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Asymptotic non-degeneracy of the solution to the mean field equation[1]

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1 Introduction

In this paper, we introduce the asymptotic non-degeneracy of the solution to the mean field equation with Dirichlet boundary condition

$$-\Delta v=\lambda\frac{V(x)e^{v}}{\int_{\Omega}V(x)e^{v}}\quad\text{in }\Omega,\qquad v=0\quad\text{on }\partial\Omega,\tag{1}$$

describing the equilibrium of the mean field of many vortices of perfect fluid in Onsager's formulation [2, 3, 9], where $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth C^2 boundary $\partial\Omega$, $V=V(x)>0$ is a C^1 function defined on $\bar{\Omega}$, and $\lambda>0$ is a constant.

We recall Ma-Wei's result [11]. If $\{(\lambda_k,v_k)\}$ $(k=1,2,\ldots)$ is a solution sequence to (1) with λ_k tend to some positive value λ_0 and $\|v_k\|_\infty\to+\infty$ then $\lambda_0=8\pi m$ for some positive integer m, and there exists a set $\mathcal{S}$ which is composed m-interior points, and $v_k\to 8\pi\sum_{x_0\in\mathcal{S}}G(\cdot,x_0)$ locally uniformly in $\bar{\Omega}\setminus\mathcal{S}$, and furthermore, it holds that

$$\frac{1}{2}\nabla R(x_0)+\sum_{x_0'\in\mathcal{S}\setminus\{x_0\}}\nabla_x G(x_0,x_0')+\frac{1}{8\pi}\nabla\log V(x_0)=0$$

for all $x_0\in\mathcal{S}$, where $G=G(x,y)$ is the Green's function of $-\Delta$ in Ω with $\cdot|_{\partial\Omega}=0$, and $R=R(x)$ is the Robin function.

We consider the case of $m=1$, that is, the singular limit is $v_k\to 8\pi G(\cdot,x_0)$, and $x_0\in\mathcal{S}$ is a critical point of $R(x)+\frac{1}{4\pi}\log V(x)$.

We will introduce some results [8, 11, 12, 15] including the case of the Liouville equation in Section 4 as the appendix.

Theorem 1. *In Ma-Wei's result, if $m=1$, $V(x)$ is C^2 near $x_0\in\mathcal{S}$, and x_0 is a non-degenerate critical point of $R(x)+\frac{1}{4\pi}\log V(x)$, then the solution (λ_k,v_k) is non-degenerate for large k, that is, the linearized operator*

$$-\Delta-\lambda_k\frac{Ve^{v_k}\cdot}{\int_{\Omega}Ve^{v_k}}+\lambda_k\frac{Ve^{v_k}\int_{\Omega}(Ve^{v_k}\cdot)}{\left(\int_{\Omega}Ve^{v_k}\right)^2}\quad in\ \Omega,\quad\cdot|_{\partial\Omega}=0$$

does not have zero eigenvalue.

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In order to prove theorem 1, we follow the argument the case of the Liouville equation [8]. In this case, we assume the existence of $w_k = w_k(x)$ satisfying

$$\begin{cases} -\Delta w_k - \lambda_k \dfrac{Ve^{v_k} w_k}{\int_\Omega Ve^{v_k}} + \lambda_k \dfrac{Ve^{v_k} \int_\Omega Ve^{v_k} w_k}{\left(\int_\Omega Ve^{v_k}\right)^2} = 0 & \text{in } \Omega, \\ w_k \not\equiv 0 & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

then we shall prove the theorem by leading a contradiction. (2) seems to be complicated (See (17) in Section 4). Therefore, we use a transformation

$$\psi_k(x) = w_k(x) - \frac{\int_\Omega Ve^{v_k} w_k}{\int_\Omega Ve^{v_k}} \tag{3}$$

called SW-transformation (see also [16]), then we have

$$\begin{cases} -\Delta \psi_k = \lambda_k \dfrac{Ve^{v_k}}{\int_\Omega Ve^{v_k}} \psi_k & \text{in } \Omega, \\ \psi_k|_{\partial\Omega} = -\dfrac{\int_\Omega Ve^{v_k} w_k}{\int_\Omega Ve^{v_k}} = c_k & \text{(unknown constant)} \\ \displaystyle\int_{\partial\Omega} \frac{\partial \psi_k}{\partial \nu} = 0 & \\ \psi_k \not\equiv 0 & \text{in } \Omega \end{cases} \tag{4}$$

where $\nu = \nu(x)$ is the outer normal vector on $\partial\Omega$. (2) and (4) is equivalent for the sake of (3). Therefore, in order to prove theorem, it is enough to lead a contradiction concerning to (4).

2 Preliminaries

We confirm several assertions for (13) and (14) in Section 4 are valid to (1) again. $\{(\lambda_k, v_k)\}$ is a solution sequence to (1) satisfying $\lambda_k \to 8\pi$, and $x_k \in \Omega$ denotes a maximum point of v_k:

$$v_k(x_k) = \|v_k\|_\infty.$$

We have $x_k \to x_0$ with $\mathcal{S} = \{x_0\}$, and the blow-up point $x_0 \in \Omega$ is a critical point of $R(x) + \frac{1}{4\pi} \log V(x)$ (See also Section 4).

The following Lemma corresponds to Lemma 6 in Section 4. This Lemma is proved by following the argument in the proof of Lemma 6 (in this case, we consider $u_k(x) = v_k(x) + \log \frac{\lambda_k}{\int_\Omega Ve^{v_k}}$).

Lemma 1. *There is a constant $C_1 > 0$ such that*

$$\left| v_k(x) - \log \frac{e^{v_k(x_k)}}{\left(1 + \frac{1}{8}\lambda_k \frac{V(x_k)e^{v_k(x_k)}}{\int_\Omega Ve^{v_k}} |x - x_k|^2\right)^2} \right| \le C_1 \tag{5}$$

for all $x \in \bar{\Omega}$ and $k = 1, 2, \dots$.

We define $\delta_k > 0$ by

$$\delta_k^2 \frac{\lambda_k e^{v_k(x_k)}}{\int_\Omega V e^{v_k}} = 1, \tag{6}$$

(similarly to Section 4). The next lemma correspond to Lemma 7 in Section 4.

Lemma 2. *It holds that $\delta_k \to 0$ as $k \to \infty$.*

We assume the existence of $w_k = w_k(x)$ satisfying (2). By using SW-transformation (3), we have the problem (4) with normalized L^∞ norm

$$\begin{cases} -\Delta\psi_k = \lambda_k \dfrac{V e^{v_k}}{\int_\Omega V e^{v_k}} \psi_k \quad \text{in } \Omega, \\ \psi_k|_{\partial\Omega} = -\dfrac{\int_\Omega V e^{v_k} w_k}{\int_\Omega V e^{v_k}} = c_k \quad \text{(unknown constant)} \\ \displaystyle\int_{\partial\Omega} \frac{\partial\psi_k}{\partial\nu} = 0 \\ \|\psi_k\|_{L^\infty(\Omega)} = 1. \end{cases} \tag{7}$$

We show a contradiction. Now, we put

$$\begin{aligned} \tilde{v}_k(x) &= v_k(\delta_k x + x_k) - \|v_k\|_\infty \\ \tilde{\psi}_k(x) &= \psi_k(\delta x + x_k) \\ \tilde{V}_k(x) &= V(\delta_k x + x_k) \end{aligned}$$

where $x \in \tilde{\Omega}_k$ which is defined by $\tilde{\Omega}_k = \{x \in \mathbb{R}^2 \mid \delta_k x + x_k \in \Omega\}$. We have

$$\begin{aligned} &-\Delta\tilde{v}_k = \tilde{V}_k e^{\tilde{v}_k}, \quad \tilde{v}_k \le 0 = \tilde{v}_k(0) \quad \text{in } \tilde{\Omega}_k, \\ &\int_{\tilde{\Omega}_k} e^{\tilde{v}_k} \le C_2 \end{aligned}$$

with a constant $C_2 > 0$ independent of k, and

$$\begin{aligned} &-\Delta\tilde{\psi}_k(x) = \tilde{V}_k e^{\tilde{v}_k} \tilde{\psi}_k \quad \text{in } \tilde{\Omega}_k, \quad \tilde{\psi}_k = c_k \quad \text{on } \partial\tilde{\Omega}_k, \\ &\int_{\tilde{\Omega}_k} \frac{\partial\tilde{\psi}_k}{\partial\nu} = 0, \quad \|\tilde{\psi}_k\|_\infty = 1. \end{aligned}$$

Concerning $\tilde{v}_k$, we can apply [1]. Passing to a subsequence, it holds $\tilde{v}_k \to \tilde{v}_0$ in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ for $0 < \alpha < 1$, with $\tilde{v}_0 = \tilde{v}_0(x)$ satisfying

$$\begin{aligned} &-\Delta\tilde{v}_0 = V(x_0) e^{\tilde{v}_0}, \quad \tilde{v}_0 \le 0 = \tilde{v}_0(0) \quad \text{in } \mathbb{R}^2, \\ &\int_{\mathbb{R}^2} e^{\tilde{v}_0} < +\infty, \end{aligned}$$

and

$$\tilde{v}_0(x) = \log \frac{1}{\left\{1 + \frac{1}{8} V(x_0)|x|^2\right\}^2}$$

by [7]. Furthermore, we have $\tilde{\psi}_k \to \tilde{\psi}_0$ in $C^{2,\alpha}_{\rm loc}(\mathbb{R}^2)$ in a subsequence, with $\tilde{\psi}_0 = \tilde{\psi}_0(x)$ such that

$$-\Delta\tilde{\psi}_0 = V(x_0)e^{\tilde{v}_0}\tilde{\psi}_0 = \frac{V(x_0)}{\left\{1+\frac{1}{8}V(x_0)|x|^2\right\}^2}\tilde{\psi}_0 \quad \text{in } \mathbb{R}^2,$$
$$\|\tilde{\psi}_0\| \le 1. \tag{8}$$

and therefore,

$$\tilde{\psi}_0(x) = \sum_{i=1}^{2} \frac{a_i x_i}{\frac{8}{c}+|x|^2} + b\cdot\frac{\frac{8}{c}-|x|^2}{\frac{8}{c}+|x|^2} \tag{9}$$

by [5] where $a_i, b \in \mathbb{R}$, and $c = V(x_0) > 0$.

We shall show $\tilde{\psi}_0 = 0$ in $\mathbb{R}^2$. If this is the case, then we obtain that $|y_k| \to +\infty$, where $y_k \in \tilde{\Omega}_k$ satisfies $\tilde{\psi}_k(y_k) = \|\tilde{\psi}_k\|_\infty = 1$, a maximum point of $\tilde{\psi}_k$. Now we use the Kelvin transformation

$$\hat{v}_k = \tilde{v}_k\left(\frac{x}{|x|^2}\right), \quad \hat{\psi}_k = \tilde{\psi}_k\left(\frac{x}{|x|^2}\right)$$

and we have

$$\|\hat{\psi}_k\|_\infty = \hat{\psi}_k\left(\frac{y_k}{|y_k|^2}\right) = 1,$$
$$-\Delta\hat{\psi}_k = \frac{1}{|x|^4}\tilde{V}_k\left(\frac{x}{|x|^2}\right)e^{\hat{v}_k}\hat{\psi}_k \quad \text{in } B_1(0)\setminus\{0\}$$

for large k. On the other hand, inequality (5) implies

$$\left|\tilde{v}_k(x) + \log\left\{1+\frac{1}{8}V(x_k)|x|^2\right\}^2\right| \le C_1 \tag{10}$$

for $x \in \tilde{\Omega}_k$, and $k = 1, 2, \ldots$, and then we obtain $e^{\tilde{v}_k(x)} = O(|x|^{-4})$ uniformly in k, that is, $|x|^{-4}e^{\hat{v}_k(x)} = O(1)$ uniformly in k, and therefore, $x = 0$ is a removable singularity of $\hat{\psi}_k$:

$$-\Delta\hat{\psi}_k = a_k(x)\hat{\psi}_k \quad \text{in } B_1(0)$$

with $a_k = a_k(x)$ satisfying $\|a_k\|_{L^\infty(B_1(0))} = O(1)$. Then, the local elliptic estimate guarantees $1 = \|\hat{\psi}_k\|_{L^\infty(B_{1/2}(0))} \le \|\hat{\psi}_k\|_{L^2(B_1(0))}$, where the right-hand side converges to 0 by the dominated convergence theorem. This is a contradiction, and the proof of Theorem 1.

3 Proof of Theorem 1

In order to show $\tilde{\psi}_0 = 0$ in $\mathbb{R}^2$, we have only to show $a_1 = a_2 = b = 0$ in (9).

Lemma 3. *If $V = V(x)$ is C^2 near $x = x_0$ in Ω and x_0 is a non-degenerate critical point of $R(x) + \frac{1}{4\pi}\log V(x)$, then it holds $a_1 = a_2 = 0$.*

We can prove $a_1 = a_2 = 0$ by using the Lemma 9 in Section 4. The next Lemma is shown shorter than Lemma 10.

Lemma 4. *Under the assumptions of the Lemma 3, it holds* $b = 0$.

Proof of Lemma 4: By Lemma 3, we obtain

$$\tilde{\psi}_k(x) \to b \cdot \frac{\frac{8}{c} - |x|^2}{\frac{8}{c} + |x|^2} \quad \text{in } C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2).$$

Now we assume $b \neq 0$. We have the following equality:

$$\begin{aligned} -\psi_k \Delta v_k &= \lambda_k \frac{Ve^{v_k}}{\int_\Omega Ve^{v_k}} \psi_k, \\ -v_k \Delta \psi_k &= \lambda_k \frac{Ve^{v_k}}{\int_\Omega Ve^{v_k}} \psi_k v_k \quad \text{in } \Omega, \end{aligned}$$

and then, we obtain

$$\begin{aligned} \frac{\lambda_k}{\int_\Omega Ve^{v_k}} \int_\Omega Ve^{v_k} (\psi_k v_k - \psi_k) &= \int_\Omega (\psi_k \Delta v_k - v_k \Delta \psi_k) \\ &= \int_{\partial\Omega} \left(\psi_k \frac{\partial v_k}{\partial \nu} - v_k \frac{\partial \psi_k}{\partial \nu} \right) \\ &= c_k \int_{\partial\Omega} \frac{\partial v_k}{\partial \nu}. \end{aligned}$$

Then, we have

$$\frac{\lambda_k}{\int_\Omega Ve^{v_k}} \int_\Omega Ve^{v_k} \psi_k v_k = \frac{\lambda_k}{\int_\Omega Ve^{v_k}} \int_\Omega Ve^{v_k} \psi_k + o(1) \tag{11}$$

as $k \to \infty$. To show (11), we use the following Lemma.

Lemma 5. *It holds that* $c_k \to 0$.

We put $c = V(x_0) > 0$ (similarly to Section 4). Concerning the left-hand side of (11), we have

$$\begin{aligned} &\frac{\lambda_k}{\int_\Omega Ve^{v_k}} \int_\Omega Ve^{v_k} v_k \psi_k \\ = \ &\frac{\lambda_k}{\int_\Omega Ve^{v_k}} \int_\Omega Ve^{v_k} (v_k - \|v_k\|_\infty) \psi_k + \frac{\lambda_k \|v_k\|_\infty}{\int_\Omega Ve^{v_k}} \int_\Omega Ve^{v_k} \psi_k \\ = \ &\frac{\lambda_k}{\int_\Omega Ve^{v_k}} \int_{\tilde{\Omega}_k} \tilde{V}_k(x) e^{v_k(\delta_k x + x_k)} \tilde{v}_k(x) \tilde{\psi}_k(x) \delta_k^2 dx + \frac{\lambda_k \|v_k\|_\infty}{\int_\Omega Ve^{v_k} \psi_k} \\ = \ &\int_{\tilde{\Omega}_k} \tilde{V}_k e^{\tilde{v}_k} \tilde{v}_k \tilde{\psi}_k + \frac{\lambda_k \|v_k\|_\infty}{\int_\Omega Ve^{v_k} \psi_k} \\ = \ &\int_{\mathbb{R}^2} \frac{c}{\{1 + \frac{c}{8}|x|^2\}^2} \cdot \log \frac{1}{\left(1 + \frac{c}{8}|x|^2\right)^2} \cdot b \cdot \frac{\frac{8}{c} - |x|^2}{\frac{8}{c} + |x|^2} dx + \frac{\lambda_k \|v_k\|_\infty}{\int_\Omega Ve^{v_k} \psi_k} + o(1) \\ = \ &8\pi b + \frac{\lambda_k \|v_k\|_\infty}{\int_\Omega Ve^{v_k} \psi_k} + o(1) \end{aligned} \tag{12}$$

as $k \to \infty$, by the dominated convergence theorem.

From the equalities (9) and (11), it holds that

$$8\pi b = \frac{\lambda_k(1 - \|v_k\|_\infty)}{\int_\Omega V e^{v_k}} \int_\Omega V e^{v_k} \psi_k + o(1)$$

as $k \to \infty$. By (7), we also have the following equalities:

$$\begin{aligned} \frac{\lambda_k}{\int_\Omega V e^{v_k}} \int_\Omega V e^{v_k} \psi_k &= \int_\Omega -\Delta \psi_k \\ &= \int_{\partial\Omega} \frac{\partial \psi_k}{\partial \nu} \\ &= 0 \end{aligned}$$

for all k, finally it holds $b = 0$. □

4 Appendix

4.1 The case of the Liouville equation

In this section we introduce a fact [15], the asymptotic non-degeneracy of the solution to the Liouville-Gel'fand problem

$$-\Delta v = \lambda V(x) e^v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{13}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $V = V(x) > 0$ is a C^1 function defined on $\bar{\Omega}$, and $\lambda > 0$ is a constant. We shall extend a result of Gladiali-Grossi [8], which is valid in the homogeneous case of $V(x) \equiv 1$;

$$-\Delta v = \lambda e^v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega \tag{14}$$

based on the following fact [12]:

Theorem 2 ([12]). *If (λ_k, v_k) $(k = 1, 2, \cdots)$ is a solution sequence for (14) satisfying $\lambda_k \to 0$, then we have a subsequence (denoted by the same symbol) such that*

$$\Sigma_k = \int_\Omega \lambda_k e^{v_k} \to 8\pi m$$

for some $m = 0, 1, 2, \cdots, +\infty$. According to this value of m, we have the following.

1. *If $m = 0$, then it holds that $\|v_k\|_\infty \to 0$.*

2. *If $0 < m < +\infty$, then the blowup set of v_k $(k = 1, 2, \cdots)$, defined by*

$$\mathcal{S} = \left\{ x_0 \in \bar{\Omega} \mid \text{there exists } x_k \to x_0 \text{ such that } v_k(x_k) \to +\infty \right\},$$

is composed of m-interior points, and $v_k \to 8\pi \sum_{x_0\in S} G(\cdot, x_0)$ *locally uniformly in* $\bar{\Omega}\backslash S$, *where* $G = G(x,y)$ *denotes the Green's function of* $-\Delta$ *in* Ω *with* $\cdot|_{\partial\Omega} = 0$. *We have* $-\Delta v_k(x)dx \rightharpoonup \sum_{x_0\in S} 8\pi\delta_{x_0}(dx)$ *in the sense of measure on* $\bar{\Omega}$ *and furthermore, it holds that*

$$\frac{1}{2}\nabla R(x_0) + \sum_{x_0'\in S\backslash\{x_0\}} \nabla_x G(x_0, x_0') = 0 \tag{15}$$

for each $x_0 \in S$, *where* $R(x) = \left[G(x,y) + \frac{1}{2\pi}\log|x-y|\right]_{y=x}$ *is the Robin function.*

3. If $m = +\infty$, *then* $v_k \to +\infty$ *locally uniformly in* Ω.

Especially, $x_0 \in S$ is a critical point of Robin function $R(x)$ if the case of $m = 1$.

Gladiali-Grossi [8] is concerned with the case $m = 1$, and study the non-degeneracy of (λ_k, v_k) for large k. From the above theorem, we have $S = \{x_0\}$ if $m = 1$ and this $x_0 \in \Omega$ is a critical point of the Robin function. What they obtained is the following theorem, motivated by the study of the detailed bifurcation diagram for (14).

Theorem 3 ([8]). *If* $m = 1$ *holds in the previous theorem and* $x_0 \in S$ *is a non-degenerate critical point of* $R(x)$, *then the solution* (λ_k, v_k) *is non-degenerate for large* k, *that is, the linearized operator* $-\Delta - \lambda_k e^{v_k}$ *in* Ω *with* $\cdot|_{\partial\Omega} = 0$ *does not have zero eigenvalue.*

Theorem 2, on the other hand, has an extension to (13). Although the results of Ma-Wei [11] are presented in the mean field formulation,

$$-\Delta v = \frac{\lambda V(x)e^v}{\int_\Omega V(x)e^v} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

it is easy to translate them into the following theorem on (13), that is, setting

$$\Lambda = \lambda \int_\Omega V(x)e^v,$$

then we have

$$-\Delta v = \Lambda \frac{Ve^v}{\int_\Omega Ve^v} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega$$

which is equal to (13) (See also [13].).

Theorem 4 ([11]). *If* (λ_k, v_k) $(k = 1, 2, \cdots)$ *is a solution sequence for (13) with* $\lambda_k \to 0$ *and* $\|v_k\|_\infty \to +\infty$ *then for some positive integer m all the second alternative results in the Theorem 2 holds, provided that* Σ_k *and (15) are replaced by*

$$\Sigma_k = \int_\Omega \lambda_k V(x)e^{v_k}$$

and

$$\frac{1}{2}\nabla R(x_0) + \sum_{x_0'\in S\backslash\{x_0\}} \nabla_x G(x_0, x_0') + \frac{1}{8\pi}\nabla \log V(x_0) = 0, \tag{16}$$

respectively.

In the case of $m = 1$ again, equation (16) means that $x_0 \in \Omega$ is a critical point of $R(x) + \frac{1}{4\pi}\log V(x)$. From this point of view, it is natural to extend Theorem 3 as follows.

Theorem 5 ([15]). *In Theorem 4, if $m = 1$, $V(x)$ is C^2 near $x_0 \in \mathcal{S}$, and x_0 is a non-degenerate critical point of $R(x) + \frac{1}{4\pi}\log V(x)$, then the solution (λ_k, v_k) is non-degenerate for large k, that is, the linearized operator $-\Delta - \lambda_k V(x)e^{v_k}$ in Ω with $\cdot|_{\partial\Omega} = 0$ does not have zero eigenvalue.*

To prove the above theorem, we follow the argument of [8], namely, the existence of $w_k = w_k(x)$ $(k = 1, 2, \cdots)$ satisfying

$$\begin{aligned} &-\Delta w_k = \lambda_k V(x) e^{v_k} w_k \quad \text{in } \Omega, \qquad w_k = 0 \quad \text{on } \partial\Omega \\ &\|w_k\|_\infty = 1, \end{aligned} \tag{17}$$

implies a contradiction. $w_k^i = \frac{\partial v_k}{\partial x_i}$ $(i = 1, 2)$ solves the linearized equation

$$-\Delta w_k^i = \lambda_k e^{v_k} w_k^i \quad \text{in } \Omega$$

(except for the boundary condition). This structure is useful to prove Theorem 3, but obviously, does not hold in (13). We will introduce new arguments to compensate this obstruction in the final section.

4.2 Preliminaries

In this section, we confirm that several assertions for (14) presented in [8] are still valid for (13). Henceforth, (λ_k, v_k) $(k = 1, 2, \ldots)$ is a solution sequence for (13) satisfying

$$\Sigma_k = \int_\Omega \lambda_k V(x) e^{v_k} \to 8\pi, \qquad \lambda_k \to 0, \tag{18}$$

and $x_k \in \Omega$ denotes a maximum point of v_k;

$$v_k(x_k) = \|v_k\|_\infty.$$

Then, we have $x_k \to x_0$ with $\mathcal{S} = \{x_0\}$, and this blowup point $x_0 \in \Omega$ is a critical point of $R(x) + \frac{1}{4\pi}\log V(x)$.

The first lemma corresponds to Theorem 6 of [8].

Lemma 6. *There is a constant $C_1 > 0$ such that*

$$\left| v_k(x) - \log \frac{e^{v_k(x_k)}}{\left\{1 + \frac{1}{8}\lambda_k V(x_k) e^{v_k(x_k)} |x - x_k|^2\right\}^2} \right| \le C_1 \tag{19}$$

for any $x \in \bar{\Omega}$ and $k = 1, 2, \cdots$.

Proof: Putting $u_k = v_k + \log\lambda_k$, we obtain

$$-\Delta u_k = V(x)e^{u_k} \quad \text{in } \Omega, \qquad u_k = \log\lambda_k \quad \text{on } \partial\Omega$$
$$\int_\Omega e^{u_k} = O(1).$$

Passing to a subsequence, we shall show that $u_k(x_k) \to +\infty$ holds. Then, Theorem 0.3 of Y.Y. Li [10] guarantees the existence of $C_1 > 0$ such that

$$\left| u_k(x) - \log \frac{e^{u_k(x_k)}}{\left\{1 + \frac{1}{8}V(x_k)e^{u_k(x_k)}\left|x - x_k\right|^2\right\}^2} \right| \leq C_1$$

for any $x \in \bar{\Omega}$ and $k = 1, 2, \cdots$, or equivalently, (19).

In fact, if $u_k(x_k) \to +\infty$ does not occur, then we may assume either $u_k(x_k) \to -\infty$ or $u_k(x_k) \to c \in \mathbb{R}$. In the first alternative, we have

$$\int_\Omega \lambda_k e^{v_k} \to 0,$$

which is impossible by (18), because there are $a, b > 0$ such that

$$a \leq V(x) \leq b \qquad (x \in \bar{\Omega}).$$

In the second alternative, on the other hand, the sequence $\{u_k\}$ is locally uniformly bounded in Ω by Brezis-Merle [1], while Theorem 4 guarantees $u_k = v_k + \log\lambda_k \to -\infty$ locally uniformly in $\bar{\Omega} \setminus \{x_0\}$. Again, we have a contradiction, and the proof is complete.

Now, we define $\delta_k > 0$ by

$$\delta_k^2 \lambda_k e^{v_k(x_k)} = 1. \tag{20}$$

The next lemma corresponds to Lemma 5 of [8].

Lemma 7. *It holds that $\delta_k \to 0$.*

Proof: Inequality (19) reads;

$$\left| v_k(x) - v_k(x_k) + \log\left\{1 + \frac{V(x_k)}{8\delta_k^2}\left|x - x_k\right|^2\right\}^2 \right| \leq C_1$$

for $x \in \bar{\Omega}$ and $k = 1, 2, \ldots$, and we have $v_k \to 8\pi G(\cdot, x_0)$ locally uniformly in $\bar{\Omega} \setminus \{x_0\}$, $V(x_k) \to V(x_0)$, and $v_k(x_k) \to +\infty$. These imply $\delta_k \to 0$, because otherwise we have a contradiction.

We assume the existence of $w_k = w_k(x)$ satisfying (17) and show a contradiction. For this purpose, we put

$$\tilde{v}_k(x) = v_k(x_k + \delta_k x) - v_k(x_k)$$
$$\tilde{w}_k(x) = w_k(x_k + \delta_k x)$$
$$\tilde{V}_k(x) = V(x_k + \delta_k x),$$

where $x \in \tilde{\Omega}_k$ for $\tilde{\Omega}_k = \{x \in \mathbb{R}^2 \mid x_k + \delta_k x \in \Omega\}$. We have

$$-\Delta \tilde{v}_k = \tilde{V}_k e^{\tilde{v}_k}, \quad \tilde{v}_k \leq 0 = \tilde{v}_k(0) \qquad \text{in } \tilde{\Omega}_k,$$
$$\int_{\tilde{\Omega}_k} e^{\tilde{v}_k} = \int_\Omega \lambda_k e^{v_k} \leq C_2$$

with a constant $C_2 > 0$ independent of k, and

$$-\Delta \tilde{w}_k = \tilde{V}_k e^{\tilde{v}_k} \tilde{w}_k \quad \text{in } \tilde{\Omega}_k, \qquad \tilde{w}_k = 0 \quad \text{on } \partial\tilde{\Omega}_k$$
$$\|\tilde{w}_k\|_\infty = 1.$$

Concerning $\tilde{v}_k$, we can apply [1]. Thus, passing to a subsequence, we obtain $\tilde{v}_k \to \tilde{v}_0$ in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$ for $0 < \alpha < 1$, with $\tilde{v}_0 = \tilde{v}_0(x)$ satisfying

$$-\Delta \tilde{v}_0 = V(x_0) e^{\tilde{v}_0}, \quad \tilde{v}_0 \leq 0 = \tilde{v}_0(0) \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{\tilde{v}_0} < +\infty,$$

and therefore,

$$\tilde{v}_0(x) = \log \frac{1}{\left\{1 + \frac{1}{8} V(x_0) |x|^2\right\}^2}$$

by [7]. This implies $\tilde{w}_k \to \tilde{w}_0$ in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$ for a subsequence, with $\tilde{w}_0 = \tilde{w}_0(x)$ satisfying

$$-\Delta \tilde{w}_0 = V(x_0) e^{\tilde{v}_0} \tilde{w}_0 = \frac{V(x_0)}{\left\{1 + \frac{1}{8} V(x_0) |x|^2\right\}^2} \tilde{w}_0 \quad \text{in } \mathbb{R}^2$$
$$\|\tilde{w}_0\|_\infty \leq 1. \tag{21}$$

We shall show $\tilde{w}_0 = 0$ in $\mathbb{R}^2$. In fact, if this is the case, then it holds that $|y_k| \to +\infty$, where $y_k \in \tilde{\Omega}_k$ denotes a maximum point of $\tilde{w}_k = \tilde{w}_k(x)$; $\tilde{w}_k(y_k) = \|\tilde{w}_k\|_\infty = 1$. We make the Kelvin transformation

$$\hat{v}_k(x) = \tilde{v}_k\left(\frac{x}{|x|^2}\right), \qquad \hat{w}_k(x) = \tilde{w}_k\left(\frac{x}{|x|^2}\right),$$

and obtain

$$\|\hat{w}_k\|_\infty = \hat{w}_k\left(\frac{y_k}{|y_k|^2}\right) = 1$$
$$-\Delta \hat{w}_k = \frac{1}{|x|^4} \tilde{V}_k\left(\frac{x}{|x|^2}\right) e^{\hat{v}_k} \hat{w}_k \qquad \text{in } B_1(0) \setminus \{0\}$$

for large k. On the other hand, inequality (19) reads;

$$\left| \tilde{v}_k(x) + \log\left\{1 + \frac{1}{8} V(x_k) |x|^2\right\}^2 \right| \leq C_1, \tag{22}$$

for $x \in \tilde{\Omega}_k$ and $k = 1, 2, \cdots$, and we have $e^{\hat{v}_k(x)} = O\left(\frac{1}{|x|^4}\right)$ uniformly in k. This means $\frac{1}{|x|^4} e^{\hat{v}_k(x)} = O(1)$ uniformly in k, and therefore, $x = 0$ is a removable singularity of $\hat{w}_k$;

$$-\Delta \hat{w}_k = a_k(x) \hat{w}_k \qquad \text{in } B_1(0)$$

with $a_k = a_k(x)$ satisfying $\|a_k\|_{L^\infty(B_1(0))} = O(1)$. Then, the local elliptic estimate guarantees $1 = \|\hat{w}_k\|_{L^\infty(B_{1/2}(0))} \leq C \|\hat{w}_k\|_{L^2(B_1(0))}$, where the right-hand side converges to 0 by the dominated convergence theorem. This is a contradiction and we obtain the proof of Theorem 5.

To prove $\tilde{w}_0 = 0$ in $\mathbb{R}^2$, we put $c = V(x_0) > 0$ and $v(x) = \tilde{w}_0(x/\sqrt{c})$ in (21). Then, this $v = v(x) \in L^\infty(\mathbb{R}^2)$ satisfies

$$-\Delta v = \frac{v}{\left\{1 + \frac{1}{8}|x|^2\right\}^2} \qquad \text{in } \mathbb{R}^2$$

and hence it holds that

$$v(x) = \sum_{i=1}^{2} \frac{a_i x_i}{8 + |x|^2} + b \cdot \frac{8 - |x|^2}{8 + |x|^2}$$

by [5], where $a_i, b \in \mathbb{R}$. Thus, we have only to derive $a_i = b = 0$ in

$$\tilde{w}_0(x) = \sum_{i=1}^{2} \frac{a_i x_i}{\frac{8}{c} + |x|^2} + b \cdot \frac{\frac{8}{c} - |x|^2}{\frac{8}{c} + |x|^2}.$$

We note that $a_i/\sqrt{c}$ (a_i in the formula for $v(x)$) is newly denoted by a_i.

To show $a_i = 0$, we use the following lemma, proven similarly to (3.13) in [8].

Lemma 8. *In case $(a_1, a_2) \neq (0, 0)$, it holds that*

$$\delta_k^{-1} w_k(x) = 2\pi \sum_{j=1}^{2} a_j \frac{\partial G}{\partial y_j}(x, x_0) + o(1) \tag{23}$$

locally uniformly in $x \in \bar{\Omega} \setminus \{x_0\}$.

Proof: In fact, we have

$$\begin{aligned} w_k(x) &= \int_\Omega G(x,y)\lambda_k V(y) e^{v_k(y)} w_k(y) dy \\ &= \int_{\tilde{\Omega}_k} G(x, x_k + \delta_k y') \tilde{V}_k(y') e^{\tilde{v}_k(y')} \tilde{w}_k(y') dy' = I_{1,k}(x) + I_{2,k}(x), \end{aligned}$$

where

$$\begin{aligned} I_{1,k}(x) &= \int_{\tilde{\Omega}_k} G(x, x_k + \delta_k y') \cdot f_k(y') dy' \\ I_{2,k}(x) &= \int_{\tilde{\Omega}_k} G(x, x_k + \delta_k y') \cdot \frac{64b}{c} \cdot \frac{\frac{8}{c} - |y'|^2}{\left(\frac{8}{c} + |y'|^2\right)^3} dy' \end{aligned}$$

with

$$f_k(y) = \tilde{V}_k(y) e^{\tilde{v}_k(y)} \tilde{w}_k(y) - \frac{64b}{c} \cdot \frac{\frac{8}{c} - |y|^2}{\left(\frac{8}{c} + |y|^2\right)^3}.$$

We have

$$\tilde{V}_k(y)e^{\tilde{v}_k(y)}\tilde{w}_k(y) \to c \cdot \frac{1}{\left(1+\frac{c}{8}|y|^2\right)^2} \cdot \left(\sum_{i=1}^{2} \frac{a_i y_i}{\frac{8}{c}+|y|^2} + b \cdot \frac{\frac{8}{c}-|y|^2}{\frac{8}{c}+|y|^2}\right),$$

or equivalently,

$$f_k(y) \to f_0(y) = \frac{64}{c}\sum_{i=1}^{2} \frac{a_i y_i}{\left(\frac{8}{c}+|y|^2\right)^3},$$

locally uniformly in $y \in \mathbb{R}^2$.

We have, on the other hand, $f_k(y) = O\left(\frac{1}{|y|^4}\right)$ uniformly in $k = 1, 2, \ldots$ by (22), and therefore, $g_k(y) \to g_0(y)$ locally uniformly in $y \in \mathbb{R}^2$ by the dominated convergence theorem, where

$$g_k(y_1, y_2) = -\int_{\frac{a_1 y_1 + a_2 y_2}{a_1^2 + a_2^2}}^{+\infty} f_k\left(a_1 t + \frac{a_2^2 y_1 - a_1 a_2 y_2}{a_1^2 + a_2^2}, a_2 t - \frac{a_1 a_2 y_1 - a_1^2 y_2}{a_1^2 + a_2^2}\right) dt$$

for $k = 0, 1, 2, \ldots$. This g_k, introduced in Lemma 6 of [8], satisfies

$$a_1 \frac{\partial g_k}{\partial y_1} + a_2 \frac{\partial g_k}{\partial y_2} = f_k$$

and therefore, it holds that

$$\begin{aligned}
I_{1,k}(x) &= \int_{\tilde{\Omega}_k} G(x, x_k + \delta_k y') f_k(y') dy' \\
&= \int_{\tilde{\Omega}_k} G(x, x_k + \delta_k y') \cdot \sum_{j=1}^{2} a_j \frac{\partial g_k}{\partial y'_j}(y') dy' \\
&= -\delta_k \sum_{j=1}^{2} a_j \int_{\tilde{\Omega}_k} \frac{\partial G}{\partial y_j}(x, x_k + \delta_k y') \cdot g_k(y') dy' \\
&= \delta_k \left\{ \sum_{j=1}^{2} a_j \frac{\partial G}{\partial y_j}(x, x_0) \int_{\mathbb{R}^2} \frac{16}{c} \cdot \frac{1}{\left(\frac{8}{c}+|y'|^2\right)^2} dy' + o(1) \right\} \\
&= \delta_k \left\{ 2\pi \sum_{j=1}^{2} a_j \frac{\partial G}{\partial y_j}(x, x_0) + o(1) \right\}
\end{aligned}$$

locally uniformly in $x \in \bar{\Omega} \setminus \{x_0\}$ by the dominated convergence theorem.

To study $I_{2,k}(x)$, we note that $u(y) = \log \frac{64}{c} \cdot \frac{1}{\left(\frac{8}{c}+|y|^2\right)^2}$ satisfies

$$\frac{\partial}{\partial y_1}\left(y_1 e^u\right) + \frac{\partial}{\partial y_2}\left(y_2 e^u\right) = \frac{128}{c} \cdot \frac{\frac{8}{c}-|y|^2}{\left(\frac{8}{c}+|y|^2\right)^3},$$

and in this case we obtain

$$
\begin{aligned}
I_{2,k}(x) &= \frac{b}{2}\int_{\tilde\Omega_k} G(x, x_k+\delta_k y')\cdot\sum_{j=1}^{2}\frac{\partial}{\partial y_j}\left(y_j e^{u(y)}\right)\bigg|_{y=y'} dy' \\
&= -\delta_k\frac{b}{2}\sum_{j=1}^{2}\int_{\tilde\Omega_k}\frac{\partial G}{\partial y_j}(x, x_k+\delta_k y')\cdot y_j' e^{u(y')}dy' \\
&= -\delta_k\frac{b}{2}\left\{\sum_{j=1}^{2}\frac{\partial G}{\partial y_j}(x, x_0)\cdot\int_{\mathbb{R}^2} y_j' e^{u(y')}dy' + o(1)\right\} = o\left(\delta_k\right)
\end{aligned}
$$

locally uniformly in $x\in\bar\Omega\setminus\{x_0\}$, again by the dominated convergence theorem. Thus, the proof of (23) is complete.

4.3 Proof of Theorem 5

We prove the following lemma, using new arguments.

Lemma 9. *If $V(x)$ is C^2 near $x = x_0 \in \Omega$ and x_0 is a non-degenerate critical point of $R(x)+\frac{1}{4\pi}\log V(x)$, then it holds that $a_1 = a_2 = 0$.*

Proof: We suppose the contrary, and then obtain (23) locally uniformly in $x\in\bar\Omega\setminus\{x_0\}$. We note

$$
-\Delta\frac{\partial v_k}{\partial x_i} = \lambda_k V e^{v_k}\frac{\partial v_k}{\partial x_i} + \lambda_k V e^{v_k}\frac{\partial \log V}{\partial x_i} \qquad \text{in } \Omega
$$

and define $h_{i,k} = h_{i,k}(x)$ by

$$
-\Delta h_{i,k} = \frac{\partial \log V}{\partial x_i}\cdot\lambda_k V e^{v_k} \quad \text{in } \Omega, \qquad h_{i,k} = 0 \quad \text{on } \partial\Omega,
$$

where $i = 1, 2$. Then, it follows that

$$
w_k\Delta\left(\frac{\partial v_k}{\partial x_i} - h_{i,k}\right) - \Delta w_k\cdot\frac{\partial v_k}{\partial x_i} = 0 \qquad \text{in } \Omega
$$

by (17), and therefore, we have

$$
\int_{\partial\Omega}\left\{w_k\frac{\partial}{\partial\nu}\left(\frac{\partial v_k}{\partial x_i} - h_{i,k}\right) - \frac{\partial w_k}{\partial\nu}\cdot\left(\frac{\partial v_k}{\partial x_i} - h_{i,k}\right)\right\} = \int_\Omega h_{i,k}\Delta w_k.
$$

Here and henceforth, ν denotes the outer unit normal vector on $\partial\Omega$. Since $w_k = h_{i,k} = 0$ on $\partial\Omega$, the above equation is reduced to

$$
\begin{aligned}
&\delta_k^{-1}\int_{\partial\Omega}\frac{\partial v_k}{\partial x_i}\frac{\partial w_k}{\partial\nu} = -\delta_k^{-1}\int_\Omega h_{i,k}\Delta w_k = -\delta_k^{-1}\int_\Omega \Delta h_{i,k}\cdot w_k \\
&= \delta_k^{-1}\int_\Omega\frac{\partial\log V}{\partial x_i}\cdot\lambda_k V e^{v_k}\cdot w_k. \qquad (24)
\end{aligned}
$$

We have

$$v_k \to 8\pi G(\cdot, x_0) \qquad \text{in } C^{2,\alpha}_{\text{loc}}(\bar{\Omega} \setminus \{x_0\})$$

$$\delta_k^{-1} w_k \to 2\pi \sum_{j=1}^{2} a_j \frac{\partial G}{\partial y_j}(\cdot, x_0) \qquad \text{in } C^{2,\alpha}_{\text{loc}}(\bar{\Omega} \setminus \{x_0\})$$

by Theorem 4 and the elliptic estimate, and therefore, the left-hand side of (24) converges to

$$16\pi^2 \sum_{j=1}^{2} a_j \int_{\partial\Omega} \frac{\partial G}{\partial x_i}(x, x_0) \frac{\partial^2 G}{\partial y_j \partial \nu_x}(x, x_0).$$

Now, we apply Lemma 7 of [8];

$$\int_{\partial\Omega} \frac{\partial G}{\partial x_i}(x, x_0) \frac{\partial^2 G}{\partial y_j \partial \nu_x}(x, x_0) = -\frac{1}{2} \frac{\partial^2 R}{\partial x_i \partial x_j}(x_0), \tag{25}$$

and then obtain

$$\lim_{k \to +\infty} \delta_k^{-1} \int_{\partial\Omega} \frac{\partial v_k}{\partial x_i} \frac{\partial w_k}{\partial \nu} = -8\pi^2 \sum_{j=1}^{2} a_j \frac{\partial^2 R}{\partial x_i \partial x_j}(x_0).$$

We here note that (25) is shown by the Pohozaev identity [14].

Therefore, if we can show

$$\lim_{k \to +\infty} \delta_k^{-1} \int_{\Omega} \frac{\partial \log V}{\partial x_i} \cdot \lambda_k V e^{v_k} \cdot w_k = 2\pi \sum_{j=1}^{2} a_j \frac{\partial^2 \log V}{\partial x_i \partial x_j}(x_0), \tag{26}$$

then

$$\sum_{j=1}^{2} a_j \left\{ \frac{\partial^2 R}{\partial x_i \partial x_j}(x_0) + \frac{1}{4\pi} \frac{\partial^2 \log V}{\partial x_i \partial x_j}(x_0) \right\} = 0$$

follows for $i = 1, 2$, and hence $a_1 = a_2 = 0$ from the assumption.

For this purpose, we use the Taylor expansion around $x_k = (x_{k1}, x_{k2})$ for large k and obtain

$$\begin{aligned} \frac{\partial \log V}{\partial x_i}(x) &= \frac{\partial \log V}{\partial x_i}(x_k) + \left[(x_1 - x_{k1}) \frac{\partial}{\partial x_1} + (x_2 - x_{k2}) \frac{\partial}{\partial x_2} \right] \\ &\cdot \frac{\partial \log V}{\partial x_i}(x_k) + R_k(x) \left| x - x_k \right| \end{aligned} \tag{27}$$

for $x = (x_1, x_2)$ with $|R_k(x)| \leq r(x, x_k)$, where $r(\cdot, x_k)$ is uniformly bounded on $\bar{\Omega}$, and near x_0,

$$r(x, x_k) = \sup_{y \in B(x_k, |x - x_k|)} \sum_{i,j} \left| \frac{\partial^2 \log V}{\partial x_i \partial x_j}(y) - \frac{\partial^2 \log V}{\partial x_i \partial x_j}(x_k) \right|.$$

Therefore, this $r(\cdot, x_k)$ is continuous there, satisfying $r(x_k, x_k) = 0$ and converging to $r(\cdot, x_0)$ uniformly. We shall show that there exists $C_3 > 0$ such that

$$\delta_k^{-1} \left| (x - x_k) w_k(x) \right| \leq C_3 \tag{28}$$

for any $x \in \bar{\Omega}$ and $k = 1, 2, \ldots$. Then, we have

$$\left| \int_\Omega R_k(x) \left| x - x_k \right| \lambda_k V e^{v_k} \delta_k^{-1} w_k \right| \le C_3 \int_\Omega r(x, x_k) \lambda_k V e^{v_k} \to 0$$

by $\lambda_k V e^{v_k} dx \rightharpoonup 8\pi \delta_{x_0}(dx)$ and $r(x_0, x_0) = 0$, and therefore, the contribution of the residual term of (27) is neglected in the limit of (24).

To show (28), we use

$$w_k(x) = I_{1,k}(x) + I_{2,k}(x)$$

with

$$\delta_k^{-1} I_{1,k}(x) = -\sum_{j=1}^{2} a_j \int_{\tilde{\Omega}_k} \frac{\partial G}{\partial y_j}(x, x_k + \delta_k y') \cdot g_k(y') dy'$$

$$\delta_k^{-1} I_{2,k}(x) = -\frac{b}{2} \sum_{j=1}^{2} \int_{\tilde{\Omega}_k} \frac{\partial G}{\partial y_j}(x, x_k + \delta_k y') \cdot y_j' e^{u(y')} dy'.$$

There is $C_4 > 0$ such that

$$\left| \frac{\partial G}{\partial y_j}(x, y) \right| \le C_4 \left| x - y \right|^{-1}$$

for any $(x, y) \in \bar{\Omega} \times \bar{\Omega}$, and therefore,

$$\delta_k^{-1} \left| w_k(x) \right| \le C_4 \left(a_1 + a_2 + \frac{b}{2} \right) \cdot \int_{\tilde{\Omega}_k} \left| x - \delta_k y' - x_k \right|^{-1} \left(\left| g_k(y') \right| + \left| y_j' \right| e^{u(y')} \right) dy'$$

holds true. It is obvious that

$$\left| g_k(y) \right| + \left| y_j \right| e^{u(y)} \le C_5 \left(1 + |y|^2 \right)^{-\frac{3}{2}}$$

with $C_5 > 0$ independent of $y \in \mathbb{R}^2$ and $k = 1, 2, \ldots$, and hence

$$\delta_k^{-1} \left| w_k(x) \right| \le C_4 C_5 \left(a_1 + a_2 + \frac{b}{2} \right) \int_{\tilde{\Omega}_k} \left| x - \delta_k y' - x_k \right|^{-1} \left(1 + |y'|^2 \right)^{-\frac{3}{2}} dy'.$$

This implies

$$\begin{aligned} & \delta_k^{-1} \left| (\delta_k x') \, w_k(x_k + \delta_k x') \right| \\ & \le C_4 C_5 \left(a_1 + a_2 + \frac{b}{2} \right) \int_{\mathbb{R}^2} \frac{|x'|}{|x' - y'|} \left(1 + |y'|^2 \right)^{-\frac{3}{2}} dy', \end{aligned}$$

but we have

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{|x'|}{|x' - y'|} \left(1 + |y'|^2 \right)^{-\frac{3}{2}} dy' \\ & = \int_0^{2\pi} d\theta \int_0^\infty |x'| \left(1 + |x' + re^{i\theta}|^2 \right)^{-\frac{3}{2}} dr \le C_6 \end{aligned}$$

with $C_6 > 0$ independent of $x' \in \mathbb{R}^2$. Hence (28) follows for $x \in \bar{\Omega}$ and $k = 1, 2, \ldots$. Thus, we have proven that the limit of the right-hand side of (24) is reduced to

$$\lim_{k\to+\infty} \delta_k^{-1} \int_\Omega \frac{\partial \log V}{\partial x_i} \cdot \lambda_k V e^{v_k} \cdot w_k = \lim_{k\to+\infty} \{II_{0,k} + II_{1,k} + II_{2,k}\},$$

where

$$II_{0,k} = \frac{\partial \log V}{\partial x_i}(x_k) \int_\Omega \lambda_k V e^{v_k} \cdot \delta_k^{-1} w_k$$

$$II_{1,k} = \frac{\partial^2 \log V}{\partial x_1 \partial x_i}(x_k) \int_\Omega (x_1 - x_{k1}) \cdot \lambda_k V e^{v_k} \cdot \delta_k^{-1} w_k$$

$$II_{2,k} = \frac{\partial^2 \log V}{\partial x_2 \partial x_i}(x_k) \int_\Omega (x_2 - x_{k2}) \lambda_k V e^{v_k} \cdot \delta_k^{-1} w_k.$$

First, we have

$$II_{0,k} = -\frac{\partial \log V}{\partial x_i}(x_k) \int_\Omega \delta_k^{-1} \Delta w_k = -\frac{\partial \log V}{\partial x_i}(x_k) \int_{\partial\Omega} \delta_k^{-1} \frac{\partial w_k}{\partial \nu}$$

$$\to -\frac{\partial \log V}{\partial x_i}(x_0) \cdot 2\pi \sum_{j=1}^{2} a_j \int_{\partial\Omega} \frac{\partial^2 G}{\partial \nu_x \partial y_j}(\cdot, x_0)$$

and

$$\int_{\partial\Omega} \frac{\partial^2 G}{\partial \nu_x \partial y_j}(\cdot, x_0) = \int_{\partial B_r(x_0)} \frac{\partial^2 G}{\partial \nu_x \partial y_j}(\cdot, x_0) = \int_{\partial B_r(x_0)} \frac{\partial^2 G_0}{\partial \nu_x \partial y_j}(\cdot, x_0) + o(1)$$

as $r \downarrow 0$, where $G_0(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|}$. Then, it holds that

$$\frac{\partial^2 G_0}{\partial \nu_x \partial y_j}(x, x_0) = -\frac{1}{2\pi} \frac{x_j - x_{0j}}{|x - x_0|^3}$$

for $x \in \partial B_r(x_0)$, and therefore,

$$\int_{\partial B_r(x_0)} \frac{\partial^2 G_0}{\partial \nu_x \partial y_j}(\cdot, x_0) = 0.$$

Thus, we have proven $\lim_{k\to+\infty} II_{0,k} = 0$.

Next, we have

$$\int_\Omega (x_\ell - x_{k\ell}) \cdot \lambda_k V e^{v_k} \cdot w_k = -\int_\Omega (x_\ell - x_{k\ell}) \Delta w_k$$

$$= \int_\Omega \frac{\partial w_k}{\partial x_\ell} - \int_{\partial\Omega} (x_\ell - x_{k\ell}) \frac{\partial w_k}{\partial \nu} = \int_{\partial\Omega} \left\{ \nu_\ell w_k - (x_\ell - x_{k\ell}) \frac{\partial w_k}{\partial \nu} \right\}$$

$$= -\int_{\partial\Omega} (x_\ell - x_{k\ell}) \frac{\partial w_k}{\partial \nu}$$

for $\ell = 1, 2$, and this implies

$$II_{\ell,k} = -\frac{\partial^2 \log V}{\partial x_\ell \partial x_i}(x_k) \int_{\partial\Omega} (x_\ell - x_{k\ell}) \delta_k^{-1} \frac{\partial w_k}{\partial \nu}$$

$$\to -\frac{\partial^2 \log V}{\partial x_\ell \partial x_i}(x_0) \cdot 2\pi \sum_{j=1}^{2} a_j \int_{\partial\Omega} (x_\ell - x_{0\ell}) \frac{\partial^2 G}{\partial \nu_x \partial y_j}(\cdot, x_0).$$

Here, we have

$$\int_{\partial\Omega} (x_\ell - x_{0\ell}) \frac{\partial^2 G}{\partial\nu_x \partial y_j}(x, x_0) = \int_{\partial\Omega} \frac{\partial}{\partial\nu_x} \left\{ (x_\ell - x_{0\ell}) \frac{\partial G}{\partial y_j}(x, x_0) \right\}$$
$$= \int_{\partial B_r(x_0)} \frac{\partial}{\partial\nu_x} \left\{ (x_\ell - x_{0\ell}) \frac{\partial G}{\partial y_j}(x, x_0) \right\}$$
$$+ \int_{\Omega\setminus B_r(x_0)} \Delta \left[(x_\ell - x_{0\ell}) \frac{\partial G}{\partial y_j}(x, x_0) \right]$$
$$= \int_{\partial B_r(x_0)} \frac{\partial}{\partial\nu_x} \left\{ (x_\ell - x_{0\ell}) \frac{\partial G}{\partial y_j}(x, x_0) \right\} + 2 \int_{\Omega\setminus B_r(x_0)} \frac{\partial^2 G}{\partial x_\ell \partial y_j}(x, x_0)$$
$$= \int_{\partial B_r(x_0)} \frac{\partial}{\partial\nu_x} \left\{ (x_\ell - x_{0\ell}) \frac{\partial G_0}{\partial y_j}(x, x_0) \right\} - 2 \int_{\partial B_r(x_0)} \nu_\ell \frac{\partial G_0}{\partial y_j}(x, x_0) + o(1)$$

as $r \downarrow 0$, and the first term of the right-hand side is equal to 0 because

$$\frac{\partial}{\partial\nu_x} \left\{ (x_\ell - x_{0\ell}) \frac{\partial G_0}{\partial y_j}(x, x_0) \right\}$$
$$= \frac{x_\ell - x_{0\ell}}{r} \left[\frac{\partial G_0}{\partial y_j}(x, x_0) + r \frac{\partial^2 G_0}{\partial r \partial y_j}(x, x_0) \right] = 0$$

in terms of $r = |x - x_0|$. On the other hand, the second term is equal to

$$-\frac{1}{\pi} \int_{\partial B_r(x_0)} \frac{(x_\ell - x_{0\ell})(x_j - x_{0j})}{r^3} = -\delta_{j\ell} = \begin{cases} -1 & (\ell = j) \\ 0 & (\ell \neq j), \end{cases}$$

and therefore,

$$\lim_{k\to+\infty} II_{\ell,k} = 2\pi a_\ell \frac{\partial^2 \log V}{\partial x_\ell \partial x_i}(x_0)$$

holds for $\ell = 1, 2$. We obtain (26), and the proof is complete.

Once $a_1 = a_2 = 0$ is obtained, then the proof of $b = 0$ is similar to [8].

Lemma 10. *Under the assumptions of the previous lemma, it holds that $b = 0$.*

The proof of Lemma 10 is omitted. See Lemma 3.2 in [15].

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