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UNIFORM $L^2$-STABILITY FOR THE BOLTZMANN EQUATION

SEUNG-YEAL HA AND SEOK-BAE YUN

ABSTRACT. We discuss a recent progress on the uniform $L^2$-stability for the Boltzmann equation in a close-to-Maxwellian regime.

1. INTRODUCTION

The purpose of this article is to present a recent formulation [6] on the uniform $L^2$-stability for the Boltzmann equation near a global Maxwellian. Consider the Boltzmann equation describing the phase space evolution of a distribution function $F = F(x, \xi, t)$ of moderately dilute gas particles with the physical position $x \in \Omega$ and the velocity $\xi \in \mathbb{R}^3$ at time $t \in \mathbb{R}_+$:

$$\partial_t F + \xi \cdot \nabla_x F = Q(F, F), \quad x \in \Omega, \quad \xi \in \mathbb{R}^3, \quad t > 0,$$

$$F(x, \xi, 0) = F^{in}(x, \xi),$$

where $Q(F, F)$ is a quadratic collision operator whose explicit form will defined below.

Let $(\xi', \xi_*)$ be the post-collisional velocities defined in terms of pre-collisional velocities $(\xi, \xi_*)$ and $\omega \in S^2_+$:

$$(1.2) \quad \xi' = \xi - [(\xi - \xi_*) \cdot \omega] \omega \quad \text{and} \quad \xi_*' = \xi_* + [(\xi - \xi_*) \cdot \omega] \omega.$$

In this case, the collision operator is given by the following form:

$$(1.3) \quad Q(F, F)(\xi) \equiv \frac{1}{\kappa} \int_{\mathbb{R}^3 \times S^2_+} q(\xi - \xi_*, \omega)(F'F_*' - FF_*)d\omega d\xi_*.$$

Here $\kappa$ is the Knudsen number which is the ratio between the mean free path and the characteristic length of the flow, $S^2_+ \equiv \{\omega \in S^2 : (\xi - \xi_*) \cdot \omega > 0\}$, and we used standard abbreviated notations:

$$F' \equiv F(x, \xi', t), \quad F_*' \equiv F(x, \xi_*, t), \quad F \equiv F(x, \xi, t) \quad \text{and} \quad F_* \equiv F(x, \xi_*, t).$$

We assume that the collision kernel $q(\cdot, \cdot)$ satisfies the inverse power law and the angular cut-off assumption:

$$q(\xi - \xi_*, \omega) = |\xi - \xi_*|^\gamma b_\gamma(\theta), \quad -\frac{3}{2} < \gamma \leq 1 \quad \text{and} \quad \frac{b_\gamma(\theta)}{\cos \theta} \leq b_* < \infty,$$

where $\theta$ is the angle between $\xi - \xi_*$ and $\omega$:

$$\theta \equiv \cos^{-1} \left( \frac{(\xi - \xi_*) \cdot \omega}{|\xi - \xi_*|} \right).$$

The spatial domain $\Omega$ is assumed to be either whole space $\mathbb{R}^3$ or a torus $T^3 = \mathbb{R}^3/L^3$ ($L$: any 3-dimensional lattice in $\mathbb{R}^3$) to focus on the initial value problem. Throughout the paper, we shall restrict ourselves to the Boltzmann equation in a maxwellian regime, and denote by $C$ the generic constant independent of time $t$. 
In a global maxwellian regime, there are many literatures available for the existence theory of solutions and convergence toward a global maxwellian (see [2, 3] for a detailed survey). We next briefly review only the global existence theory of solutions to (1.1). In [10], Ukai first established the global existence of mild solutions to the Boltzmann equation for hard potential and hard sphere models combining a spectral analysis and a bootstrapping argument. Later Caflisch [1] and Ukai-Asano [11] further extended Ukai’s seminal work to the moderately soft potentials $\gamma \in (-1, 0]$ on a periodic domain and whole space respectively. For the general case of $\gamma \in (-3, 0]$, the global existence of classical solutions was finally settled by Guo [5] employing an energy method. A global existence theory in an energy space $H_s^\#(\mathbb{R}_{\xi}^3)$ ($s \geq 3$) became available only in recent years due to Liu-Yang-Yu [8] and Guo [4]. In particular, Liu, Yang and Yu in [7] introduced a macro-microscopic decomposition of the solution so that the Boltzmann equation can be rewritten as a new fluid type system and an equation for a non-fluid component. Hence the existence theory for (1.1) in a global Maxwellian regime is now in a good shape for small perturbations.

The rest of this paper is organized as follows. In Section 2, we review the basic properties of the linearized collision operator and micro-macro decomposition of a solution and the Boltzmann equation, and key trilinear estimates for the stability analysis. In Section 3, we discuss a priori uniform $L^2$-stability estimates [6] for the Boltzmann equation with moderately soft potentials $-\frac{3}{2} < \gamma \leq 1$.

**Notations:** Throughout the paper, we use various local and global norms on $\Omega$, $\mathbb{R}_{\xi}^3$ and $\Omega \times \mathbb{R}_{\xi}^3$. Let $h = g(x, t, \xi)$ be a measurable function on $\Omega \times \mathbb{R}_{t} \times \mathbb{R}_{\xi}^3$. Below, $p, q \in [1, \infty]$:

$$\|h(x, t)\|_{L^q} \equiv \left\{ \begin{array}{ll} \left( \int_{\mathbb{R}^3} |f(x, \xi, t)|^q dx \right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\
\text{esssup}_{\xi \in \mathbb{R}^3} |f(x, \xi, t)|, & q = \infty, \end{array} \right.$$  

$$\|h(t)\|_{L^p(L^q_{\xi})} \equiv \left\{ \begin{array}{ll} \left( \int_{\mathbb{R}^3} \|h(x, t)\|_{L^q_{\xi}}^{p} dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\
\text{esssup}_{x \in \mathbb{R}^3} \|h(x, t)\|_{L^q_{\xi}}, & p = \infty, \end{array} \right.$$  

$$\|h(t)\|_{L^p} \equiv \|h(t)\|_{L^p(L^2_{\xi})}.$$  

2. PRELIMINARIES

In this section, we review the basic properties of collision operators around a global Maxwellian, and micro-macro decomposition introduced in [7, 8]. Consider the Boltzmann equation:

$$\partial_t F + \xi \cdot \nabla x F = Q(F, F), \quad x \in \Omega, \xi \in \mathbb{R}^3, t \in \mathbb{R}_+,$$

$$F(0, x, \xi) = F_0(x, \xi).$$

We now introduce a symmetric bilinear operator $Q[F, G]$ associated with $Q(F, F)$:

$$Q[F, G](\xi) \equiv \frac{1}{2\kappa} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} q(\xi - \xi_*, \omega) (F'G_* + F_*G' - FG_* - F_*G) d\omega d\xi_*.$$  

Then it is easy to see that $$Q[F, F] \equiv Q(F, F).$$
UNIFORM $L^{2}$-STABILITY FOR THE BOLTZMANN EQUATION

2.1. The Boltzmann equation near $M$. In this part, we study the linearization of the Boltzmann equation around a global Maxwellian. We first introduce the perturbation $f$ as

$$F = M + M^{\frac{1}{2}}f, \quad M \equiv \frac{1}{\sqrt{(2\pi)^3}} e^{-\frac{|\xi|^2}{2}}.$$  

Then the perturbation $f$ satisfies the linearized Boltzmann equation:

$$\partial_t f + \xi \cdot \nabla_x f = L(f) + \Gamma(f, f),$$  

where $L(\cdot)$ and $\Gamma(\cdot, \cdot)$ are linear and nonlinear collision operators

$$L(f) \equiv 2\Lambda f^{-}\tau Q[M,M^{\frac{1}{2}}f]$$  

and

$$\Gamma(f, f) \equiv M^{-f}Q[M^{\frac{1}{2}}f,M^{\frac{1}{2}}f].$$  

We formally define a quadratic form $\Gamma[\cdot, \cdot]$ associated with $\Gamma(\cdot, \cdot)$:

$$\Gamma[g, h] \equiv M^{-1}Q[M^{\frac{1}{2}}g, M^{\frac{1}{2}}h].$$  

Proposition 2.1. [2] For the Boltzmann equation (2.2), there exist positive constants $\nu_{1} = \nu_{1}(\gamma), \nu_{2} = \nu_{2}(\gamma), \sigma, k_{1}, k_{2}, k_{3}, k_{4}$ such that

(1) $L$ has the decomposition

$$L = -\nu(\xi)I + K,$$

where $I$ is an identity operator and $\nu(\xi)$ is a collision frequency satisfying

$$\nu_{1}(\xi)^{\gamma} \leq \nu(\xi) \leq \nu_{2}(\xi)^{\gamma}, \quad \langle \xi \rangle = 1 + |\xi|, \quad \xi \in \mathbb{R}^{3},$$

and $K$ is a compact operator.

(2) $L$ is a non-positive self-adjoint operator on $L^{2}_{\xi}$ with the estimate

$$\langle Lh, h \rangle \leq -\sigma \nu^{\frac{1}{2}} P_{1}h, P_{1}h).$$

where $\langle \cdot, \cdot \rangle$ is a usual $L^{2}$-inner product.

2.2. Micro-macro decomposition. In this part, we briefly present the micro-macro decomposition which enable us to see the multi-scale nature of the Boltzmann equation. This beautiful idea of decompose the solution and the Boltzmann equation to see its corresponding fluid part and non-fluid part directly at a time was introduced by Liu and Yu in [7] to the study of the positivity of Boltzmann shock. This micro-macro decomposition will play a key role in our $L^{2}$-stability analysis for hard potential case in Section 3.2.

The linear collision operator $L$ defines an unbounded symmetric operator on $L^{2}_{\xi}$:

$$L^{2}_{\xi} \equiv (L^{2}_{\xi}(\mathbb{R}^{3}), \langle \cdot, \cdot \rangle) \quad \text{and} \quad \langle f, g \rangle \equiv \int_{\mathbb{R}^{3}} f(\xi)g(\xi)d\xi \quad \text{for} \quad f, g \in L^{2}_{\xi}.$$  

The null space $\mathcal{N}$ of $L$ is a five-dimensional vector space spanned by an orthonormal basis \{\chi_{i}\}_{i=0}^{4}:

$$\mathcal{N} \equiv \text{span}\{\chi_{0}, \chi_{1}, \chi_{2}, \chi_{3}, \chi_{4}\},$$

$$\chi_{0} = M^{\frac{1}{2}}, \quad \chi_{i} = \xi_{i}M^{\frac{1}{2}}, \quad \chi_{4} = \frac{1}{\sqrt{6}}(|\xi|^{2} - 3)M^{\frac{1}{2}}, \quad \langle \chi_{i}, \chi_{j} \rangle = \delta_{ij}, \quad i = 1, 2, 3.$$
We decompose Hilbert space $L_{\xi}^{2}$ as a direct sum of $\mathcal{N}$ and its orthogonal component $\mathcal{N}^\perp$, and we denote by $P_0$ the projection on this null space and $P_1$ the complementary projection:

$$
\begin{cases}
  f = P_0 f + P_1 f = f_0 + f_1, \\
  f_0 = P_0 f \equiv \rho(x, t) \chi_0 + \sum_{i=1}^{3} m_i(x, t) \chi_i + e(x, t) \chi_4,
  \\
  \rho(x, t) = \langle f, \chi_0 \rangle, \quad m_i(x, t) = \langle f, \chi_i \rangle \quad (i = 1, 2, 3), \\
  e(x, t) = \langle f, \chi_4 \rangle,
\end{cases}
$$

We next present trilinear estimates for nonlinear term $\Gamma[f + g, f - g](f - g)$. The property of $\Gamma[f + g, f - g] \in \mathcal{N}^\perp$ and Cauchy-Schwarz yield the following estimates.

**Lemma 2.1.** [6] Let $-\frac{3}{2} \leq \gamma \leq 1$, and $f, g$ be measurable functions in $\mathbb{R}_{x}^{3} \times \mathbb{R}_{\xi}^{3}$ satisfying

$$
\|\nu^{\frac{1}{2}}(f + g)(t)\|_{L^{2}(L_{\xi}^{2})} < \infty, \quad \|f - g\|_{L^{2}} + \|\nu^{\frac{1}{2}} P_1 (f - g)\|_{L^{2}} < \infty.
$$

Then there exists a positive constant $C$ independent of $t$ such that

(i) $-\frac{3}{2} \leq \gamma \leq 0$;

$$
\left| \int_{\mathbb{R}^{3}} \langle \Gamma[f + g, f - g], f - g \rangle (x) dx \right| 
\leq C \left( \|\nu^{\frac{1}{2}} f(t)\|_{L_{\xi}^{2}(L_{\xi}^{2})}^{2} + \|\nu^{\frac{1}{2}} g(t)\|_{L_{\xi}^{2}(L_{\xi}^{2})}^{2} \right) \|f(t) - g(t)\|_{L^{2}}^{2} + \frac{\sigma}{2} \|\nu^{\frac{1}{2}} P_1 (f(t) - g(t))\|_{L^{2}}^{2}.
$$

(ii) $0 < \gamma \leq 1$;

$$
\left| \int_{\mathbb{R}^{3}} \langle \Gamma[f + g, f - g], f - g \rangle (x) dx \right| 
\leq C \left( \|\nu^{\frac{1}{2}} f(t)\|_{L_{\xi}^{2}(L_{\xi}^{2})}^{2} + \|\nu^{\frac{1}{2}} g(t)\|_{L_{\xi}^{2}(L_{\xi}^{2})}^{2} \right) \|f(t) - g(t)\|_{L^{2}}^{2}
+ \left[ C \|f(t)\|_{L_{\xi}^{2}(L_{\xi}^{2})} + \|g(t)\|_{L_{\xi}^{2}(L_{\xi}^{2})} \right] + \frac{\sigma}{2} \|\nu^{\frac{1}{2}} P_1 (f(t) - g(t))\|_{L^{2}}^{2}.
$$

### 3. A PRIORI UNIFORM $L^2$-STABILITY

In this section, we briefly present a priori uniform $L^2$-stability estimates. For details, we refer to [6]. Let $f$ and $g$ be two classical solutions to the Boltzmann equation (2.2) and $f, g \in L^\infty(\mathbb{R}_{+}; L_{x,\xi}^{2} \cap L_{x}^{\infty} L_{\xi}^{2})$. Then $f$ and $g$ satisfy

(3.1) \quad \partial_t f + \xi \cdot \nabla_x f = L(f) + \Gamma(f, f),

(3.2) \quad \partial_t g + \xi \cdot \nabla_x g = L(g) + \Gamma(g, g).

We subtract (3.2) from (3.1) and multiply $(f - g)$ to both sides to find

(3.3) \quad \partial_t |f - g|^2 + \xi \cdot \nabla_x |f - g|^2 = L(f - g)(f - g) + \Gamma[f + g, f - g](f - g).

We now integrate (3.3) with respect to $(x, \xi)$ using the boundary condition and Proposition 2.1 to see

$$
\frac{d}{dt} \|f(t) - g(t)\|_{L^{2}}^{2} = \int_\Omega \langle L(f - g), f - g \rangle dx + \int_\Omega \langle \Gamma[f + g, f - g], f - g \rangle dx
\leq -\sigma \|\nu^{\frac{1}{2}} P_1 (f(t) - g(t))\|_{L^{2}} + \int_\Omega \langle \Gamma[f + g, f - g], f - g \rangle dx.
$$

We set the uniform $L^2$-stability criterion as follows.

(3.5) \quad \int_0^\infty \left( \|\nu^{\frac{1}{2}} f(s)\|_{L_{\xi}^{2}(L_{\xi}^{2})}^{2} + \|\nu^{\frac{1}{2}} g(t)\|_{L_{\xi}^{2}(L_{\xi}^{2})}^{2} \right) ds < \infty.
3.1. Soft potential and Maxwellian molecule: $-\frac{3}{2} < \gamma \leq 0$. Suppose two smooth perturbations $f$ and $g$ satisfy the stability condition (3.5). In (3.4), we use Lemma 2.1 to derive a Gronwall type inequality:

$$\frac{d}{dt} \|f(t) - g(t)\|_{L^2}^2 \leq -\frac{\sigma}{2} \|\nu^{\frac{1}{2}} P_1 (f(t) - g(t))\|_{L^2}^2 + C \left( \|\nu^{\frac{1}{2}} f(t)\|_{L^2_{\xi}(L^2_\xi)}^2 + \|\nu^{\frac{1}{2}} g(t)\|_{L^2_{\xi}(L^2_\xi)}^2 \right) \|f(t) - g(t)\|_{L^2}^2.$$ 

Then Gronwall's lemma yields

$$\|f(t) - g(t)\|_{L^2}^2 + \frac{\sigma}{2} \int_0^t \|\nu^{\frac{1}{2}} P_1 (f(s) - g(s))\|_{L^2}^2 ds \leq \exp \left[ C \int_0^t \left( \|\nu^{\frac{1}{2}} f(s)\|_{L^2_{\xi}(L^2_\xi)}^2 + \|\nu^{\frac{1}{2}} g(t)\|_{L^2_{\xi}(L^2_\xi)}^2 \right) dt \right] \|f^{in} - g^{in}\|_{L^2}^2.$$ 

This yields the uniform $L^2$-stability estimate.

**Theorem 3.1.** [6] For $\gamma \in (-\frac{3}{2}, 0]$ and let $F$ and $G$ be two classical solutions to (1.1) in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac{1}{2}}d\xi dx) \cap L_{x}^\infty(L^2(M^{-\frac{1}{2}}d\xi)))$ corresponding to initial data $F^{in}$, $G^{in}$ respectively. Suppose the smooth perturbations $f$ and $g$ satisfy the condition (3.5). Then we have

$$\sup_{0 \leq t < \infty} \|F(t) - G(t)\|_{L^2(M^{-1/2}d\xi dx)} \leq C \|F^{in} - G^{in}\|_{L^2(M^{-1/2}d\xi dx)},$$

where $C$ is a positive constant independent of $t$.

**Remark 3.1.** As a direct application of the above theorem, the classical solutions in [1, 5, 11] are uniformly $L^2$-stable.

3.2. Hard potential and hard sphere model: $0 < \gamma \leq 1$. Suppose two smooth perturbations $f$ and $g$ satisfy the stability condition (3.5) and the smallness condition:

$$\|f(t)\|_{L^\infty_\xi(L^2_\xi)} + \|g(t)\|_{L^\infty_\xi(L^2_\xi)} \ll \frac{\sigma}{4}.$$ 

In (3.4), we use Lemma 2.1 to get

$$\frac{d}{dt} \|f(t) - g(t)\|_{L^2}^2 \leq C \left( \|\nu^{\frac{1}{2}} f(t)\|_{L^2_{\xi}(L^2_\xi)}^2 + \|\nu^{\frac{1}{2}} g(t)\|_{L^2_{\xi}(L^2_\xi)}^2 \right) \|f(t) - g(t)\|_{L^2}^2 + \frac{\sigma}{2} + C \left( \|f(t)\|_{L^\infty_\xi(L^2_\xi)}^2 + \|g(t)\|_{L^\infty_\xi(L^2_\xi)}^2 \right) \|\nu^{\frac{1}{2}} P_1 (f(t) - g(t))\|_{L^2}^2.$$ 

We use (3.7) to find

$$\frac{d}{dt} \|f(t) - g(t)\|_{L^2}^2 \leq C \left( \|\nu^{\frac{1}{2}} f(t)\|_{L^2_{\xi}(L^2_\xi)}^2 + \|\nu^{\frac{1}{2}} g(t)\|_{L^2_{\xi}(L^2_\xi)}^2 \right) \|f(t) - g(t)\|_{L^2}^2 - \frac{C}{4} \|\nu^{\frac{1}{2}} P_1 (f(t) - g(t))\|_{L^2}^2.$$ 

Then Gronwall's lemma yield the following stability estimate.

**Theorem 3.2.** [6] For $\gamma \in (0, 1)$ and let $F$ and $G$ be two small classical solutions to (1.1) in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac{1}{2}}d\xi dx) \cap L_{x}^\infty(L^2(M^{-\frac{1}{2}}d\xi)))$ corresponding to small initial data $F^{in}$, $G^{in}$
respectively. Suppose the smooth perturbations $f$ and $g$ satisfy (3.5) and (3.6). Then we have
\[
\sup_{0 \leq t < \infty} \|F(t) - G(t)\|_{L^2(M^{-1/2}d\xi dx)} \leq C\|F^{in} - G^{in}\|_{L^2(M^{-1/2}d\xi dx)},
\]
where $C$ is a positive constant independent of $t$.

**Remark 3.2.** As a direct application of this theorem, the classical solutions in [12] are uniformly $L^2$-stable.

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