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UNIFORM $L^2$-STABILITY FOR THE BOLTZMANN EQUATION

SEUNG-YEAL HA AND SEOK-BAE YUN

ABSTRACT. We discuss a recent progress on the uniform $L^2$-stability for the Boltzmann equation in a close-to-Maxwellian regime.

1. INTRODUCTION

The purpose of this article is to present a recent formulation [6] on the uniform $L^2$-stability for the Boltzmann equation near a global Maxwellian. Consider the Boltzmann equation describing the phase space evolution of a distribution function $F = F(x, \xi, t)$ of moderately dilute gas particles with the physical position $x \in \Omega$ and the velocity $\xi \in \mathbb{R}^3$ at time $t \in \mathbb{R}_+$:

$$\partial_t F + \xi \cdot \nabla_x F = Q(F, F), \quad x \in \Omega, \quad \xi \in \mathbb{R}^3, \quad t > 0,$$

$$F(x, \xi, 0) = F^{in}(x, \xi),$$

where $Q(F, F)$ is a quadratic collision operator whose explicit form will defined below.

Let $(\xi', \xi_*)$ be the post-collisional velocities defined in terms of pre-collisional velocities $(\xi, \xi_*)$ and $\omega \in S^2_+:

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega] \omega \quad \text{and} \quad \xi_*' = \xi_* + [(\xi - \xi_*) \cdot \omega] \omega.$$ 

In this case, the collision operator is given by the following form:

$$Q(F, F)(\xi) \equiv \frac{1}{\kappa} \int_{\mathbb{R}^3 \times S^2_+} q(\xi - \xi_*, \omega)(F' F_*' - F F_*) d\omega d\xi_*.$$ 

Here $\kappa$ is the Knudsen number which is the ratio between the mean free path and the characteristic length of the flow, $S^2_+ \equiv \{ \omega \in S^2 : (\xi - \xi_*) \cdot \omega > 0 \}$, and we used standard abbreviated notations:

$$F' \equiv F(x, \xi', t), \quad F_*' \equiv F(x, \xi_*, t), \quad F \equiv F(x, \xi, t) \quad \text{and} \quad F_* \equiv F(x, \xi_*, t).$$ 

We assume that the collision kernel $q(\cdot, \cdot)$ satisfies the inverse power law and the angular cut-off assumption:

$$q(\xi - \xi_*, \omega) = |\xi - \xi_*|^\gamma b_\gamma(\theta), \quad -\frac{3}{2} < \gamma \leq 1 \quad \text{and} \quad \frac{b_\gamma(\theta)}{\cos \theta} \leq b_* < \infty,$$

where $\theta$ is the angle between $\xi - \xi_*$ and $\omega$:

$$\theta \equiv \cos^{-1}\left(\frac{(\xi - \xi_*) \cdot \omega}{|\xi - \xi_*|}\right).$$ 

The spatial domain $\Omega$ is assumed to be either whole space $\mathbb{R}^3$ or a torus $T^3 = \mathbb{R}^3 / L^3$ ($L$ : any 3-dimensional lattice in $\mathbb{R}^3$) to focus on the initial value problem. Throughout the paper, we shall restrict ourselves to the Boltzmann equation in a maxwellian regime, and denote by $C$ the generic constant independent of time $t$. 
In a global maxwellian regime, there are many literatures available for the existence theory of solutions and convergence toward a global maxwellian (see [2, 3] for a detailed survey). We next briefly review only the global existence theory of solutions to (1.1). In [10], Ukai first established the global existence of mild solutions to the Boltzmann equation for hard potential and hard sphere models combining a spectral analysis and a bootstrapping argument. Later Caffiisch[1] and Ukai-Asano [11] further extended Ukai’s seminal work to the moderately soft potentials \( \gamma \in (-1, 0) \) on a periodic domain and whole space respectively. For the general case of \( \gamma \in (-3, 0] \), the global existence of classical solutions is finally settled by Guo [5] employing an energy method. A global existence theory in an energy space \( H^s_\xi(L^2_\xi) \) \( (s \geq 8) \) became available only in recent years due to Liu-Yang-Yu [8] and Guo [4]. In particular, Liu, Yang and Yu in [7] introduced a macro-microscopic decomposition of the solution so that the Boltzmann equation can be rewritten as a new fluid type system and an equation for a non-fluid component. Hence the existence theory for (1.1) in a global Maxwellian regime is now in a good shape for small perturbations.

The rest of this paper is organized as follows. In Section 2, we review the basic properties of the linearized collision operator and micro-macro decomposition of a solution and the Boltzmann equation, and key trilinear estimates for the stability analysis. In Section 3, we discuss an a priori uniform \( L^2 \)-stability estimates [6] for the Boltzmann equation with moderately soft potentials \( -\frac{3}{2} < \gamma \leq 1 \).

Notations: Throughout the paper, we use various local and global norms on \( \Omega, \mathbb{R}^3 \) and \( \Omega \times \mathbb{R}^3 \). Let \( h = g(x, t, \xi) \) be a measurable function on \( \Omega \times \mathbb{R} \times \mathbb{R}^3_\xi \). Below, \( p, q \in [1, \infty] \):

\[
\|h(x, t)\|_{L^q_\xi} \equiv \left\{ \begin{array}{ll} \left( \int_{\mathbb{R}^3} |h(x, t)|^q \, dx \right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\
\sup_{\xi \in \mathbb{R}^3} |h(x, t)|, & q = \infty, \end{array} \right.
\]

\[
\|h(t)\|_{L^p_\xi(L^q_\xi)} \equiv \left\{ \begin{array}{ll} \left( \int_{\mathbb{R}^3} \|h(x, t)\|^p_{L^q_\xi} \, dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\
\sup_{x \in \mathbb{R}^3} \|h(x, t)\|_{L^q_\xi}, & p = \infty, \end{array} \right.
\]

\[
\|h(t)\|_{L^p_\xi} \equiv \|h(t)\|_{L^p_\xi(L^q_\xi)}. \]

2. Preliminaries

In this section, we review the basic properties of collision operators around a global Maxwellian, and micro-macro decomposition introduced in [7, 8]. Consider the Boltzmann equation

\[
\partial_t F + \xi \cdot \nabla_x F = Q(F, F), \quad x \in \Omega, \ \xi \in \mathbb{R}^3, \ t \in \mathbb{R}_+,
\]

\[
F(0, x, \xi) = F_0(x, \xi).
\]

We now introduce a symmetric bilinear operator \( Q[F, G] \) associated with \( Q(F, F) \):

\[
Q[F, G](\xi) \equiv \frac{1}{2\kappa} \int_{\mathbb{R}^3 \times S^2} q(\xi - \xi_*, \omega) \left( F'G' + F'G - FG - F_0G \right) \, d\omega d\xi_*.
\]

Then it is easy to see that

\[
Q[F, F] \equiv Q(F, F).
\]
2.1. The Boltzmann equation near $M$. In this part, we study the linearization of the Boltzmann equation around a global Maxwellian. We first introduce the perturbation $f$ as

$F = M + M^{\frac{1}{2}}f, \quad M \equiv \frac{1}{\sqrt{(2\pi)^3}}e^{-\frac{|\xi|^2}{2}}.$

Then the perturbation $f$ satisfies the linearized Boltzmann equation:

$\partial_t f + \xi \cdot \nabla_x f = L(f) + \Gamma(f, f), \quad (2.2)$

where $L(\cdot)$ and $\Gamma(\cdot, \cdot)$ are linear and nonlinear collision operators

$L(f) \equiv 2M^{-\frac{1}{2}}Q[M, M^\frac{1}{2}f]$ and $\Gamma(f, f) \equiv M^{-\frac{1}{2}}Q[Mf, M^\frac{1}{2}f].$

We formally define a quadratic form $\Gamma[\cdot, \cdot]$ associated with $\Gamma(\cdot, \cdot)$:

$\Gamma[g, h] \equiv M^{-\frac{1}{2}}Q[Mg, Mh].$

Proposition 2.1. [2] For the Boltzmann equation (2.2), there exist positive constants $\nu_1 = \nu_1(\gamma), \nu_2 = \nu_2(\gamma), \sigma, k_1, k_2, k_3, k_4$ such that

1. $L$ has the decomposition

$L = -\nu(\xi)I + K,$

where $I$ is an identity operator and $\nu(\xi)$ is a collision frequency satisfying

$\nu_1(\xi)^\gamma \leq \nu(\xi) \leq \nu_2(\xi)^\gamma, \quad \langle \xi \rangle = 1 + |\xi|, \quad \xi \in \mathbb{R}^3,$

and $K$ is a compact operator.

2. $L$ is a non-positive self-adjoint operator on $L^2_\xi$ with the estimate

$\langle Lh, h \rangle \leq -\sigma(\nu^{\frac{1}{2}}P_1 h, P_1 h).$ 

where $\langle \cdot, \cdot \rangle$ is a usual $L^2$-inner product.

2.2. Micro-macro decomposition. In this part, we briefly present the micro-macro decomposition which enable us to see the multi-scale nature of the Boltzmann equation. This beautiful idea of decompose the solution and the Boltzmann equation to see its corresponding fluid part and non-fluid part directly at a time was introduced by Liu and Yu in [7] to the study of the positivity of Boltzmann shock. This micro-macro decomposition will play a key role in our $L^2$-stability analysis for hard potential case in Section 3.2.

The linear collision operator $L$ defines an unbounded symmetric operator on $L^2_\xi$:

$L^2_\xi \equiv (L^2_\xi(\mathbb{R}^3), \langle \cdot, \cdot \rangle) \quad \text{and} \quad \langle f, g \rangle \equiv \int_{\mathbb{R}^3} f(\xi)g(\xi)d\xi \quad \text{for} \quad f, g \in L^2_\xi.$

The null space $\mathcal{N}$ of $L$ is a five-dimensional vector space spanned by an orthonormal basis \{$x_i$\}_{i=0}^{4}:

$\mathcal{N} \equiv \text{span}\{x_0, x_1, x_2, x_3, x_4\}, \quad x_0 = M^\frac{1}{2}, \quad x_i = \xi_i M^\frac{1}{2}, \quad x_4 = \frac{1}{\sqrt{6}}(|\xi|^2 - 3)M^\frac{1}{2}, \quad \langle x_i, x_j \rangle = \delta_{ij}, \quad i = 1, 2, 3.$
We decompose Hilbert space $L_{2}^{2}$ as a direct sum of $\mathcal{N}$ and its orthogonal component $\mathcal{N}^\perp$, and we denote by $P_{0}$ the projection on this null space and $P_{1}$ the complementary projection:

\[
\begin{align*}
&f = P_{0}f + P_{1}f = f_{0} + f_{1}, \\
n&f_{0} = P_{0}f \equiv \rho(x, t)\chi_{0} + \sum_{i=1}^{3} m_{i}(x, t)\chi_{i} + e(x, t)\chi_{4}, \\
n&\rho(x, t) = \langle f, \chi_{0} \rangle, \ m_{i}(x, t) = \langle f, \chi_{i} \rangle \ (i = 1, 2, 3) , \ e(x, t) = \langle f, \chi_{4} \rangle , \\
n&f_{1} = P_{1}f = f - f_{0},
\end{align*}
\]

We next present trilinear estimates for nonlinear term $\Gamma[f + g, f - g](f - g)$. The property of $\Gamma[f + g, f - g] \in \mathcal{N}^\perp$ and Cauchy-Schwarz yield the following estimates.

Lemma 2.1. [6] Let $-\frac{3}{2} \leq \gamma \leq 1$, and $f, g$ be measurable functions in $\mathbb{R}^{3} \times \mathbb{R}^{3}$ satisfying

\[
||\nu^\frac{1}{2}(f + g)||_{L^\infty} < \infty , \ ||f - g||_{L^{2}} + ||\nu^\frac{1}{2}P_{1}(f - g)||_{L^{2}} < \infty .
\]

Then there exists a positive constant $C$ independent of $t$ such that

(i) $-\frac{3}{2} \leq \gamma \leq 0$;

\[
\int_{\mathbb{R}^{3}} \langle \Gamma[f + g, f - g], f - g \rangle (x) dx \leq C \left( ||\nu^\frac{1}{2}f(t)||_{L^{2}}^{2} + ||\nu^\frac{1}{2}g(t)||_{L^{2}}^{2} + \frac{\sigma}{2} ||\nu^\frac{1}{2}P_{1}(f(t) - g(t))||_{L^{2}}^{2} \right) .
\]

(ii) $0 < \gamma \leq 1$;

\[
\int_{\mathbb{R}^{3}} \langle \Gamma[f + g, f - g], f - g \rangle (x) dx \leq C \left( ||\nu^\frac{1}{2}f(t)||_{L^{2}}^{2} + ||\nu^\frac{1}{2}g(t)||_{L^{2}}^{2} \right) ||f(t) - g(t)||_{L^{2}}^{2} + \frac{\sigma}{2} ||\nu^\frac{1}{2}P_{1}(f(t) - g(t))||_{L^{2}}^{2} .
\]

3. A PRIORI UNIFORM $L^{2}$-STABILITY

In this section, we briefly present a priori uniform $L^{2}$-stability estimates. For details, we refer to [6]. Let $f$ and $g$ be two classical solutions to the Boltzmann equation (2.2) and $f, g \in L^\infty(\mathbb{R}^{\mathbb{R}_{x}}; L^{2}_{x,\xi} \cap L^{\infty}_{x}(L^{2}_{\xi}))$. Then $f$ and $g$ satisfy

\[
\begin{align*}
(3.1) \quad &\partial_{t}f + \xi \cdot \nabla_{x}f = L(f) + \Gamma(f, f), \\
n(3.2) \quad &\partial_{t}g + \xi \cdot \nabla_{x}g = L(g) + \Gamma(g, g).
\end{align*}
\]

We subtract (3.2) from (3.1) and multiply $(f - g)$ to both sides to find

\[
(3.3) \quad \partial_{t}||f - g||^{2} + \xi \cdot \nabla_{x}||f - g||^{2} = \langle \Gamma[f + g, f - g], f - g \rangle .
\]

We now integrate (3.3) with respect to $(x, \xi)$ using the boundary condition and Proposition 2.1 to see

\[
\int \frac{d}{dt}||f(t) - g(t)||_{L^{2}}^{2} = \int_{\Omega} \langle L(f - g), f - g \rangle dx + \int_{\Omega} \langle \Gamma[f + g, f - g], f - g \rangle dx
\]

\[
\leq -\sigma ||\nu^\frac{1}{2}P_{1}(f(t) - g(t))||_{L^{2}}^{2} + \int_{\Omega} \langle \Gamma[f + g, f - g], f - g \rangle dx .
\]

We set the uniform $L^{2}$-stability criterion as follows.

\[
(3.5) \quad \int_{0}^{\infty} \left( ||\nu^\frac{1}{2}f(s)||_{L^{2}_{x,\xi}}^{2} + ||\nu^\frac{1}{2}g(t)||_{L^{2}_{x,\xi}}^{2} \right) dt < \infty .
\]
3.1. Soft potential and Maxwellian molecule: $-\frac{3}{2} < \gamma \leq 0$. Suppose two smooth perturbations $f$ and $g$ satisfy the stability condition (3.5). In (3.4), we use Lemma 2.1 to derive a Gronwall type inequality:

$$\frac{d}{dt} \| f(t) - g(t) \|_{L^2}^2 \leq -\sigma \| \nu^{\frac{1}{2}} P_1 ( f(t) - g(t) ) \|_{L^2}^2 + C \left( \| \nu^{\frac{1}{2}} f(t) \|_{L^2 \mathbb{F}(L^2)}^2 + \| \nu^{\frac{1}{2}} g(t) \|_{L^2 \mathbb{F}(L^2)}^2 \right) \| f(t) - g(t) \|_{L^2}^2.$$

Then Gronwall's lemma yields

$$\| f(t) - g(t) \|_{L^2}^2 + \frac{\sigma}{2} \int_0^t \| \nu^{\frac{1}{2}} P_1 ( f(s) - g(s) ) \|_{L^2}^2 \, ds \leq \frac{\sigma}{2} \| f^{in} - g^{in} \|_{L^2}^2$$

This yields the uniform $L^2$-stability estimate.

**Theorem 3.1.** [6] For $\gamma \in (-\frac{3}{2}, 0]$ and let $F$ and $G$ be two classical solutions to (1.1) in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac{1}{2}} d\xi dx) \cap L^\infty_x(L^2(M^{-\frac{1}{2}} d\xi)))$ corresponding to initial data $F^{in}, G^{in}$ respectively. Suppose the smooth perturbations $f$ and $g$ satisfy the condition (3.5). Then we have

$$\sup_{0 \leq t < \infty} \| F(t) - G(t) \|_{L^2(M^{-1/2} d\xi dx)} \leq C \| F^{in} - G^{in} \|_{L^2(M^{-1/2} d\xi dx)},$$

where $C$ is a positive constant independent of $t$.

**Remark 3.1.** As a direct application of the above theorem, the classical solutions in [1, 5, 11] are uniformly $L^2$-stable.

3.2. Hard potential and hard sphere model: $0 < \gamma \leq 1$. Suppose two smooth perturbations $f$ and $g$ satisfy the stability condition (3.5) and the smallness condition:

$$\| f(t) \|_{L^2 \mathbb{F}(L^2)} + \| g(t) \|_{L^2 \mathbb{F}(L^2)} \ll \frac{\sigma}{4}.$$

In (3.4), we use Lemma 2.1 to get

$$\frac{d}{dt} \| f(t) - g(t) \|_{L^2}^2 \leq C \left( \| \nu^{\frac{1}{2}} f(t) \|_{L^2 \mathbb{F}(L^2)}^2 + \| \nu^{\frac{1}{2}} g(t) \|_{L^2 \mathbb{F}(L^2)}^2 \right) \| f(t) - g(t) \|_{L^2}^2 + \left[ -\sigma \right] \| \nu^{\frac{1}{2}} P_1 ( f(t) - g(t) ) \|_{L^2}^2.$$

We use (3.7) to find

$$\frac{d}{dt} \| f(t) - g(t) \|_{L^2}^2 \leq C \left( \| \nu^{\frac{1}{2}} f(t) \|_{L^2 \mathbb{F}(L^2)}^2 + \| \nu^{\frac{1}{2}} g(t) \|_{L^2 \mathbb{F}(L^2)}^2 \right) \| f(t) - g(t) \|_{L^2}^2 - \frac{\sigma}{4} \| \nu^{\frac{1}{2}} P_1 ( f(t) - g(t) ) \|_{L^2}^2.$$

Then Gronwall’s lemma yield the following stability estimate.

**Theorem 3.2.** [6] For $\gamma \in (0, 1]$ and let $F$ and $G$ be two small classical solutions to (1.1) in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac{1}{2}} d\xi dx) \cap L^\infty_x(L^2(M^{-\frac{1}{2}} d\xi)))$ corresponding to small initial data $F^{in}, G^{in}$
respectively. Suppose the smooth perturbations $f$ and $g$ satisfy (3.5) and (3.6). Then we have
\[
\sup_{0 \leq t < \infty} \| F(t) - G(t) \|_{L^2(M^{-1/2}d\xi dx)} \leq C \| F^{in} - G^{in} \|_{L^2(M^{-1/2}d\xi dx)},
\]
where $C$ is a positive constant independent of $t$.

Remark 3.2. As a direct application of this theorem, the classical solutions in [12] are uniformly $L^2$-stable.

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