数理解析研究所講究録
第 1591 巻 2008 年 11-28



# On the structure of the critical set in the minimax theorem without the Palais-Smale condition[1]

宮崎大学・工学部 大塚 浩史 (Hiroshi Ohtsuka)[2]
Faculty of Applied Physics, Department of Engineering,
University of Miyazaki

**Abstract**

In this note, we are concerned with the variational problems of minimax type relating to the mean field equation appears in the vortex system in two dimensional fluids. The variational problems are not known to satisfy the Palais-Smale condition and solutions are obtained by using an indirect method called *the Struwe's monotonicity trick* for each cases. Our interest is to discriminate between the critical points obtained by different variational problems. To this purpose, we try to study the local structures around the critical points, but standard methods seem not to be applicable also because of the lack of the Palais-Smale conditions. Under these situation, we noticed recently that the abstract refinement of the Struwe's monotonicity trick by Jeanjean is applicable to study the local structure around the critical points. We review here some known facts on the existence of solutions rather in detail and describe our result and scopes.

This is based on the joint work with Prof. Takashi Suzuki of Osaka University.

# 1 Preliminaries

We are concerned with the following equation:

$$-\Delta_g v = \lambda \left( \frac{e^v}{\int_M e^v dv_g} - \frac{1}{|M|} \right), \tag{1}$$

where $(M, g)$ is a two-dimensional compact orientable Riemannian manifold without boundary and $\lambda$ is a non-negative constant. $\Delta_g$, $dv_g$, and $|M|$ are the Laplace-Beltrami operator, the volume form, and the volume of $M$, respectively. The equation (1) is invariant under the replacement of $v$ by $v + (constant)$, and henceforth we take the normalization

$$\int_M v dv_g = 0. \tag{2}$$

[1]This work is supported by Grant-in-Aid for Scientific Research (No.19540222), Japan Society for the Promotion of Science.

[2]`ohtsuka@cc.miyazaki-u.ac.jp`

The equation like (1) is sometimes called *the mean field equation* because it appears in the mean field limit of the equilibrium states for the statistical mechanics of the vortex system of one species [5, 6, 18], see also general references of this field [24, 21, 27]. Associating $-\lambda$ and $v/\lambda$ with the inverse temperature of the state and the Hamiltonian of the system, we are indeed able to see that the non-linear term of (1) resembles the canonical Gibbs measure as follows:

$$\frac{e^v}{\int_M e^v dv_g} = \frac{e^{-(-\lambda)\frac{v}{\lambda}}}{\int_M e^{-(-\lambda)\frac{v}{\lambda}} dv_g}.$$

We note that there are many other roots of the equation (1), for example, the conformal changes of metrics on surfaces [2], the self-dual gauge field theories [36], and describing the stationary states of chemotaxis or self-interacting particles [34]. We also note that similar problems are considered on a two-dimensional bounded domain $\Omega$ under several boundary conditions according to the motivations of the problems [10, 9, 32, 28, 23, 22], but to simplify the presentation we only consider on $(M, g)$ under (2).

In this note, we are concerned with the variational solution to the problem (1) and (2). To this purpose, we take

$$E = \left\{ v \in W^{1,2}(M) \mid \int_M v dv_g = 0 \right\},$$

which forms a Hilbert space with the inner product $\langle v, w \rangle = \int_M \nabla_g v \cdot \nabla_g w dv_g$ and the norm $\|\cdot\|_E = \{\langle \cdot, \cdot \rangle\}^{1/2}$. From the following fact, that is one version of the Trudinger-Moser inequality, the right-hand side of (1) is well-defined for each $v \in E$:

**Fact 1** ([12, Theorem 1.7]). *There is a constant $C$ determined by $M$ such that*

$$\int_M e^{4\pi v^2} dv_g \leq C$$

*holds for every $v \in E$ satisfying $\|v\|_E \leq 1$.*

The problem (1) and (2) is the Euler-Lagrange equation of

$$I_\lambda(v) = \frac{1}{2}\|v\|_E^2 - \lambda \log \left( \frac{1}{|M|} \int_M e^{v(x)} dv_g \right)$$

defined on $E$. The elementary inequality

$$v(x) \leq \frac{1}{16\pi}\|v\|_E^2 + 4\pi \left( \frac{v(x)}{\|v\|_E} \right)^2$$

implies

$$\int_M e^v dv_g \le e^{\frac{1}{16\pi}\|v\|_E^2} \int_M e^{4\pi\left(\frac{v(x)}{\|v\|_E}\right)^2} dv_g.$$

Therefore, Fact 1 assures that

$$\inf_{v\in E} I_\lambda(v) > -\infty$$

for $0 < \lambda \le 8\pi$. On the other hand,

$$I_\lambda(v) = \frac{\lambda}{8\pi} I_{8\pi}(v) + \frac{1}{2}\left(1 - \frac{\lambda}{8\pi}\right)\|v\|_E^2$$

holds and hence the functional $I_\lambda(\cdot)$ is coercive on $E$ if $0 < \lambda < 8\pi$. Therefore we have the following from the standard direct method of calculus of variations:

**Fact 2** (cf. [5, Proposition 7.3] or [18, Theorem 3]). *If* $0 < \lambda < 8\pi$, *the minimization problem* $\inf_{v\in E} I_\lambda(v)$ *is attained.*

On the contrary, $I_\lambda(v)$ becomes not coercive on $E$ when $\lambda \ge 8\pi$ and even unbounded from the blow when $\lambda > 8\pi$. Moreover $I_\lambda$ is not known to satisfy *the Palais-Smale condition* (see Section 2) for $\lambda > 8\pi$, see [20, 28, 29] and the references therein. Therefore finding solutions to (1) and (2) becomes a delicate problem when $\lambda \ge 8\pi$.

In this note, first we review several known variational schemes to the problem (1) and (2), all of which are based on the combinatorially use of the so-called *Struwe's monotonicity trick* (see Section 4 for detailed description) and the blow-up analysis of the solution sequence to (1) and (2) (see Fact 5). The main interested to us is in the differences between the solutions obtained by different variational schemes. To this purpose, we are now try to study the local structure around the solutions as the critical points of $I_\lambda$, such as the Morse indices of them. There are indeed several standard method, but they seems not to be applicable to our cases also because of the lack of the Palais-Smale conditions. Under these situations, recently we noticed that the abstract refinement of the Struwe's monotonicity trick by Jeanjean [16] is also applicable to study the local structure around the critical points.

In the following, we review several variational schemes to the problem (1) and (2) and present our recent result and scopes on the local structure of the critical points of $I_\lambda$.

# 2 Minimax variational schemes

We review here the following two variational solutions to the problem (1) and (2) obtained by the different variational schemes:

- Struwe-Tarantello solution based on the mountain-pass theory [33].
- Ding-Jost-Li-Wang solution based on the linking theory [10].

**Struwe-Tarantello solution** First, we recall the standard mountain-pass theorem, given a real Banach space $(X, \|\cdot\|)$, a $C^1$ functional $I : X \to \mathbf{R}$, and $u_0, u_1 \in X$ with $u_0 \neq u_1$. Then, taking the path space

$$\Gamma := \{\gamma \in C([0,1], X) \mid \gamma(0) = u_0,\ \gamma(1) = u_1\}$$

joining $u_0$ and $u_1$, we assume $(I, u_0, u_1)$ is a triplet satisfying the mountain-pass structure,

$$c_I > \max\{I(u_0), I(u_1)\}, \tag{3}$$

where $c_I$ is the mountain-pass value of $I$ defined by

$$c_I := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)). \tag{4}$$

We call $\{u_k\} \subset X$ a Palais-Smale sequence if

$$I(u_k) \to c \qquad \text{and} \qquad I'(u_k) \to 0 \text{ in } X^*$$

for some $c \in \mathbf{R}$, and such a sequence is called the $(\mathrm{PS})_c$ sequence in short. *The Palais-Smale condition*, denoted by the (PS) condition, indicates that any $(\mathrm{PS})_c$ sequence admits a subsequence converging strongly in $X$, where $c \in \mathbf{R}$ is arbitrary.

A form of the mountain-pass theorem originated by Ambrosetti-Rabinowitz [3] is stated as follows:

**Fact 3** ([13]). *Suppose the mountain-pass structure (3) and the (PS) condition. Then, the mountain-pass value $c_I$ defined by (4) is a critical value of $I$, i.e., there is $v \in X$ satisfying $I'(v) = 0$ and $I(v) = c_I$.*

We can weaken the above required (PS) condition to the local Palais-Smale condition denoted by $(\mathrm{PS})_{c_I}$; any $(\mathrm{PS})_{c_I}$ sequence has a strongly converging subsequence, see, e.g., [35].

Obviously we have a trivial solution $v = 0$ to the problem (1) and (2), and we are able to observed that

$$I''_\lambda(0)(v,v) = \|v\|_E^2 - \frac{\lambda}{|M|}\int_M v^2 dv_g \geq \left(1 - \frac{\lambda}{\nu_2|M|}\right)\|v\|_E^2$$

for each $v \in E$, where $\nu_2$ is the second eigenvalue of $-\Delta_g$ because we assumed $\int_M v dv_g = 0$ on $E$. Therefore the trivial solution $v = 0$ is a local minimum of $I_\lambda$ when $\lambda < \nu_2|M|$. On the other hand we know that $I_\lambda$ is unbounded from the below if $\lambda > 8\pi$. Consequently $(I_\lambda, 0, v_1)$ for some $v_1 \in E$ satisfies the mountain-pass structure

$$c(\lambda) > \max\{I_\lambda(0), I_\lambda(v_1)\}, \tag{5}$$

if $8\pi < \lambda < \nu_2|M|$, where $c(\lambda)$ is the mountain-pass value for $(I_\lambda, 0, v_1)$:

$$c(\lambda) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)). \tag{6}$$

The case $8\pi < \nu_2\,|M|$ needed here actually arises when $M$ is a flat torus with the fundamental cell domain $[0,1]\times[0,1]$, i.e., $\nu_2|M| = 4\pi^2$, and henceforth, we are always concerned with such $(M, g)$. There is, however, the other case of $8\pi \geq \nu_2|M|$, e.g., the example attributed to Calabi, i.e., the dumbbell surface homeomorphic to $S^2$ with a slender pipe, see [8].

Since $(I_\lambda, 0, v_1)$ for some $v_1$ has the mountain pass structure, the only requirement is the $(\mathrm{PS})_{c(\lambda)}$ condition: any sequence $\{u_k\}$ satisfying

$$I_\lambda(u_k) \to c \qquad \text{and} \qquad I'_\lambda(u_k) \to 0 \text{ in } E^*. \tag{7}$$

has converging subsequence. Unfortunately, our $I_\lambda$ is not known to satisfy $(\mathrm{PS})_{c(\lambda)}$ condition.

To overcome this difficulty, the following fact is observed. From the Jensen's inequality

$$\log\left(\frac{1}{|M|}\int_M e^u\right) \geq \log e^{\frac{1}{|M|}\int_M u} = 0 \tag{8}$$

and hence $\lambda \mapsto I_\lambda(v)$ is non-increasing for each $v$. Therefore the inequality (8) implies also the uniform mountain-pass structure, i.e., we obtain (3) for any $\lambda \in [\lambda_0, \lambda_1]$ with fixed $v_1 \in E$, where $8\pi < \lambda_0 < \lambda_1 < \nu_2|M|$ are arbitrary. Consequently, $\lambda \mapsto c(\lambda)$ is non-increasing, and $c'(\lambda) \equiv \frac{d}{d\lambda}c(\lambda)$ exists for a.e. $\lambda$.

The existence of $c'(\lambda)$ induces the existence of a bounded $(\mathrm{PS})_{c(\lambda)}$ sequence [33, Lemma 3.5], see Section 4 for more details. Then, we can use the bounded Palais-Smale $c$ condition denoted by $(\mathrm{BPS})_c$ condition satisfied by $I_\lambda$; every *bounded* $(\mathrm{PS})_c$ sequence to $I_\lambda$ has a convergence subsequence. This $(\mathrm{BPS})_c$ condition to $I_\lambda$ is a consequence of the Trudinger-Moser inequality (Fact 1) and the elliptic estimate. In this way, we obtain the following theorem.

**Fact 4** ([33, Lemma 3.3]). *If $\lambda \mapsto c(\lambda)$ is differentiable at $\lambda \in (8\pi, \nu_2 |M|)$, then this $c(\lambda)$ is a critical value of $I_\lambda$.*

These arguments are sometimes called the Struwe's monotonicity trick [16]. Concerning the existence of the non-trivial solution, the residual set of $\lambda$ is compensated by the blowup analysis [20, 19] originated in [26, 4]. One conclusion of these results is as follows:

**Fact 5** ([19, Theorem 0.2]). *Let $\{\lambda_n\}$ be a sequence satisfying $\lambda_n \longrightarrow \lambda \geq 0$ and $\{(v_n, \lambda_n)\}$ be a sequence of solutions of (1) and (2). Then $\{v_n\}$ is relatively compact in $E$ if $\lambda \notin 8\pi\mathbf{N}$.*

Consequently, any $\lambda \in (8\pi, \nu_2 |M|) \setminus 8\pi\mathbf{N}$ admits a non-trivial solution.

**Ding-Jost-Li-Wang solution** Another variational scheme to get a solution to (1) and (2) is based on the following observation. Take an isometric embedding $(M, g)$ into $\mathbf{R}^N$ with sufficiently large $N$ by Nash's theorem, see [2, Theorem 4.34] for example, and let

$$m(v) = \frac{\int_M xe^v}{\int_M e^v} \in \mathbf{R}^N$$

denote the center of mass of $v \in E$. The following lemma, which is essentially used in [10], describes the concentration of a sequence in $E$ satisfying $I_\lambda \to -\infty$:

**Fact 6** ([7, Lemma 1]). *Let $\{v_n\} \subset E$ satisfy $I_\lambda(v_n) \to -\infty$ and $x_n \equiv m(v_n) \to x_\infty \in \mathbf{R}^N$ for $\lambda \in (8\pi, 16\pi)$. Then $x_\infty \in M$ and*

$$\frac{e^{v_n}}{\int_M e^{v_n}} \rightharpoonup \delta_{x_\infty} \qquad \text{weakly-}* \text{ in } \quad \mathcal{M}(M) = C(M)'. \tag{9}$$

The origin of this fact is in the notion of the improved Trudinger-Moser inequality established by Aubin [1]. Fact 6 says that $I_\lambda^{-1}(-\infty)$ represents the topology of the base space $M$ and we are able to use the linking theory if genus$(M) > 0$.

Suppose genus$(M) > 0$ and choose a Jordan curve $\Gamma_1 \subset M$ and a closed curve $\Gamma_2 \subset \mathbf{R}^N \setminus M$ that links $\Gamma_1$. We denote the two-dimensional unit disc as $D = \{(r, \theta) \mid 0 \leq r < 1, 0 \leq \theta < 2\pi\}$ and consider a family

$$\begin{aligned} D_\lambda = \{h \in C(D; E) | \\ & m(h(\cdot, \cdot)) \text{ can be extended continuously to } \overline{D}, \\ & m(h(1, \cdot)) : S^1 \to \Gamma_1 \text{ has degree } 1, \\ & \lim_{r \to 1} \sup_{0 \leq \theta < 2\pi} I_\lambda(h(r, \theta)) = -\infty\}. \end{aligned}$$

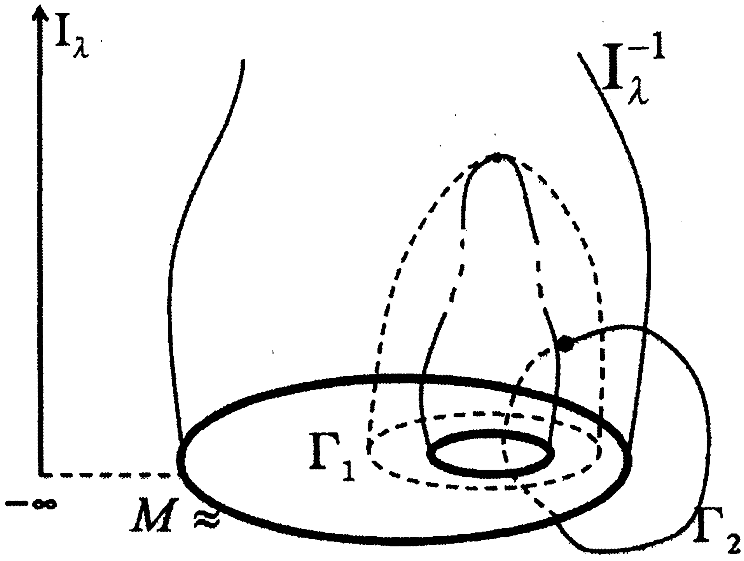


Figure 1: The linking structure

From Fact 6, we have

$$\alpha(\lambda) := \inf_{h \in D_\lambda} \max_{(r,\theta) \in D} I_\lambda(h(r,\theta)) > -\infty$$

if $8\pi < \lambda < 16\pi$ and genus$(M) > 0$, see Figure 1. On the other hand, $\lambda \mapsto \alpha(\lambda)$ is non-increasing and $\alpha'(\lambda)$ exists for a.e. $\lambda$ similar to the case of the mountain-pass value $c(\lambda)$. Using the Struwe's monotonicity trick as above, we get the following fact:

**Fact 7** ([10, Theorem 1.2]). *If $\lambda \mapsto \alpha(\lambda)$ is differentiable at $\lambda \in (8\pi, 16\pi)$, then this $\alpha(\lambda)$ is a critical value of $I_\lambda$.*

The residual set of $\lambda$ is also compensated by the blowup analysis (Fact 5) and consequently any $\lambda \in (8\pi, 16\pi)$ admits a solution to (1) and (2). Nevertheless it may happen that this solution is the trivial one $v = 0$, the solution obtained in Fact 4, or the other solution recently obtained by Djadli[11], which we mention briefly in Section 3, see Fact 11. So the next objective is the discrimination of these solutions.

# 3 The result and scopes

One method to discriminate between the solutions is to calculate *the Morse index* of each solution as the critical point of $I_\lambda$. The Morse index is defined

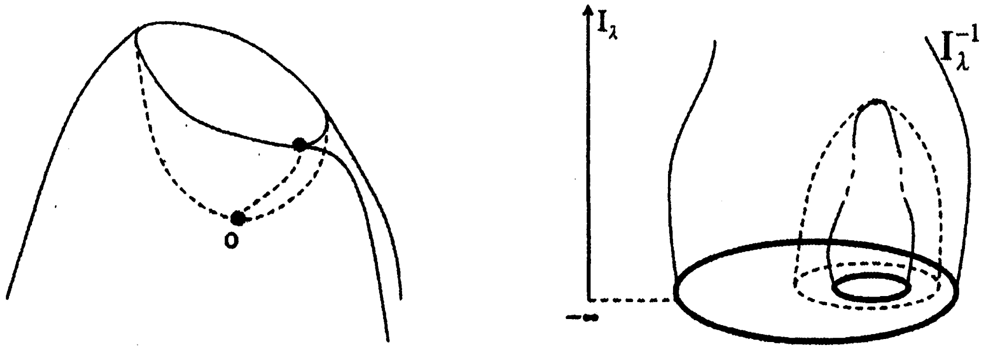


Figure 2: Critical points by Fact 4 and Fact 7.

for a critical point of a functional $I(\cdot) \in C^2(H, \mathbf{R})$ for a Hibert space $H$. Assume that $u$ is a critical point of $I$, that is, $u$ satisfies $I'(u) = 0$, and the Morse index of $u$ is defined as the supremum of the dimensions of the vector subspaces of $H$ on which $I''(u)$ is negative definite, see, e.g., [25, p.185].

In our cases, Fact 4 and Fact 7 seems to give generically critical points with the Morse index 1 and 2, respectively, see Figuer 2. But the standard argument, e.g., [13, 31], seems to require the Palais-Smale condition. So we also need to overcome this difficulty here. To this purpose we get at present the following fact (Theorem 10) for the solutions obtained by Fact 4.

For a general functional $I \in C^1(X, \mathbf{R})$ on a real Banach space $X$ and $c \in \mathbf{R}$, we set

$$\mathrm{Cr}(I, c) := \{v \in X | I(v) = c, I'(v) = 0\},$$
$$I^c := \{u \in X \mid I(u) \leq c\}, \quad \dot{I}^c := \{u \in X \mid I(u) < c\}.$$

To describe the geometric structure around critical points, Hofer introduced the following concepts:

**Definition 8** ([13]). *Given $I \in C^1(X, \mathbf{R})$ and $v \in \mathrm{Cr}(I, c)$, we say the following:*

*(i) $v$ is a local minimum if there is an open neighbourhood $V$ of $v$ such that $I(u) \geq I(v)$ for any $u \in V$.*

*(ii) $v$ is of mountain-pass type if any open neighbourhood $U$ of $v$ has the properties that $U \cap \dot{I}^c \neq \emptyset$ (that is, $v$ is not a local minimum) and $U \cap \dot{I}^c$ is not path-connected.*

Concerning the above concept, Hofer established the following fact for $I \in C^1(X, \mathbf{R})$ satisfying the (PS) condition:

**Fact 9** ([14]). *Let $c_I$ be the mountain-pass value in Fact 3. Then, there exists a critical point in* $\mathrm{Cr}(I, c_I)$, *either a local minimum or of mountain-pass type. If all the critical points in* $\mathrm{Cr}(I, c_I)$ *are isolated in $X$, furthermore, the set* $\mathrm{Cr}(I, c_I)$ *contains a critical point of mountain-pass type.*

Roughly speaking, the concept "mountain-pass type" seems to be a $C^1$ version of the situation described by the Morse index $\leq 1$. Indeed assuming that the functional $I$ belongs to $C^2(H, \mathbf{R})$ for some Hilbert space $H$ and $I'$ has the form identity-compact, Hofer proved the Morse index of the isolated mountain-pass critical point is $\leq 1$, see the proof of [13, Theorem 2] (see also [15, Thorem 2]). In general, the estimate is not improved to the Morse index $= 1$ because we do not assume the non-degeneracy of $I''$. Therefore to determine the exact Morse index of the critical point of mountain-pass type is another problem. We note that in the same papers Hofer calculates the exact topological index at the isolated mountain-pass critical point assuming the spectral assumption that the first eigenvalue $\lambda_1$ of $I''$ is simple provided $\lambda_1 = 0$, which seems not to be satisfied by our $I_\lambda$.

Recently we extend the above result to our cases $I_\lambda$ for $\lambda > 8\pi$ not satisfying the (PS) condition:

**Theorem 10** ([30]). *In Fact 4, if $c'(\lambda)$ exists and $\lambda \notin 8\pi\mathbf{N}$, then there exists a critical point in* $\mathrm{Cr}(I_\lambda, c(\lambda))$, *either a local minimum or of mountain-pass type. If all the critical points in* $\mathrm{Cr}(I_\lambda, c(\lambda))$ *are isolated, furthermore,* $\mathrm{Cr}(I_\lambda, c(\lambda))$ *contains a critical point of mountain-pass type.*

The Palais-Smale condition is used in twofold in the proof of the original result by Hofer [14], that is, the compactness of $\mathrm{Cr}(I_\lambda, c(\lambda))$ and the deformation of the sub-level set of $I_\lambda$. We can avoid the first issue by the blowup analysis (Fact 5) under the cost of $\lambda \notin 8\pi\mathbf{N}$. The second issue is compensated by the combination of the abstract setting of the Struwe's monotonicity trick by Jeanjean [16] (see Fact 12) and the quantitative deformation lemma of Willem [35] (see Fact 13), which is a deformation lemma not assuming the $(\mathrm{PS})_c$ condition a priori. In Section 4, we present this theorem in an abstract form (Theorem 16) and sketch the proof of it.

The Hofer's calculation of the Morse index for the critical point of mountain-pass type seems to be applicable for Theorem 10 and we think that it is $\leq 1$. Similar result seems to hold for the solutions obtained by Fact 7. These will be discussed in the forth coming paper.

Recently we are informed that another variational scheme based on the similar argument to Fact 7 is established, which is applicable to all $\lambda \in (8\pi, \infty)\backslash 8\pi\mathbf{N}$ *without* any topological assumption like $\mathrm{genus}(M) > 0$ [11].

For every $k = 1, 2, 3, \cdots$, assume

$$\lambda \in (8k\pi, 8(k+1)\pi).$$

Let $\Sigma_k$ be the family of formal sums of Dirac measures on $M$:

$$\Sigma_k := \left\{ \Sigma_{i=1}^k l_i \delta_{x_i} \mid l_i \geq 0, \quad \Sigma_{i=1}^k l_i = 1 \right\}$$

This is known as the formal set of barycenters of $M$ of order $k$ and $\Sigma_1$ represents nothing but $M$. It is observed that $\Sigma_k$ represents $I_\lambda^{-1}$ for $\lambda \in (8k\pi, 8(k+1)\pi)$ in an appropriate sense (cf. Fact 6) and that $\Sigma_k$ is non-contractible for any $k \geq 1$.

Let $\widehat{\Sigma}_k = \Sigma_k \times [0,1]$ be the cone over $\Sigma_k$ with $\Sigma_k \times \{0\}$ collapse to a single point. Taking an appropriate family $\Gamma \subset C(\widehat{\Sigma}_k, H_1(M))$, the minimax value

$$\Gamma(\lambda) := \inf_{\gamma \in \Gamma} \max_{m \in \widehat{\Sigma}_k} I_\lambda(\gamma(m))$$

is proved to be finite and the following is obtained by Djadli:

**Fact 11** ([11, Theorem 1.1]). *If $\lambda \mapsto \Gamma(\lambda)$ is differentiable at $\lambda \in (8k\pi, 8(k+1)\pi)$, then this $\Gamma(\lambda)$ is a critical value of $I_\lambda$.*

It seems interesting to calculate the Morse index of the critical point obtained by this variational scheme, which seems to be $\leq 3k$.

# 4 Sketch of the proof of the main result

We start with recalling the Jeanjean's abstract refinement of the Struwe's monotonicity trick [16]:

**(H1)** $(X, \|\cdot\|)$ is a real Banach space and $\Lambda \subset (0, \infty)$ is a non-void interval,

**(H2)** $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of $C^1$ functionals on $X$ with the form

$$I_\lambda(u) = A(u) - \lambda B(u)$$

for $\lambda \in \Lambda$, where $B(u) \geq 0$ for any $u \in X$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to +\infty$,

**(H3)** The mountain-pass structure holds uniformly in $\lambda \in \Lambda$:

$$c(\lambda) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(u_0), I_\lambda(u_1)\},$$

where $u_0 \neq u_1$.

As we have seen in Section 2, the functional associated with the mean field equation satisfies the above assumptions, where $X = E$,

$$A(u) = \frac{1}{2}\|\nabla u\|_2^2, \qquad B(u) = \log\left(\frac{1}{|M|}\int_M e^u\right),$$

$u_0 = 0$, and $\|u_1\|_E \gg 1$.

Thanks to $B(u) \geq 0$, the function $\lambda \in \Lambda \mapsto c(\lambda)$ is non-increasing and $c'(\lambda)$ exists for a.e $\lambda$. Then, there is a *mini-maximizing* sequence accompanied with paths of which tops are contained in a bounded set. We obtain, more precisely, the following fact.

**Fact 12** ([16, Proposition 2.1]). *If $c'(\lambda)$ exists, then any $\lambda_k \uparrow \lambda$ takes $\{\gamma_k\} \subset \Gamma$ and $K = K(c'(\lambda)) > 0$ such that*

*(i)* $\|\gamma_k(t)\| \leq K$ *if* $I_\lambda(\gamma_k(t)) \geq c(\lambda) - (\lambda - \lambda_k)$, *where* $t \in (0,1)$.

*(ii)* $\max_{t\in[0,1]} I_\lambda(\gamma_k(t)) \leq c(\lambda) + (-c'(\lambda) + 2)(\lambda - \lambda_k)$.

Here, we confirm the difference between Fact 12 and the other arguments. First, similarly to the original assertion [33], the above sequence $\{\gamma_k\} \subset \Gamma$ is taken by

$$\max_{t\in[0,1]} I_{\lambda_k}(\gamma_k(t)) \leq c(\lambda_k) + (\lambda - \lambda_k). \tag{10}$$

In Fact 12, however, this mini-maximizing sequence $\{\gamma_k\} \subset \Gamma$ is controlled in accordance with $I_\lambda$. It follows from (10) that $I_\lambda \leq I_{\lambda_k}$ and hence $c(\lambda) \leq c(\lambda_k)$, but Fact 12 (ii) is more delicate. Actually, the derivation of Lemma 12 (ii) from (10) is not trivial. Second, the monotonicity assumption (H2) and the existence of $c'(\lambda)$ are not essential. These conditions can be replaced by the existence of a strict increasing sequence $\lambda_k \uparrow \lambda$ such that

$$\frac{c(\lambda_k) - c(\lambda)}{\lambda - \lambda_k} \leq M(\lambda)$$

with $M(\lambda) < \infty$ under the cost of an additional assumption to $I_\lambda$. Then, Denjoy's theorem is applicable to infer that the residual set of such $\lambda$ is measure zero, see [17, Lemma 2.1].

Since the tops of $\{\gamma_k\}$ obtained by Fact 12 are bounded, we are able to make a meaningful deformation of them, using the (BPS) condition for the (PS) condition. This is done by the quantitative deformation lemma of Willem [35] stated as follows.

**Fact 13** ([35, Lemma 2.3]). *Given a real Banach space $(X, \|\cdot\|)$ and $\varphi = \varphi(x) \in C^1(X, \mathbf{R})$, we suppose that $S \subset X$, $c \in \mathbf{R}$, $\varepsilon > 0$, and $\delta > 0$ satisfy*

$$\|\varphi'(u)\| \geq \frac{8\varepsilon}{\delta}$$

*for every $u \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$, where*

$$S_r := \{u \in X \mid \operatorname{dist}(u, S) \leq r\}.$$

*Then, there exists $\eta \in C([0,1] \times X, X)$ such that*

*(i)* $\eta(t,u) = u$ *if either* $t = 0$ *or* $u \notin \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$,

*(ii)* $\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$,

*(iii)* $\eta(t, \cdot)$ *is a homeomorphism of* $X$ *for every* $t \in [0,1]$,

*(iv)* $\|\eta(t,u) - u\| \leq \delta$ *for every* $u \in X$ *and* $t \in [0,1]$,

*(v)* $\varphi(\eta(\cdot, u))$ *is non-increasing for every* $u \in X$,

*(vi)* $\varphi(\eta(t,u)) < c$ *for every* $u \in \varphi^c \cap S_\delta$ *and* $t \in (0,1]$.

Under these preparations, we can show the following deformation lemma à la Hofer [14, Lemma 2] (or [13, Lemma 1], [15, Lemma 1]) suitable for our case:

**Lemma 14.** *Let $I \in C^1(X, \mathbf{R})$ satisfy (BPS)$_c$ for $c \in \mathbf{R}$. Suppose that $\operatorname{Cr}(I, c)$ is bounded and contained in an open neighbourhood $W \subset B_R(0)$, where $R > 0$ and $2\overline{\delta} \equiv \operatorname{dist}(\partial W, \operatorname{Cr}(I,c)) > 0$. Then, each $\overline{\varepsilon} > 0$ and $\delta \in (0, \overline{\delta})$ admit $\varepsilon \in (0, \overline{\varepsilon}]$ and $\eta \in C([0,1] \times X, X)$ such that*

*(i)* $\eta(0,u) = u$ *and* $I(\eta(\cdot, u))$ *is non-increasing for every* $u \in X$

*(ii)* $\eta(1, (I^{c+\varepsilon} \backslash W) \cap B_R(0)) \subset I^{c-\varepsilon}$

*(iii)* $\|\eta(t,u) - u\| \leq \delta$ *for every* $u \in \overline{W}$ *and* $t \in [0,1]$

*(iv)* $\eta(t,u) = u$ *for every* $t \in [0,1]$ *and* $u \in I^{-1}((-\infty, c-\overline{\varepsilon}]) \cup I^{-1}([c+\overline{\varepsilon}, \infty)) \cup B_{R+2\delta}(0)^c$.

*Proof.* Putting $S = B_R(0) \backslash W$, we have $\overline{S_{2\delta}} \cap \operatorname{Cr}(I,c) = \emptyset$ and $S_{2\delta} \subset B_{R+2\overline{\delta}}(0)$ for $\delta \in (0, \overline{\delta})$. By (BPS)$_c$, on the other hand, there are $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $\|I'(u)\| \geq \delta_0$ for every $u \in I^{-1}([c-2\varepsilon_0, c+2\varepsilon_0]) \cap S_{2\delta}$. Taking $\varepsilon \in (0, \min(\varepsilon_0, \delta_0\delta/8, \overline{\varepsilon}/3))$, therefore, the conclusion is obtained by Fact 13 with these $S$, $c$, $\varepsilon$, and $\delta$. □

If the $(\mathrm{PS})_c$ condition arises to $I \in C^1(X, \mathbf{R})$, then the $(\mathrm{BPS})_c$ condition holds and $\mathrm{Cr}(I, c)$ is compact. This compactness of $\mathrm{Cr}(I, c)$ implies its boundedness, and also the positivity of $2\overline{\delta}$. Lemma 14 has thus decomposed the $(\mathrm{PS})_c$ condition into the $(\mathrm{BPS})_c$ condition, the boundedness of $\mathrm{Cr}(I, c)$, and $2\overline{\delta} > 0$.

Now, we shall state the topological device that is used for the proof of Theorem 10 and contains another necessity of the compactness of $\mathrm{Cr}(I, c)$.

**Fact 15** ([14, Lemma 1]). *Let $(X, d)$ be a metric space and $\Sigma, K \subset X$ be non-empty subsets such that $K$ is compact and $K \subset \overline{\Sigma}$. We assume that there is an open cover $\{U_\kappa\}_{\kappa \in K}$ of $K$ such that $\kappa \in U_\kappa$ and $U_\kappa \cap \Sigma$ is path-connected. Then there is a finite disjoint open cover $\{V_i\}_{i=1,2,\dots,m}$ of $K$ in $X$ such that $V_i \cap \Sigma$ is contained in a path-connected component of $U \cap \Sigma$, where $U = \bigcup_{\kappa \in K} U_\kappa$.*

We need to use this Fact with $K = \mathrm{Cr}(I, c)$ in the proof like Hofer [14].

We are now able to present the following abstract result that derives Theorem 10 , because $\mathrm{Cr}(I_\lambda, c(\lambda))$ is compact in (1) if $\lambda \notin 8\pi\mathbf{N}$, see [33]:

**Theorem 16.** *Suppose (H1)-(H3) and the existence of $c'(\lambda)$. Then, the $(BPS)_{c(\lambda)}$ condition implies $\mathrm{Cr}(I_\lambda, c(\lambda)) \neq \emptyset$. If $\mathrm{Cr}(I_\lambda, c(\lambda))$ is compact, moreover, there is an element in $\mathrm{Cr}(I_\lambda, c(\lambda))$, either a local minimum or a mountain-pass type. If all the critical points in $\mathrm{Cr}(I_\lambda, c(\lambda))$ are isolated, finally, then $\mathrm{Cr}(I_\lambda, c(\lambda))$ contains a critical point of mountain-pass type.*

Here we only sketch the proof of the special case to clarify the idea behind the general proof; assuming $\mathrm{Cr}(I_\lambda, c(\lambda)) = \{v\}$, we shall show that $v$ is a critical point of mountain-pass type. For this purpose we need not use the topological Fact 15.

Suppose the contrary; $v$ is not a critical point of mountain-pass type. We are able to find an open neighbourhood $U$ of $v$ such that $U \cap \dot{I}_\lambda^{c(\lambda)}$ is either empty or path-connected. We set, as in [13, Theorem 1] (or [15, Theorem 1]),

$$\overline{\varepsilon} := \frac{1}{2}(c(\lambda) - \max\{I_\lambda(u_0), I_\lambda(u_1)\}), \tag{11}$$

$$\overline{\delta} := \frac{1}{8}\min\{\mathrm{dist}((\partial U) \cup \{u_0, u_1\}, \mathrm{Cr}(I_\lambda, c(\lambda)))\},$$

$$W := \{u \in X \mid \mathrm{dist}(u, \mathrm{Cr}(I_\lambda, c(\lambda))) < \overline{\delta}\}. \tag{12}$$

Given $\lambda_k \uparrow \lambda$, now we take $\{\gamma_k\}$ and $K = K(c'(\lambda))$ of Fact 12. We may assume $W \subset B_R(0)$ for some $R \geq K(c'(\lambda))$. Applying Lemma 14 with these

$\bar{\varepsilon}$, $c = c(\lambda)$, and $W \subset B_R(0)$, we obtain $\varepsilon \subset (0, \bar{\varepsilon}]$ and $\eta \in C([0,1] \times X, X)$ for each $\delta \in (0, \bar{\delta}/2)$. This $\eta$ satisfies

$$\eta(1, \overline{W}) \subset (\overline{W})_\delta \subset U \tag{13}$$

by Lemma 14 (iii).

It holds that

$$\max_{t \in [0,1]} I_\lambda(\gamma_k(t)) \leq c(\lambda) + \varepsilon \tag{14}$$

for $k \gg 1$. Now, we derive a contradiction by deforming this $\gamma_k$ into a path in $\dot{I}_\lambda^{c(\lambda)}$, taking regards that $W$ is a residual set of $\eta$ in Lemma 14 (ii). Thus, we define

$$M := \{t \in [0,1] \mid \gamma_k(t) \notin W\} \tag{15}$$

$$B := \left(U \cap \dot{I}_\lambda^{c(\lambda)}\right) \bigcup \eta(1, \gamma_k(M)). \tag{16}$$

First, we confirm $B \subset \dot{I}_\lambda^{c(\lambda)}$. In fact,

$$\eta(1, \gamma_k(M) \cap B_R(0)) \subset I_\lambda^{c(\lambda)-\varepsilon} \subset \dot{I}_\lambda^{c(\lambda)}$$

by (14) and Lemma 14 (ii), while $\gamma_k(t) \in B_R(0)^c \subset B_K(0)^c$ implies

$$I_\lambda(\gamma_k(t)) < c(\lambda) - (\lambda - \lambda_k) < c(\lambda)$$

by Lemma 12 (i) and

$$\eta(1, \gamma_k(M) \backslash B_R(0)) \subset \dot{I}_\lambda^{c(\lambda)}$$

from the monotonicity of $I_\lambda(\eta(\cdot, u))$. This proves $B \subset \dot{I}_\lambda^{c(\lambda)}$.

Next, noting that $B (\supset \eta(1, \gamma_k(M)))$ contains $u_0$ and $u_1$, we take the path-component of $B$ containing $u_0$, denoted by $\tilde{B}$. We shall derive $u_1 \in \tilde{B} (\subset B \subset \dot{I}_\lambda^{c(\lambda)})$, which contradicts the definition of $c(\lambda)$. This proof is based on [13, Theorem 1] (or [15, Theorem 1]).

It suffices to prove $t_0 = 1$, where

$$t_0 := \sup\{t \in M \mid \eta(1, \gamma_k(t)) \in \tilde{B}\}.$$

In fact, we may assume $M \neq [0,1]$, and therefore, if $t_0 = 1$, then it holds that $\eta(1, \gamma_k(t)) \in \tilde{B}$ for a family of $\{t\}$ converging to 1. We have $I_\lambda(\gamma_k(t)) < c(\lambda) - \bar{\varepsilon}$ for such $t$, and hence $\eta(1, \gamma_k(t)) = \gamma_k(t) \in \tilde{B}$. This fact implies the desired $u_1 \in \tilde{B}$, because $\tilde{B}$ is path-connected and $u_1 = \gamma_k(1) \in B$.

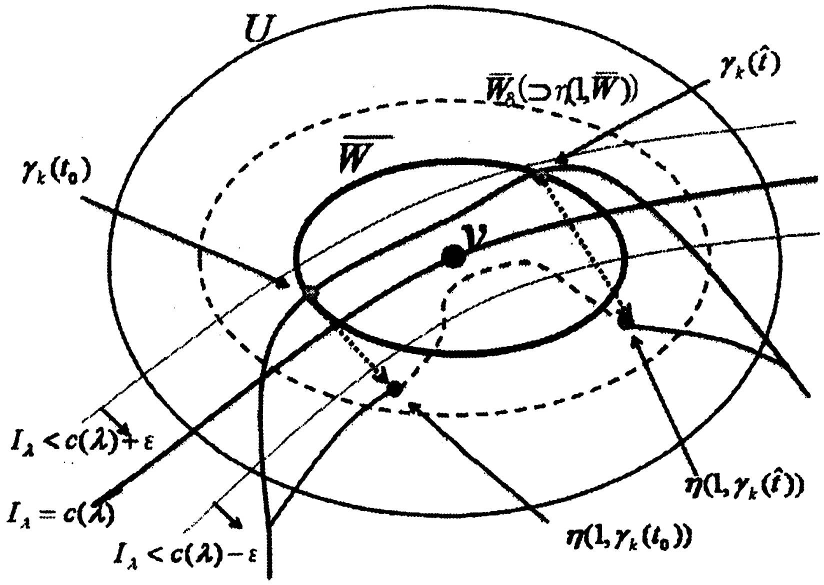


Figure 3: Local deformation of mini-maxmizing path.

Let $[t^-, t^+]$ denote the component of the closed set $M$ containing $t_0$, then it follows that $t_0 = t^+$ using some continuity argument. Therefore we are able to picture $\gamma_k(t)$ come into $\overline{W}$ at $\gamma_k(t_0) \in \partial W$, see Figure 3.

On the other hand, we obtain $\hat{t} \in (t_0, 1)$ by $t_0 = t^+$, where

$$\hat{t} = \sup\{t \in [0,1] \mid \gamma_k(t) \in \overline{W}\}. \tag{17}$$

We have $\gamma_k(\hat{t}) \in \partial W \subset B_R(0)$ and we are able to picture $\gamma_k(t)$ leave $\overline{W}$ at $\gamma_k(\hat{t})$, see Figure 3. Consequently we have

$$\eta(1, \gamma_k(\hat{t})) \in (\overline{W})_\delta \cap \dot{I}_\lambda^{c(\lambda)} \subset U \cap \dot{I}_\lambda^{c(\lambda)}$$

by (13), (14), and Lemma 14 (ii). In particular, $U \cap \dot{I}_\lambda^{c(\lambda)}$ is path-connected because it it not empty. Similarly it follows that $\eta(1, \gamma_k(t_0)) \in U \cap \dot{I}_\lambda^{c(\lambda)}$, and thus, $\eta(1, \gamma_k(\hat{t}))$ and $\eta(1, \gamma_k(t_0))$ are in the same path-component of $B \supset U \cap \dot{I}_\lambda^{c(\lambda)}$, see Figure 3. This implies $t_0 < \hat{t} \leq t_0$, a contradiction. □

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