Psi^{\mathrm{A}}(\boldsymbol{z}; x_{L+i}) \right\rangle_{n,4} = \prod_{i=1}^{L} x_i^{-n} \cdot P^{\mathrm{Jack}}_{(n^L)}(x_1, \dots, x_{L+K}; 2).$$

For $\xi \in \mathbb{T}$, we obtain

$$\langle |\det(I + \xi M)|^{2m} \rangle_{S^{\mathrm{A\,II}}(n)} = \prod_{j=0}^{2m-1} \frac{\Gamma(\frac{j+1}{2})\Gamma(n+m+\frac{j+1}{2})}{\Gamma(m+\frac{j+1}{2})\Gamma(n+\frac{j+1}{2})} \sim \frac{2^m}{(2m-1)!! \prod_{j=1}^{2m-1}(2j-1)!!} \cdot n^{2m^2}.$$

5.4 $SO(2n+1)$ – type B

Consider the special orthogonal group $SO(2n+1)$. An element M in $SO(2n+1)$ is an orthogonal matrix in $SL(2n+1,\mathbb{R})$, with eigenvalues given by $z_1, z_1^{-1}, \cdots, z_n, z_n^{-1}, 1$. The pdf for $z_1, z_2, \ldots, z_n$ is proportional to $\Delta^{\mathrm{HO}}(\boldsymbol{z}; 1,0,1)$, and it therefore follows from Theorem 4.2 that ([CFKRS1] and [BG, Proposition 16])

$$\left\langle \prod_{i=1}^{m} \det(I + x_i M) \right\rangle_{SO(2n+1)}$$
$$= \prod_{i=1}^{m}(1+x_i)\cdot \left\langle \prod_{i=1}^{m} \Psi^{\mathrm{BC}}(\boldsymbol{z}; x_i) \right\rangle_n^{1,0,1} = \prod_{i=1}^{m} x_i^n (1+x_i)\cdot P^{\mathrm{HO}}_{(n^m)}(x_1,\ldots,x_m;1,0,1).$$

Here $P^{\mathrm{HO}}_{\lambda}(x_1,\ldots,x_m;1,0,1)$ is the irreducible character of $SO(2m+1)$ associated with the partition λ. Corollary 4.3 and a simple calculation lead to ([KS2, BG])

$$\langle \det(I+M)^m \rangle_{SO(2n+1)} = 2^m \prod_{j=0}^{m-1} \frac{\Gamma(2n+2j+2)}{2^j (2j+1)!!\, \Gamma(2n+j+1)} \sim \frac{2^{2m}}{\prod_{j=1}^{m}(2j-1)!!} n^{m^2/2+m/2}.$$

5.5 $Sp(2n)$ – type C

Consider the symplectic group

$$Sp(2n) = \{ M \in U(2n) \mid MJM^{\mathrm{T}} = J \},$$

where $J = \left(\begin{smallmatrix} O_n & I_n \\ -I_n & O_n \end{smallmatrix}\right)$. The eigenvalues are given by $z_1, z_1^{-1}, \cdots, z_n, z_n^{-1}$. The corresponding pdf of $z_1, z_2, \ldots, z_n$ is proportional to $\Delta^{\mathrm{HO}}(\boldsymbol{z}; 0,1,1)$ and therefore we have ([CFKRS1] and [BG, Proposition 11])

$$\left\langle \prod_{i=1}^{m} \det(I + x_i M) \right\rangle_{Sp(2n)} = \left\langle \prod_{i=1}^{m} \Psi^{\mathrm{BC}}(\boldsymbol{z}; x_i) \right\rangle_n^{0,1,1} = \prod_{i=1}^{m} x_i^n \cdot P^{\mathrm{HO}}_{(n^m)}(x_1,\ldots,x_m;0,1,1).$$

Here $P^{\mathrm{HO}}_{\lambda}(x_1,\ldots,x_m;0,1,1)$ is the irreducible character of $Sp(2m)$ associated with the partition λ. We obtain ([KS2, BG])

$$\langle \det(I+M)^m \rangle_{Sp(2n)} = \prod_{j=0}^{m-1} \frac{\Gamma(2n+2j+3)}{2^{j+1}\cdot (2j+1)!!\, \Gamma(2n+j+2)} \sim \frac{1}{\prod_{j=1}^{m}(2j-1)!!} \cdot n^{m^2/2+m/2}.$$

5.6 $SO(2n)$ – type D

Consider the special orthogonal group $SO(2n)$. The eigenvalues of a matrix $M \in SO(2n)$ are of the form $z_1, z_1^{-1}, \cdots, z_n, z_n^{-1}$. The corresponding pdf of $z_1, z_2, \dots, z_n$ is proportional to $\Delta^{\mathrm{HO}}(\boldsymbol{z}; 0, 0, 1)$, and therefore we have ([CFKRS1] and [BG, Proposition 13])

$$\left\langle \prod_{i=1}^{m} \det(I + x_i M) \right\rangle_{SO(2n)} = \left\langle \prod_{i=1}^{m} \Psi^{\mathrm{BC}}(\boldsymbol{z}; x_i) \right\rangle_n^{0,0,1} = \prod_{i=1}^{m} x_i^n \cdot P_{(n^m)}^{\mathrm{HO}}(x_1, \dots, x_m; 0, 0, 1).$$

Here $P_\lambda^{\mathrm{HO}}(x_1, \dots, x_m; 0, 0, 1)$ is simply the irreducible character of $O(2m)$ (not $SO(2m)$) associated with the partition λ. We have ([KS2, BG])

$$\langle \det(I + M)^m \rangle_{SO(2n)} = \prod_{j=0}^{m-1} \frac{\Gamma(2n + 2j)}{2^{j-1} (2j-1)!! \, \Gamma(2n + j)} \sim \frac{2^m}{\prod_{j=1}^{m-1} (2j-1)!!} \cdot n^{m^2/2 - m/2}.$$

5.7 $U(2n + r)/(U(n + r) \times U(n))$ – type A III

Let r be a nonnegative integer and let

$$G^{\mathrm{A\,III}}(n, r) = \left\{ M = H \cdot \begin{pmatrix} I_{n+r} & O \\ O & -I_n \end{pmatrix} \;\middle|\; \begin{matrix} H \in U(2n+r) \text{ is Hermitian} \\ \text{of signature } (n+r, n) \end{matrix} \right\}.$$

Then $G^{\mathrm{A\,III}}(n, r) \cong U(2n + r)/(U(n + r) \times U(n))$, see [Du]. The eigenvalues of a matrix $M \in G^{\mathrm{A\,III}}(n, r) \subset U(2n + r)$ are of the form

$$z_1, z_1^{-1}, \cdots, z_n, z_n^{-1}, \underbrace{1, 1, \dots, 1}_{r}. \tag{5.1}$$

The corresponding pdf of $z_1, z_2, \dots, z_n$ is proportional to $\Delta^{\mathrm{HO}}(\boldsymbol{z}; r, \frac{1}{2}, 1)$, and therefore we have

$$\begin{aligned} &\left\langle \prod_{i=1}^{m} \det(I + x_i M) \right\rangle_{G^{\mathrm{A\,III}}(n,r)} \\ =& \prod_{i=1}^{m} (1 + x_i)^r \cdot \left\langle \prod_{i=1}^{m} \Psi^{\mathrm{BC}}(\boldsymbol{z}; x_i) \right\rangle_n^{r, \frac{1}{2}, 1} = \prod_{i=1}^{m} (1 + x_i)^r x_i^n \cdot P_{(n^m)}^{\mathrm{HO}}(x_1, \dots, x_m; r, \frac{1}{2}, 1). \end{aligned}$$

We obtain

$$\begin{aligned} &\langle \det(I + M)^m \rangle_{G^{\mathrm{A\,III}}(n,r)} = 2^{mr} \left\langle \Psi^{\mathrm{BC}}(\boldsymbol{z}; 1) \right\rangle_n^{r, \frac{1}{2}, 1} \\ =& \frac{\pi^{m/2}}{\prod_{j=0}^{m-1} 2^j (r+j)!} \prod_{j=0}^{m-1} \frac{\Gamma(n + r + j + 1)^2}{\Gamma(n + \frac{r+j+1}{2}) \Gamma(n + \frac{r+j}{2} + 1)} \sim \frac{\pi^{m/2}}{2^{m(m-1)/2} \prod_{j=0}^{m-1} (r+j)!} \cdot n^{m^2/2 + rm}. \end{aligned}$$

5.8 $O(2n+r)/(O(n+r)\times O(n))$ – type BD I

Let r be a nonnegative integer and let

$$G^{\mathrm{BD\,I}}(n,r)=\left\{M=H\cdot\left(\begin{smallmatrix}I_{n+r} & O\\ O & -I_n\end{smallmatrix}\right)\;\middle|\;\begin{smallmatrix}H\in O(2n+r)\text{ is symmetric}\\ \text{of signature }(n+r,n)\end{smallmatrix}\right\}.$$

Then $G^{\mathrm{BD\,I}}(n,r)\cong O(2n+r)/(O(n+r)\times O(n))$. The eigenvalues of a matrix $M\in G^{\mathrm{BD\,I}}(n,r)\subset O(2n+r)$ are of the form (5.1). The corresponding pdf of $z_1,z_2,\dots,z_n$ is proportional to $\Delta^{\mathrm{HO}}(\boldsymbol{z};\frac{r}{2},0,\frac{1}{2})$, and therefore we have

$$\left\langle\prod_{i=1}^{m}\det(I+x_iM)\right\rangle_{G^{\mathrm{BD\,I}}(n,r)}$$
$$=\prod_{i=1}^{m}(1+x_i)^r\cdot\left\langle\prod_{i=1}^{m}\Psi^{\mathrm{BC}}(\boldsymbol{z};x_i)\right\rangle_n^{\frac{r}{2},0,\frac{1}{2}}=\prod_{i=1}^{m}(1+x_i)^r x_i^n\cdot P^{\mathrm{HO}}_{(n^m)}(x_1,\dots,x_m;r,1,2).$$

We obtain

$$\langle\det(I+M)^m\rangle_{G^{\mathrm{BD\,I}}(n,r)}$$
$$=2^{mr}\prod_{j=0}^{m-1}\frac{\Gamma(2n+4j+2r+3)}{2^{2j+r+1}(4j+2r+1)!!\,\Gamma(2n+2j+r+2)}\sim\frac{2^{mr}}{\prod_{j=0}^{m-1}(4j+2r+1)!!}\cdot n^{m^2+rm}.$$

5.9 $Sp(2n)/U(n)$ – type C I

Let

$$S^{\mathrm{C\,I}}(n)=\left\{M=H\cdot\left(\begin{smallmatrix}I_n & O\\ O & -I_n\end{smallmatrix}\right)\;\middle|\;\begin{smallmatrix}H\in U(2n)\text{ is Hermitian}\\ \text{and }JH=-\overline{H}J\end{smallmatrix}\right\}.$$

Then $S^{\mathrm{C\,I}}(n)\subset Sp(2n)$ and $S^{\mathrm{C\,I}}(n)\cong Sp(2n)/U(n)\cong Sp(2n)/(Sp(2n)\cap SO(2n))$. The eigenvalues of a matrix $M\in S^{\mathrm{C\,I}}(n)$ are of the form $z_1,z_1^{-1},\cdots,z_n,z_n^{-1}$. The corresponding pdf of $z_1,z_2,\dots,z_n$ is proportional to $\Delta^{\mathrm{HO}}(\boldsymbol{z};0,\frac{1}{2},\frac{1}{2})$, and therefore we have

$$\left\langle\prod_{i=1}^{m}\det(I+x_iM)\right\rangle_{S^{\mathrm{C\,I}}(n)}=\left\langle\prod_{i=1}^{m}\Psi^{\mathrm{BC}}(\boldsymbol{z};x_i)\right\rangle_n^{0,\frac{1}{2},\frac{1}{2}}=\prod_{i=1}^{m}x_i^n\cdot P^{\mathrm{HO}}_{(n^m)}(x_1,\dots,x_m;0,2,2).$$

We obtain

$$\langle\det(I+M)^m\rangle_{S^{\mathrm{C\,I}}(n)}=\prod_{j=0}^{m-1}\frac{(n+2j+3)\Gamma(2n+4j+5)}{2^{2j+2}(4j+3)!!\,\Gamma(2n+2j+4)}\sim\frac{1}{2^m\prod_{j=1}^{m}(4j-1)!!}\cdot n^{m^2+m}.$$

5.10 $Sp(4n+2r)/(Sp(2n+2r)\times Sp(2n))$ – type C II

Let r be a nonnegative integer and let

$$G^{\mathrm{C\,II}}(n,r)=\left\{M=H\cdot\begin{pmatrix} I'_{n+r,n} & O\\ O & I'_{n+r,n}\end{pmatrix}\;\middle|\;\begin{matrix}H\in Sp(4n+2r)\text{ is Hermitian}\\ \text{of signature }(n+r,n)\end{matrix}\right\}$$

with $I'_{n+r,n}=\begin{pmatrix} I_{n+r} & O\\ O & -I_n\end{pmatrix}$. Then $G^{\mathrm{C\,II}}(n,r)\subset Sp(4n+2r)$ and $G^{\mathrm{C\,II}}(n,r)\cong Sp(4n+2r)/(Sp(2n+2r)\times Sp(2n))$. The eigenvalues of a matrix $M\in G^{\mathrm{C\,II}}(n,r)$ are of the form

$$z_1,z_1,z_1^{-1},z_1^{-1},\cdots,z_n,z_n,z_n^{-1},z_n^{-1},\underbrace{1,\ldots,1}_{2r}.$$

The corresponding pdf of $z_1,z_2,\ldots,z_n$ is proportional to $\Delta^{\mathrm{HO}}(\boldsymbol{z};2r,\frac{3}{2},2)$, and therefore we have

$$\begin{aligned}\left\langle\prod_{i=1}^m\det(I+x_iM)^{1/2}\right\rangle_{G^{\mathrm{C\,II}}(n,r)}&=\prod_{i=1}^m(1+x_i)^r\left\langle\prod_{i=1}^m\Psi^{\mathrm{BC}}(\boldsymbol{z};x_i)\right\rangle_n^{2r,\frac{3}{2},2}\\&=\prod_{i=1}^m(1+x_i)^r x_i^n\cdot P^{\mathrm{HO}}_{(n^m)}(x_1,\ldots,x_m;r,\frac{1}{4},\frac{1}{2}).\end{aligned}$$

We obtain

$$\begin{aligned}\langle\det(I+M)^m\rangle_{G^{\mathrm{C\,II}}(n,r)}&=\frac{2^{4mr+m^2+m}}{\prod_{j=0}^{m-1}(4j+4r+1)!!}\cdot\frac{\prod_{p=1}^{4m}\Gamma(n+r+\frac{p+1}{4})}{\prod_{j=1}^{2m}\Gamma(n+\frac{r}{2}+\frac{j}{4})\Gamma(n+\frac{r+1}{2}+\frac{j}{4})}\\&\sim\frac{2^{4mr+m^2+m}}{\prod_{j=0}^{m-1}(4j+4r+1)!!}n^{m^2+2mr}.\end{aligned}$$

5.11 $SO(4n+2)/U(2n+1)$ – type D III-odd

Let

$$S^{\mathrm{D\,III}}(n)=\{M\in SO(2n)\mid MJ\text{ is dexter skewsymmetric}\}.$$

We omit a definition of a dexter matrix here, but the details may be found in [Du]. We have that $S^{\mathrm{D\,III}}(n)\subset SO(2n)$ and $S^{\mathrm{D\,III}}(n)\cong SO(2n)/(SO(2n)\cap Sp(2n))\cong SO(2n)/U(n)$.

Consider $S^{\mathrm{D\,III}}(2n+1)$. This set is a "half" of $\{M\in SO(2n)\mid MJ\text{ is skewsymmetric}\}$. The eigenvalues of a matrix $M\in S^{\mathrm{D\,III}}(2n+1)\subset SO(4n+2)$ are of the form

$$z_1,z_1,z_1^{-1},z_1^{-1},\cdots,z_n,z_n,z_n^{-1},z_n^{-1},1,1.$$

The corresponding pdf of $z_1, z_2, \ldots, z_n$ is proportional to $\Delta^{\mathrm{HO}}(\boldsymbol{z}; 2, \frac{1}{2}, 2)$ and therefore we have

$$\left\langle \prod_{i=1}^{m} \det(I + x_i M)^{1/2} \right\rangle_{S^{\mathrm{D\,III}}(2n+1)} = \prod_{i=1}^{m}(1+x_i) \left\langle \prod_{i=1}^{m} \Psi^{\mathrm{BC}}(\boldsymbol{z}; x_i) \right\rangle_n^{2,\frac{1}{2},2}$$
$$= \prod_{i=1}^{m}(1+x_i)x_i^n \cdot P^{\mathrm{HO}}_{(n^m)}(x_1, \ldots, x_m; 1, -\frac{1}{4}, \frac{1}{2}).$$

We obtain

$$\langle \det(I+M)^m \rangle_{S^{\mathrm{D\,III}}(2n+1)} = \frac{2^{m^2+5m}}{\prod_{j=1}^{m}(4j-1)!!} \cdot \prod_{j=1}^{2m} \frac{\Gamma(n+\frac{j}{2}+\frac{3}{4})\Gamma(n+\frac{j}{2})}{\Gamma(n+\frac{j}{4})\Gamma(n+\frac{j}{4}+\frac{1}{2})}$$
$$\sim \frac{2^{m^2+5m}}{\prod_{j=1}^{m}(4j-1)!!} \cdot n^{m^2+m}.$$

5.12 $SO(4n)/U(2n)$ – type D III-even

Consider $S^{\mathrm{D\,III}}(2n)$. Since all skewsymmetric matrices of even size are dexter, we have

$$S^{\mathrm{D\,III}}(2n) = \{M \in SO(4n) \mid MJ \text{ is skewsymmetric}\}.$$

The eigenvalues of the matrix $M \in S^{\mathrm{D\,III}}(2n) \subset SO(4n)$ are of the form

$$z_1, z_1, z_1^{-1}, z_1^{-1}, \cdots, z_n, z_n, z_n^{-1}, z_n^{-1}.$$

The corresponding pdf of $z_1, z_2, \ldots, z_n$ is proportional to $\Delta^{\mathrm{HO}}(\boldsymbol{z}; 0, \frac{1}{2}, 2)$ and therefore we have

$$\left\langle \prod_{i=1}^{m} \det(I + x_i M)^{1/2} \right\rangle_{S^{\mathrm{D\,III}}(2n)} = \left\langle \prod_{i=1}^{m} \Psi^{\mathrm{BC}}(\boldsymbol{z}; x_i) \right\rangle_n^{0,\frac{1}{2},2} = P^{\mathrm{HO}}_{(n^m)}(x_1, \ldots, x_m; 0, -\frac{1}{4}, \frac{1}{2}).$$

Hence we obtain

$$\langle \det(I+M)^m \rangle_{S^{\mathrm{D\,III}}(2n)} = \frac{2^{m^2+m}}{\prod_{j=1}^{m-1}(4j-1)!!} \cdot \prod_{j=0}^{2m-1} \frac{\Gamma(n+\frac{j}{2}+\frac{1}{4})\Gamma(n+\frac{j-1}{2})}{\Gamma(n+\frac{j-1}{4})\Gamma(n+\frac{j+1}{4})}$$
$$\sim \frac{2^{m^2+m}}{\prod_{j=1}^{m-1}(4j-1)!!} \cdot n^{m^2-m}.$$

6 Conclusions and related works

6.1 Conclusion

We have considered random matrix ensembles S associated with classical groups and compact symmetric spaces.

The pdf for the eigenvalues in S is given by $\Delta^{\text{Jack}}(\boldsymbol{z}; 2/\beta)$ or $\Delta^{\text{HO}}(\boldsymbol{z}; k_1, k_2, k_3)$, where β or (k_1, k_2, k_3) are "parameter(s)" tabulated in List 2. We have proved that the average of the product of characteristic polynomials on S is given by a simple factor times a Jack polynomial or Jacobi polynomial with a rectangular-shaped partition, and with corresponding "dual parameter(s)".

List 2.

type	matrix set S	parameter(s)	dual parameter(s)
A (CUE)	$U(n)$	$\beta = 2$ $(\alpha = 1)$	$\beta/2 = 1/\alpha = 1$
A I (COE)	$S^{\text{A I}}(n)$	$\beta = 1$ $(\alpha = 2)$	$\beta/2 = 1/\alpha = 1/2$
A II (CSE)	$S^{\text{A II}}(n)$	$\beta = 4$ $(\alpha = 1/2)$	$\beta/2 = 1/\alpha = 2$
B	$SO(2n+1)$	$(k_1, k_2, k_3) = (1, 0, 1)$	$(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = (1, 0, 1)$
C	$Sp(2n)$	$(k_1, k_2, k_3) = (0, 1, 1)$	$(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = (0, 1, 1)$
D	$SO(2n)$	$(k_1, k_2, k_3) = (0, 0, 1)$	$(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = (0, 0, 1)$
A III - r	$G^{\text{A III}}(n, r)$	$(k_1, k_2, k_3) = (r, \frac{1}{2}, 1)$	$(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = (r, \frac{1}{2}, 1)$
BD I - r	$G^{\text{BD I}}(n, r)$	$(k_1, k_2, k_3) = (\frac{r}{2}, 0, \frac{1}{2})$	$(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = (r, 1, 2)$
C I	$S^{\text{C I}}(n)$	$(k_1, k_2, k_3) = (0, \frac{1}{2}, \frac{1}{2})$	$(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = (0, 2, 2)$
C II - r	$G^{\text{C II}}(n, r)$	$(k_1, k_2, k_3) = (2r, \frac{3}{2}, 2)$	$(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = (r, \frac{1}{4}, \frac{1}{2})$
D III - odd	$S^{\text{D III}}(2n+1)$	$(k_1, k_2, k_3) = (2, \frac{1}{2}, 2)$	$(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = (1, -\frac{1}{4}, \frac{1}{2})$
D III - even	$S^{\text{D III}}(2n)$	$(k_1, k_2, k_3) = (0, \frac{1}{2}, 2)$	$(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) = (0, -\frac{1}{4}, \frac{1}{2})$

6.2 Explicit expansions for the averages

Consider classical groups. Then the average of the product of characteristic polynomials is given by an irreducible character. Corresponding irreducible characters have determinantal expressions (Weyl's character formula) and hence, via the Laplace expansion for determinants, we obtained the explicit expansion of the average. For example, the average on $U(n)$ is given by expression (2.4). See [CFKRS1, BG].

The author could not obtain any similar expressions for symmetric spaces, because Jack and Jacobi polynomials with general parameters do not have determinantal (determinant-like) expressions.

6.3 Ratio cases

For classical groups G, the averages of the ratios of characteristic polynomials are calculated in [CFS, BG, HPZ]. For example, for $Sp(2n)$, we have that

$$\left\langle \frac{\prod_{j=1}^{m} \det(I + x_j M)}{\prod_{i=1}^{l} \det(I + y_i M)} \right\rangle_{Sp(2n)}$$
$$= \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{\pm 1\}^m} \prod_{j=1}^{m} x_j^{n(1-\epsilon_j)} \frac{\prod_{j=1}^{m} \prod_{i=1}^{l} (1 - x_j^{\epsilon_j} y_i)}{\prod_{1 \le i \le j \le m} (1 - x_i^{\epsilon_i} x_j^{\epsilon_j}) \prod_{1 \le i < j \le l} (1 - y_i y_j)}.$$

However, the problem of calculating the averages of the ratios over symmetric spaces remains open.

6.4 Hermitian matrices

We have considered unitary matrices. Results for the case of Hermitian matrices (GUE etc.) are seen in [BH] and the references of [BG].

6.5 Exceptional Lie groups

Consider the exceptional Lie groups E_6, E_7, E_8, F_4, G_2. These groups can not be realized in matrix groups, so our problem is not formulated directly. Keating, Linden, and Rudnick [KLR] studied unitary matrix representations of these groups. Given a unitary matrix representation $\rho : G \to U(n)$ of the compact Lie group G, we can define the characteristic polynomial average

$$\int_G |\det(I + x\rho(g))|^m dg,$$

where dg is the Haar measure on G. In [KLR], the average is calculated only for the cases with 7- and 14-dimensional representations of G_2 and with $x = 1$. Other cases are open problems.

6.6 Corresponding zeta functions

As we have seen in §2 (also [KS1, KS2, CFKRS1, CFKRS2]), the characteristic polynomial averages for the classical groups are closely related to L-functions. In addition, the average for the exceptional group G_2 (§6.5) corresponds to the zeta function over finite fields [KLR].

How about symmetric spaces? As far as the author knows, the corresponding zeta functions associated with symmetric spaces have not been found (even for the much-studied COE). Farmer, Mezzadri and Snaith [FMS] suggest that if the zeta function associated with the COE (type A I) exists, it should have the functional equation but no Euler product.

6.7 Study due to Yor et al.

Bourgade, Hughes, Nikeghbali, and Yor [BHNY] propose a probabilistic approach to the characteristic polynomial averages over the unitary group. They proved the following statement by such a probabilistic approach: Let $Z_n := \det(I_n - M)$, where M is distributed with the Haar measure on $U(n)$. Then,

$$\frac{\log Z_n}{\sqrt{\frac{1}{2}\log n}} \xrightarrow{\text{law}} \mathcal{N}_1 + i\mathcal{N}_2$$

in the limit as $n \to \infty$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent standard normal variables. This statement has previously been derived by Keating and Snaith [KS1] using Selberg's integrals.

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References

[BHNY] P. Bourgade, C. Hughes, A. Nikeghbali, and M. Yor, The characteristic polynomial of a random unitary matrix: a probabilistic approach, arXiv:0706.0333v1.

[BH] E. Brezin and S. Hikami, Characteristic polynomials of random matrices, Comm. Math. Phys. **214** (2000), 111–135.

[BG] D. Bump and A. Gamburd, On the averages of characteristic polynomials from classical groups, Comm. Math. Phys. **265** (2006), 227–274.

[CFKRS1] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, Autocorrelation of random matrix polynomials, Comm. Math. Phys. **237** (2003), 365–395.

[CFKRS2] ——, Integral moments of zeta and L-functions, Proc. London Math. Soc. **91** (2004), 33–104.

[CFS] J. B. Conrey, P. J. Forrester, N. C. Snaith, Averages of ratios of characteristic polynomials for the compact classical groups, Int. Math. Res. Not. (2005), no. 7, 397–431.

[Di] J. F. van Diejen, Properties of some families of hypergeometric orthogonal polynomials in several variables, Trans. Amer. Math. Soc. **351** (1999), 233–270.

[Du] E. Dueñez, Random matrix ensembles associated to compact symmetric spaces, Comm. Math. Phys. **244** (2004), 29–61.

[FMS] D. W. Farmer, F. Mezzadri, and N. C. Snaith, Random polynomials, random matrices and L-functions, Nonlinearity **19** (2006), 919–936.

[HPZ] A. Huckleberry, A. Püttmann, and M. R. Zirnbauer, Haar expectations of ratios of random characteristic polynomials, arXiv:0709.1215v1.

[KLR] J. P. Keating, N. Linden, Z. Rudnick, Random matrix theory, the exceptional Lie groups and L-functions, J. Phys. A **36** (2003), no. 12, 2933–2944.

[KS1] J. P. Keating and N. C. Snaith, Random matrix theory and $\zeta(1/2+it)$, Comm. Math. Phys. **214** (2000), 57–89.

[KS2] ——, Random matrix theory and L-functions at $s = 1/2$, Comm. Math. Phys. **214** (2000), 91–110.

[Mac] I. G. Macdonald, "Symmetric Functions and Hall Polynomials", 2nd ed., Oxford University Press, Oxford, 1995.

[Mat] S. Matsumoto, Moments of characteristic polynomials for compact symmetric spaces and Jack polynomials, J. Phys. A: Math. Theor. **40** (2007), 13567–13586.

[Me] M. L. Mehta, "Random Matrices", 3rd ed., Academic Press, 2004.

[Mi] K. Mimachi, A duality of Macdonald-Koornwinder polynomials and its application to integral representations, Duke Math. J. **107** (2001), 265–281.

[T] E. C. Titchmarsh, "The Theory of the Riemann Zeta Functions", 2nd ed., The Clarendon Press, Oxford University Press, 1986.