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Balanced C_6-Bowtie Designs
– p-Orbits and L-orbits –

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1. Introduction

Let K_n denote *the complete graph* of n vertices. *The complete multi-graph* λK_n is the complete graph K_n in which every edge is taken λ times. Let C_6 be *the* 6*-cycle* (or *the cycle* on 6 vertices). *The* C_6*-bowtie* is a graph of 2 edge-disjoint C_6's with a common vertex and the common vertex is called *the center of the* C_6*-bowtie.*
When λK_n is decomposed into edge-disjoint sum of C_6-bowties, we say that λK_n *has a* C_6*-bowtie decomposition.* Moreover, when every vertex of λK_n appears in the same number of C_6-bowties, we say that λK_n *has a balanced* C_6*-bowtie decomposition* and this number is called *the replication number.* This balanced C_6-bowtie decomposition of λK_n is called *a balanced* C_6*-bowtie design.*
In this paper, it is shown that the necessary condition for the existence of a balanced C_6-bowtie decomposition of λK_n is $\lambda(n-1) \equiv 0 \pmod{24}$ and $n \geq 11$. Sufficient conditions and decomposition algorithms are also given.
It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a* C_3*-bowtie system.*
For combinatorial designs, see [1,4,5,15]. Another type of foil-decompositions, see [6–14].

2. Balanced C_6-bowtie decomposition of λK_n

Notation. We consider the vertex set V of λK_n as $V = \{1, 2, ..., n\}$. We denote a C_6-bowtie passing through $1-2-3-4-5-6-1$, $1-7-8-9-10-11-1$ by $\{(1,2,3,4,5,6),(1,7,8,9,10,11)\}$. In the followings, the vertex additions $i+x$ are taken modulo n with residues $1, 2, ..., n$.

Theorem 1. If λK_n has a balanced C_6-bowtie decomposition, then $\lambda(n-1) \equiv 0 \pmod{24}$ and $n \geq 11$.
Proof. Suppose that λK_n has a balanced C_6-bowtie decomposition. Let b be the number of C_6-bowties and r be the replication number. Then $b = \lambda n(n-1)/24$ and $r = 11\lambda(n-1)/24$. Among r C_6-bowties having a vertex v of λK_n, let r_1 and r_2 be the numbers of C_6-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4r_1 + 2r_2 = \lambda(n-1)$. From these relations, $r_1 = \lambda(n-1)/24$ and $r_2 = 10\lambda(n-1)/24$. Thus, $\lambda(n-1) \equiv 0 \pmod{24}$. Since a C_6-bowtie is a subgraph of λK_n, $n \geq 11$.

Note. The condition $\lambda(n-1) \equiv 0 \pmod{24}$ and $n \geq 11$ in Theorem 1 can be classified as follows:
(i) $\lambda \geq 1$ and $n \equiv 1 \pmod{24}$, $n \geq 25$,
(ii) $\lambda \equiv 0 \pmod 2$ and $n \equiv 1 \pmod{12}$, $n \geq 13$,
(iii) $\lambda \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 8$, $n \geq 17$,
(iv) $\lambda \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 6$, $n \geq 13$,
(v) $\lambda \equiv 0 \pmod 6$ and $n \equiv 1 \pmod 4$, $n \geq 13$,

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(vi) $\lambda \equiv 0 \pmod 8$ and $n \equiv 1 \pmod 3$, $n \geq 13$,
(vii) $\lambda \equiv 0 \pmod{12}$ and $n \equiv 1 \pmod 2$, $n \geq 11$, and
(viii) $\lambda \equiv 0 \pmod{24}$ and $n \geq 11$.

Theorem 2. If λK_n has a balanced C_6-bowtie decomposition, then $(s\lambda)K_n$ has a balanced C_6-bowtie decomposition for every s.

Definition. *The C_6-t-foil* is a graph of t edge-disjoint C_6's with a common vertex and *the C_6-t-foiloid* is a multi-graph of t C_6's with a common vertex.
For example, $\{(1,2,3,4,5,6),(1,7,8,9,10,11),(1,12,13,14,15,16),(1,17,18,19,20,21)\}$ is a C_6-4-foil. $\{(1,2,3,4,5,6),(1,7,8,9,10,11),(1,2,3,5,7,8),(1,6,8,10,12,18)\}$ is a C_6-4-foiloid.

Theorem 3. When $\lambda \geq 1$, $n \equiv 1 \pmod{24}$, and $n \geq 25$, λK_n has a balanced C_6-bowtie decomposition.

Example 3.1. Balanced C_6-bowtie decomposition of K_{25}.
$\{(i,i+1,i+6,i+19,i+10,i+3),(i,i+2,i+8,i+22,i+12,i+4)\}$ $(i = 1,2,...,25)$.

Example 3.2. Balanced C_6-bowtie decomposition of K_{49}.
$\{(i,i+1,i+10,i+35,i+18,i+5),(i,i+2,i+12,i+38,i+20,i+6)\}$,
$\{(i,i+3,i+14,i+41,i+22,i+7),(i,i+4,i+16,i+44,i+24,i+8)\}$ $(i = 1,2,...,49)$.

Example 3.3. Balanced C_6-bowtie decomposition of K_{73}.
$\{(i,i+1,i+14,i+51,i+26,i+7),(i,i+2,i+16,i+54,i+28,i+8)\}$,
$\{(i,i+3,i+18,i+57,i+30,i+9),(i,i+4,i+20,i+60,i+32,i+10)\}$,
$\{(i,i+5,i+22,i+63,i+34,i+11),(i,i+6,i+24,i+66,i+36,i+12)\}$ $(i = 1,2,...,73)$.

Theorem 4. When $\lambda \equiv 0 \pmod 2$, $n \equiv 1 \pmod{12}$, and $n \geq 13$, λK_n has a balanced C_6-bowtie decomposition.

Example 4.1. Balanced C_6-bowtie decomposition of $2K_{13}$.
$\{(i,i+1,i+9,i+5,i+4,i+8),(i,i+2,i+12,i+10,i+3,i+6)\}$ $(i = 1,2,...,13)$.

Example 4.2. Balanced C_6-bowtie decomposition of $2K_{25}$.
$\{(i,i+1,i+10,i+2,i+18,i+17),(i,i+4,i+16,i+11,i+24,i+20)\}$,
$\{(i,i+2,i+12,i+5,i+20,i+18),(i,i+3,i+14,i+8,i+22,i+19)\}$ $(i = 1,2,...,25)$.

Example 4.3. Balanced C_6-bowtie decomposition of $2K_{37}$.
$\{(i,i+1,i+14,i+2,i+26,i+25),(i,i+6,i+24,i+17,i+36,i+30)\}$,
$\{(i,i+2,i+16,i+5,i+28,i+26),(i,i+3,i+18,i+8,i+30,i+27)\}$,
$\{(i,i+4,i+20,i+11,i+32,i+28),(i,i+5,i+22,i+14,i+34,i+29)\}$ $(i = 1,2,...,37)$.

Theorem 5. When $\lambda \equiv 0 \pmod 3$, $n \equiv 1 \pmod 8$, and $n \geq 17$, λK_n has a balanced C_6-bowtie decomposition.

Example 5.1. Balanced C_6-bowtie decomposition of $3K_{17}$.
$\{(i,i+1,i+6,i+11,i+10,i+3),(i,i+2,i+8,i+16,i+12,i+4)\}$,
$\{(i,i+1,i+6,i+13,i+10,i+3),(i,i+2,i+8,i+14,i+12,i+4)\}$ $(i = 1,2,...,17)$.

Example 5.2. Balanced C_6-bowtie decomposition of $3K_{25}$.

$\{(i, i+1, i+8, i+15, i+14, i+4), (i, i+2, i+10, i+19, i+16, i+5)\}$,
$\{(i, i+3, i+12, i+23, i+18, i+6), (i, i+1, i+8, i+16, i+14, i+4)\}$,
$\{(i, i+2, i+10, i+20, i+16, i+5), (i, i+3, i+12, i+24, i+18, i+6)\}$ $(i = 1, 2, ..., 25)$.

Example 5.3. Balanced C_6-bowtie decomposition of $3K_{33}$.
$\{(i, i+1, i+10, i+19, i+18, i+5), (i, i+4, i+16, i+32, i+24, i+8)\}$,
$\{(i, i+2, i+12, i+23, i+20, i+6), (i, i+3, i+14, i+27, i+22, i+7)\}$,
$\{(i, i+4, i+16, i+31, i+24, i+8), (i, i+1, i+10, i+20, i+18, i+5)\}$,
$\{(i, i+2, i+12, i+24, i+20, i+6), (i, i+3, i+14, i+28, i+22, i+7)\}$ $(i = 1, 2, ..., 33)$.

Theorem 6. When $\lambda \equiv 0 \pmod 4$, $n \equiv 1 \pmod 6$, and $n \geq 13$, λK_n has a balanced C_6-bowtie decomposition.

Example 6.1. Balanced C_6-bowtie decomposition of $4K_{19}$.
$\{(i, i+7, i+17, i+1, i+14, i+13), (i, i+8, i+3, i+5, i+6, i+4)\}$,
$\{(i, i+9, i+2, i+18, i+10, i+15), (i, i+7, i+17, i+1, i+14, i+13)\}$,
$\{(i, i+8, i+3, i+5, i+6, i+4), (i, i+9, i+2, i+18, i+10, i+15)\}$ $(i = 1, 2, ..., 19)$.

Example 6.2. Balanced C_6-bowtie decomposition of $4K_{31}$.
$\{(i, i+1, i+12, i+2, i+22, i+21), (i, i+4, i+18, i+11, i+28, i+24)\}$,
$\{(i, i+2, i+14, i+5, i+24, i+22), (i, i+3, i+16, i+8, i+26, i+23)\}$,
$\{(i, i+5, i+20, i+14, i+30, i+25), (i, i+1, i+12, i+2, i+22, i+21)\}$,
$\{(i, i+2, i+14, i+5, i+24, i+22), (i, i+3, i+16, i+8, i+26, i+23)\}$,
$\{(i, i+4, i+18, i+11, i+28, i+24), (i, i+5, i+20, i+14, i+30, i+25)\}$ $(i = 1, 2, ..., 31)$.

Example 6.3. Balanced C_6-bowtie decomposition of $4K_{43}$.
$\{(i, i+1, i+16, i+2, i+30, i+29), (i, i+6, i+26, i+17, i+40, i+34)\}$,
$\{(i, i+2, i+18, i+5, i+32, i+30), (i, i+3, i+20, i+8, i+34, i+31)\}$,
$\{(i, i+4, i+22, i+11, i+36, i+32), (i, i+5, i+24, i+14, i+38, i+33)\}$,
$\{(i, i+7, i+28, i+20, i+42, i+35), (i, i+1, i+16, i+2, i+30, i+29)\}$,
$\{(i, i+2, i+18, i+5, i+32, i+30), (i, i+3, i+20, i+8, i+34, i+31)\}$,
$\{(i, i+4, i+22, i+11, i+36, i+32), (i, i+5, i+24, i+14, i+38, i+33)\}$,
$\{(i, i+6, i+26, i+17, i+40, i+34), (i, i+7, i+28, i+20, i+42, i+35)\}$ $(i = 1, 2, ..., 43)$.

Theorem 7. When $\lambda \equiv 0 \pmod 6$, $n \equiv 1 \pmod 4$, and $n \geq 13$, λK_n has a balanced C_6-bowtie decomposition.

Example 7.1. Balanced C_6-bowtie decomposition of $6K_{21}$.
$\{(i, i+1, i+2, i+3, i+13, i+11), (i, i+10, i+8, i+18, i+19, i+20)\}$,
$\{(i, i+2, i+4, i+7, i+3, i+12), (i, i+9, i+18, i+14, i+17, i+19)\}$,
$\{(i, i+3, i+6, i+11, i+5, i+13), (i, i+4, i+8, i+15, i+7, i+14)\}$,
$\{(i, i+5, i+10, i+19, i+9, i+15), (i, i+6, i+12, i+2, i+11, i+16)\}$,
$\{(i, i+7, i+14, i+6, i+13, i+17), (i, i+8, i+16, i+10, i+15, i+18)\}$ $(i = 1, 2, ..., 21)$.

Example 7.2. Balanced C_6-bowtie decomposition of $6K_{29}$.
$\{(i, i+1, i+2, i+3, i+17, i+15), (i, i+14, i+12, i+26, i+27, i+28)\}$,
$\{(i, i+2, i+4, i+7, i+3, i+16), (i, i+13, i+26, i+22, i+25, i+27)\}$,
$\{(i, i+3, i+6, i+11, i+5, i+17), (i, i+12, i+24, i+18, i+23, i+26)\}$,
$\{(i, i+4, i+8, i+15, i+7, i+18), (i, i+11, i+22, i+14, i+21, i+25)\}$,
$\{(i, i+5, i+14, i+4, i+9, i+19), (i, i+6, i+12, i+23, i+11, i+20)\}$,

$\{(i,i+7,i+14,i+27,i+13,i+21),(i,i+8,i+16,i+2,i+15,i+22)\}$,
$\{(i,i+9,i+18,i+6,i+17,i+23),(i,i+10,i+20,i+25,i+15,i+24)\}$ $(i=1,2,...,29)$.

Example 7.3. Balanced C_6-bowtie decomposition of $6K_{45}$.
$\{(i,i+1,i+2,i+3,i+25,i+23),(i,i+22,i+20,i+42,i+43,i+44)\}$,
$\{(i,i+2,i+4,i+7,i+3,i+24),(i,i+21,i+42,i+38,i+41,i+43)\}$,
$\{(i,i+3,i+6,i+11,i+5,i+25),(i,i+20,i+40,i+34,i+39,i+42)\}$,
$\{(i,i+4,i+8,i+15,i+7,i+26),(i,i+19,i+38,i+30,i+37,i+41)\}$,
$\{(i,i+5,i+10,i+19,i+9,i+27),(i,i+18,i+36,i+26,i+35,i+40)\}$,
$\{(i,i+6,i+12,i+23,i+11,i+28),(i,i+17,i+34,i+22,i+33,i+39)\}$,
$\{(i,i+7,i+14,i+27,i+13,i+29),(i,i+8,i+16,i+31,i+15,i+30)\}$,
$\{(i,i+9,i+18,i+35,i+17,i+31),(i,i+14,i+28,i+10,i+27,i+36)\}$,
$\{(i,i+10,i+20,i+39,i+19,i+32),(i,i+13,i+26,i+6,i+25,i+35)\}$,
$\{(i,i+11,i+22,i+43,i+21,i+33),(i,i+12,i+24,i+2,i+23,i+34)\}$,
$\{(i,i+15,i+30,i+14,i+29,i+37),(i,i+16,i+32,i+18,i+31,i+38)\}$ $(i=1,2,...,45)$.

Theorem 8. When $\lambda \equiv 0 \pmod 8$, $n \equiv 1 \pmod 3$, and $n \geq 13$, λK_n has a balanced C_6-bowtie decomposition.

Example 8.1. Balanced C_6-bowtie decomposition of $8K_{16}$.
$\{(i,i+1,i+12,i+2,i+7,i+6),(i,i+3,i+11,i+8,i+5,i+13)\}$,
$\{(i,i+12,i+14,i+5,i+3,i+7),(i,i+4,i+2,i+11,i+13,i+9)\}$,
$\{(i,i+5,i+4,i+14,i+15,i+10),(i,i+1,i+12,i+2,i+7,i+6)\}$,
$\{(i,i+12,i+14,i+2,i+9,i+7),(i,i+3,i+11,i+8,i+5,i+13)\}$,
$\{(i,i+14,i+2,i+4,i+13,i+9),(i,i+1,i+6,i+5,i+15,i+10)\}$ $(i=1,2,...,16)$.

Example 8.2. Balanced C_6-bowtie decomposition of $8K_{22}$.
$\{(i,i+1,i+16,i+2,i+9,i+8),(i,i+7,i+6,i+20,i+21,i+14)\}$,
$\{(i,i+3,i+20,i+8,i+13,i+10),(i,i+4,i+15,i+11,i+7,i+18)\}$,
$\{(i,i+5,i+2,i+14,i+17,i+12),(i,i+3,i+20,i+8,i+13,i+10)\}$,
$\{(i,i+7,i+6,i+20,i+21,i+14),(i,i+1,i+16,i+2,i+9,i+8)\}$,
$\{(i,i+2,i+18,i+5,i+11,i+9),(i,i+6,i+4,i+17,i+19,i+13)\}$,
$\{(i,i+4,i+15,i+11,i+7,i+18),(i,i+5,i+2,i+14,i+17,i+12)\}$,
$\{(i,i+6,i+4,i+17,i+19,i+13),(i,i+2,i+18,i+5,i+11,i+9)\}$ $(i=1,2,...,22)$.

Example 8.3. Balanced C_6-bowtie decomposition of $8K_{28}$.
$\{(i,i+1,i+20,i+2,i+11,i+10),(i,i+9,i+8,i+26,i+27,i+18)\}$,
$\{(i,i+3,i+24,i+8,i+15,i+12),(i,i+4,i+26,i+11,i+17,i+13)\}$,
$\{(i,i+5,i+19,i+14,i+9,i+23),(i,i+6,i+2,i+17,i+21,i+15)\}$,
$\{(i,i+7,i+4,i+20,i+23,i+16),(i,i+3,i+24,i+8,i+15,i+12)\}$,
$\{(i,i+9,i+8,i+26,i+27,i+18),(i,i+1,i+20,i+2,i+11,i+10)\}$,
$\{(i,i+2,i+22,i+5,i+13,i+11),(i,i+8,i+6,i+23,i+25,i+17)\}$,
$\{(i,i+4,i+26,i+11,i+17,i+13),(i,i+5,i+19,i+14,i+9,i+23)\}$,
$\{(i,i+6,i+2,i+17,i+21,i+15),(i,i+7,i+4,i+20,i+23,i+16)\}$,
$\{(i,i+8,i+6,i+23,i+25,i+17),(i,i+2,i+22,i+5,i+13,i+11)\}$ $(i=1,2,...,28)$.

Theorem 9. When $\lambda \equiv 0 \pmod{12}$, $n \equiv 1 \pmod 2$, and $n \geq 11$, λK_n has a balanced C_6-bowtie decomposition.

Example 9.1.p. Balanced C_6-bowtie decomposition of $12K_{11}$.

p-orbit : 1 2 4 8 5 10 9 7 3 6 1 $(L = 11, g = 2)$
$\{(i, i+1, i+2, i+4, i+8, i+5), (i, i+10, i+9, i+7, i+3, i+6)\}$,
$\{(i, i+2, i+4, i+8, i+5, i+10), (i, i+9, i+7, i+3, i+6, i+1)\}$,
$\{(i, i+3, i+6, i+1, i+2, i+4), (i, i+8, i+5, i+10, i+9, i+7)\}$,
$\{(i, i+4, i+8, i+5, i+10, i+9), (i, i+7, i+3, i+6, i+1, i+2)\}$,
$\{(i, i+5, i+10, i+9, i+7, i+3), (i, i+6, i+1, i+2, i+4, i+8)\}$ $(i = 1, 2, ..., 11)$.

Example 9.2.L. Balanced C_6-bowtie decomposition of $12K_{15}$.
L-orbit : 1 2 9 4 10 7 3 1 $(L = 8)$
L-orbit : 5 8 12 14 13 6 11 5 $(L = 8)$
$\{(i, i+1, i+2, i+9, i+4, i+10), (i, i+14, i+13, i+6, i+11, i+5)\}$,
$\{(i, i+2, i+9, i+4, i+10, i+7), (i, i+13, i+6, i+11, i+5, i+8)\}$,
$\{(i, i+3, i+1, i+2, i+9, i+4), (i, i+12, i+14, i+13, i+6, i+11)\}$,
$\{(i, i+4, i+10, i+7, i+3, i+1), (i, i+11, i+5, i+8, i+12, i+14)\}$,
$\{(i, i+5, i+8, i+12, i+14, i+13), (i, i+10, i+7, i+3, i+1, i+2)\}$,
$\{(i, i+6, i+11, i+5, i+8, i+12), (i, i+9, i+4, i+10, i+7, i+3)\}$,
$\{(i, i+7, i+3, i+1, i+2, i+9), (i, i+8, i+12, i+14, i+13, i+6)\}$ $(i = 1, 2, ..., 15)$.

Example 9.3.p. Balanced C_6-bowtie decomposition of $12K_{23}$.
p-orbit : 1 5 2 10 4 20 8 17 16 11 9 22 18 21 13 19 3 15 6 7 12 14 1 $(L = 23, g = 5)$
$\{(i, i+1, i+5, i+2, i+10, i+4), (i, i+22, i+18, i+21, i+13, i+19)\}$,
$\{(i, i+2, i+10, i+4, i+20, i+8), (i, i+21, i+13, i+19, i+3, i+15)\}$,
$\{(i, i+3, i+15, i+6, i+7, i+12), (i, i+20, i+8, i+17, i+16, i+11)\}$,
$\{(i, i+4, i+20, i+8, i+17, i+16), (i, i+19, i+3, i+15, i+6, i+7)\}$,
$\{(i, i+5, i+2, i+10, i+4, i+20), (i, i+18, i+21, i+13, i+19, i+3)\}$,
$\{(i, i+6, i+7, i+12, i+14, i+1), (i, i+17, i+16, i+11, i+9, i+22)\}$,
$\{(i, i+7, i+12, i+14, i+1, i+5), (i, i+16, i+11, i+9, i+22, i+18)\}$,
$\{(i, i+8, i+17, i+16, i+11, i+9), (i, i+15, i+6, i+7, i+12, i+14)\}$,
$\{(i, i+9, i+22, i+18, i+21, i+13), (i, i+14, i+1, i+5, i+2, i+10)\}$,
$\{(i, i+10, i+4, i+20, i+8, i+17), (i, i+13, i+19, i+3, i+15, i+6)\}$,
$\{(i, i+11, i+9, i+22, i+18, i+21), (i, i+12, i+14, i+1, i+5, i+2)\}$ $(i = 1, 2, ..., 23)$.

Example 9.3. Balanced C_6-bowtie decomposition of $12K_{23}$.
$\{(i, i+2, i+1, i+12, i+11, i+22), (i, i+10, i+5, i+9, i+4, i+8)\}$,
$\{(i, i+4, i+2, i+3, i+1, i+22), (i, i+16, i+8, i+15, i+7, i+14)\}$,
$\{(i, i+6, i+3, i+5, i+2, i+4), (i, i+18, i+9, i+17, i+8, i+16)\}$,
$\{(i, i+8, i+4, i+7, i+3, i+6), (i, i+20, i+10, i+19, i+9, i+18)\}$,
$\{(i, i+10, i+5, i+9, i+4, i+8), (i, i+2, i+1, i+12, i+11, i+22)\}$,
$\{(i, i+12, i+6, i+11, i+5, i+10), (i, i+4, i+2, i+3, i+1, i+22)\}$,
$\{(i, i+14, i+7, i+13, i+6, i+12), (i, i+22, i+11, i+21, i+10, i+20)\}$,
$\{(i, i+16, i+8, i+15, i+7, i+14), (i, i+6, i+3, i+5, i+2, i+4)\}$,
$\{(i, i+18, i+9, i+17, i+8, i+16), (i, i+12, i+6, i+11, i+5, i+10)\}$,
$\{(i, i+20, i+10, i+19, i+9, i+18), (i, i+8, i+4, i+7, i+3, i+6)\}$,
$\{(i, i+22, i+11, i+21, i+10, i+20), (i, i+14, i+7, i+13, i+6, i+12)\}$ $(i = 1, 2, ..., 23)$.

Conjecture 10. When $\lambda \equiv 0 \pmod{24}$ and $n \geq 11$, λK_n has a balanced C_6-bowtie decomposition.

Example 10.1.LA Balanced C_6-bowtie decomposition of $24K_{12}$.
L-orbit : 1 7 8 10 2 9 6 5 3 11 4 1 $(L = 12)$

$\{(i,i+1,i+7,i+8,i+10,i+2),(i,i+9,i+6,i+5,i+3,i+11)\}$,
$\{(i,i+2,i+9,i+6,i+5,i+3),(i,i+11,i+4,i+1,i+7,i+8)\}$,
$\{(i,i+3,i+11,i+4,i+1,i+7),(i,i+8,i+10,i+2,i+9,i+6)\}$,
$\{(i,i+4,i+1,i+7,i+8,i+10),(i,i+2,i+9,i+6,i+5,i+3)\}$,
$\{(i,i+5,i+3,i+11,i+4,i+1),(i,i+7,i+8,i+10,i+2,i+9)\}$,
$\{(i,i+6,i+5,i+3,i+11,i+4),(i,i+1,i+7,i+8,i+10,i+2)\}$,
$\{(i,i+7,i+8,i+10,i+2,i+9),(i,i+6,i+5,i+3,i+11,i+4)\}$,
$\{(i,i+8,i+10,i+2,i+9,i+6),(i,i+5,i+3,i+11,i+4,i+1)\}$,
$\{(i,i+9,i+6,i+5,i+3,i+11),(i,i+4,i+1,i+7,i+8,i+10)\}$,
$\{(i,i+10,i+2,i+9,i+6,i+5),(i,i+3,i+11,i+4,i+1,i+7)\}$,
$\{(i,i+11,i+4,i+1,i+7,i+8),(i,i+10,i+2,i+9,i+6,i+5)\}$ $(i=1,2,...,12)$.

Example 10.1.LB Balanced C_6-bowtie decomposition of $24K_{12}$.
L-orbit : 1 7 2 3 5 9 6 11 10 8 4 1 $(L=12)$
$\{(i,i+1,i+7,i+2,i+3,i+5),(i,i+9,i+6,i+11,i+10,i+8)\}$,
$\{(i,i+2,i+3,i+5,i+9,i+6),(i,i+11,i+10,i+8,i+4,i+1)\}$,
$\{(i,i+3,i+5,i+9,i+6,i+11),(i,i+10,i+8,i+4,i+1,i+7)\}$,
$\{(i,i+4,i+1,i+7,i+2,i+3),(i,i+5,i+9,i+6,i+11,i+10)\}$,
$\{(i,i+5,i+9,i+6,i+11,i+10),(i,i+8,i+4,i+1,i+7,i+2)\}$,
$\{(i,i+6,i+11,i+10,i+8,i+4),(i,i+1,i+7,i+2,i+3,i+5)\}$,
$\{(i,i+7,i+2,i+3,i+5,i+9),(i,i+6,i+11,i+10,i+8,i+4)\}$,
$\{(i,i+8,i+4,i+1,i+7,i+2),(i,i+3,i+5,i+9,i+6,i+11)\}$,
$\{(i,i+9,i+6,i+11,i+10,i+8),(i,i+4,i+1,i+7,i+2,i+3)\}$,
$\{(i,i+10,i+8,i+4,i+1,i+7),(i,i+2,i+3,i+5,i+9,i+6)\}$,
$\{(i,i+11,i+10,i+8,i+4,i+1),(i,i+7,i+2,i+3,i+5,i+9)\}$ $(i=1,2,...,12)$.

Example 10.2. Balanced C_6-bowtie decomposition of $24K_{14}$.
$\{(i,i+1,i+8,i+2,i+10,i+6),(i,i+4,i+7,i+13,i+5,i+9)\}$,
$\{(i,i+2,i+10,i+6,i+4,i+7),(i,i+13,i+5,i+9,i+11,i+1)\}$,
$\{(i,i+5,i+2,i+3,i+12,i+10),(i,i+7,i+6,i+11,i+1,i+8)\}$,
$\{(i,i+8,i+9,i+4,i+7,i+13),(i,i+12,i+14,i+5,i+2,i+3)\}$,
$\{(i,i+9,i+11,i+1,i+8,i+2),(i,i+12,i+10,i+13,i+4,i+3)\}$,
$\{(i,i+10,i+13,i+12,i+3,i+5),(i,i+11,i+1,i+8,i+9,i+4)\}$,
$\{(i,i+1,i+8,i+9,i+4,i+7),(i,i+2,i+3,i+12,i+10,i+13)\}$,
$\{(i,i+3,i+5,i+2,i+11,i+12),(i,i+10,i+8,i+4,i+7,i+6)\}$,
$\{(i,i+4,i+7,i+6,i+11,i+1),(i,i+13,i+12,i+3,i+5,i+2)\}$,
$\{(i,i+7,i+13,i+5,i+9,i+11),(i,i+8,i+2,i+10,i+6,i+2)\}$,
$\{(i,i+9,i+4,i+7,i+13,i+5),(i,i+11,i+1,i+8,i+2,i+10)\}$,
$\{(i,i+3,i+1,i+10,i+13,i+12),(i,i+6,i+11,i+4,i+8,i+9)\}$,
$\{(i,i+6,i+4,i+7,i+12,i+11),(i,i+5,i+9,i+13,i+1,i+8)\}$ $(i=1,2,...,14)$.

Example 10.3.LA Balanced C_6-bowtie decomposition of $24K_{20}$.
L-orbit : 1 11 12 14 18 6 1 $(L=7)$
L-orbit : 2 13 16 2 $(L=4)$
L-orbit : 3 15 10 9 7 3 $(L=6)$
L-orbit : 4 17 4 $(L=3)$
L-orbit : 5 19 8 5 $(L=4)$
$\{(i,i+1,i+11,i+12,i+14,i+18),(i,i+3,i+15,i+10,i+9,i+7)\}$,
$\{(i,i+2,i+5,i+16,i+7,i+13),(i,i+6,i+1,i+11,i+12,i+14)\}$,
$\{(i,i+3,i+15,i+10,i+9,i+7),(i,i+1,i+11,i+12,i+14,i+18)\}$,

$\{(i, i+7, i+11, i+4, i+17, i+13), (i, i+19, i+16, i+5, i+14, i+8)\}$,
$\{(i, i+5, i+14, i+8, i+2, i+19), (i, i+7, i+3, i+15, i+10, i+9)\}$,
$\{(i, i+6, i+1, i+11, i+12, i+14), (i, i+2, i+5, i+16, i+7, i+13)\}$,
$\{(i, i+7, i+3, i+15, i+10, i+9), (i, i+5, i+14, i+8, i+2, i+19)\}$,
$\{(i, i+8, i+2, i+19, i+16, i+5), (i, i+18, i+6, i+1, i+11, i+12)\}$,
$\{(i, i+9, i+7, i+3, i+15, i+10), (i, i+8, i+2, i+19, i+16, i+5)\}$,
$\{(i, i+10, i+9, i+7, i+3, i+15), (i, i+12, i+14, i+18, i+6, i+1)\}$,
$\{(i, i+11, i+12, i+14, i+18, i+6), (i, i+15, i+10, i+9, i+7, i+3)\}$,
$\{(i, i+12, i+14, i+18, i+6, i+1), (i, i+10, i+9, i+7, i+3, i+15)\}$,
$\{(i, i+13, i+19, i+2, i+5, i+16), (i, i+17, i+4, i+11, i+14, i+7)\}$,
$\{(i, i+14, i+18, i+6, i+1, i+11), (i, i+9, i+7, i+3, i+15, i+10)\}$,
$\{(i, i+15, i+10, i+9, i+7, i+3), (i, i+11, i+12, i+14, i+18, i+6)\}$,
$\{(i, i+16, i+7, i+13, i+19, i+2), (i, i+14, i+18, i+6, i+1, i+11)\}$,
$\{(i, i+17, i+4, i+11, i+14, i+7), (i, i+13, i+19, i+2, i+5, i+16)\}$,
$\{(i, i+18, i+6, i+1, i+11, i+12), (i, i+16, i+7, i+13, i+19, i+2)\}$,
$\{(i, i+19, i+16, i+5, i+14, i+8), (i, i+7, i+11, i+4, i+17, i+13)\}$ $(i = 1, 2, ..., 20)$.

Example 10.3.LB Balanced C_6-bowtie decomposition of $24K_{20}$.
L-orbit : 1 11 2 3 5 9 17 14 8 15 10 19 18 16 12 4 7 13 6 1 ($L = 20$)
$\{(i, i+1, i+11, i+2, i+3, i+5), (i, i+9, i+17, i+14, i+8, i+15)\}$,
$\{(i, i+2, i+3, i+5, i+9, i+17), (i, i+14, i+8, i+15, i+10, i+19)\}$,
$\{(i, i+3, i+5, i+9, i+17, i+14), (i, i+8, i+15, i+10, i+19, i+18)\}$,
$\{(i, i+4, i+7, i+13, i+6, i+1), (i, i+11, i+2, i+3, i+5, i+9)\}$,
$\{(i, i+5, i+9, i+17, i+14, i+8), (i, i+15, i+10, i+19, i+18, i+16)\}$,
$\{(i, i+6, i+1, i+11, i+2, i+3), (i, i+5, i+9, i+17, i+14, i+8)\}$,
$\{(i, i+7, i+13, i+6, i+1, i+11), (i, i+2, i+3, i+5, i+9, i+17)\}$,
$\{(i, i+8, i+15, i+10, i+19, i+18), (i, i+16, i+12, i+4, i+7, i+13)\}$,
$\{(i, i+9, i+17, i+14, i+8, i+15), (i, i+10, i+19, i+18, i+16, i+12)\}$,
$\{(i, i+10, i+19, i+18, i+16, i+12), (i, i+4, i+7, i+13, i+6, i+1)\}$,
$\{(i, i+11, i+2, i+3, i+5, i+9), (i, i+17, i+14, i+8, i+15, i+10)\}$,
$\{(i, i+12, i+4, i+7, i+13, i+6), (i, i+1, i+11, i+2, i+3, i+5)\}$,
$\{(i, i+13, i+6, i+1, i+11, i+2), (i, i+3, i+5, i+9, i+17, i+14)\}$,
$\{(i, i+14, i+8, i+15, i+10, i+19), (i, i+18, i+16, i+12, i+4, i+7)\}$,
$\{(i, i+15, i+10, i+19, i+18, i+16), (i, i+12, i+4, i+7, i+13, i+6)\}$,
$\{(i, i+16, i+12, i+4, i+7, i+13), (i, i+6, i+1, i+11, i+2, i+3)\}$,
$\{(i, i+17, i+14, i+8, i+15, i+10), (i, i+19, i+18, i+16, i+12, i+4)\}$,
$\{(i, i+18, i+16, i+12, i+4, i+7), (i, i+13, i+6, i+1, i+11, i+2)\}$,
$\{(i, i+19, i+18, i+16, i+12, i+4), (i, i+7, i+13, i+6, i+1, i+11)\}$ $(i = 1, 2, ..., 20)$.

Main Conjecture. λK_n has a balanced C_6-bowtie decomposition if and only if $\lambda(n-1) \equiv 0 \pmod{24}$ and $n \geq 11$.

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