

Title	On Terwilliger algebras with respect to subsets in Hamming graphs and Johnson graphs (Finite Groups and Algebraic Combinatorics)
Author(s)	Hosoya, Rie
Citation	数理解析研究所講究録 (2008), 1593: 87-92
Issue Date	2008-04
URL	http://hdl.handle.net/2433/81654
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

On Terwilliger algebras with respect to subsets in Hamming graphs and Johnson graphs

国際基督教大学 細谷 利恵 (Rie Hosoya)
International Christian University

In this talk, we determine irreducible modules of the Terwilliger algebra of a Q -polynomial distance-regular graph Γ with respect to a subset with a special condition. Here we focus on the case where Γ is the Johnson graph. We construct irreducible modules of the Terwilliger algebra of Γ from those of binary Hamming graphs. This is a joint work with Hajime Tanaka.

1 Width and dual width

Let Γ be a Q -polynomial distance-regular graph of diameter D with vertex set X . We refer the reader to [1], [2] for terminology and background materials on Q -polynomial distance-regular graphs. Let C be a nonempty subset of X . Let $\chi \in \mathbb{C}^X$ be the *characteristic vector* of C , i.e.,

$$(\chi)_x = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Let A_0, \dots, A_D be distance matrices of Γ . We write $A = A_1$. Let E_0, \dots, E_D be primitive idempotents of Γ . Brouwer, Godsil, Koolen and Martin [3] introduced two parameters of C . The *width* w of C is defined as

$$w = \max\{i \mid \chi^T A_i \chi \neq 0\}.$$

Dually, the *dual width* w^* of C is defined as

$$w^* = \max\{i \mid \chi^T E_i \chi \neq 0\}.$$

We can verify that $w = \max\{\partial(x, y) \mid x, y \in C\}$, i.e., the maximum distance between two vertices in C . Obviously, $w = 0$ if and only if $C = \{x\}$ ($x \in X$). The following fundamental bound holds.

Theorem 1 [3]

$$w + w^* \geq D.$$

When the above bound is attained, Brouwer et.al. showed that some good properties hold:

Theorem 2 [3] *Suppose $w + w^* = D$. Then*

(i) C is completely regular.

(ii) C induces a Q -polynomial distance-regular graph whenever C is connected.

Recently, Tanaka proved the following:

Theorem 3 [8] *Suppose $w + w^* = D$. Then*

(i) C induces a Q -polynomial distance-regular graph whenever $q \neq -1$.

(ii) C is convex if and only if Γ has classical parameters.

The subsets with $w + w^* = D$ were classified for some Q -polynomial distance-regular graphs (see [3], [8]). Our current goal is to characterize Q -polynomial distance-regular graphs having subsets with $w + w^* = D$ in terms of Terwilliger algebras. We will see the definitions and basic terminology on Terwilliger algebras in the next section.

2 Terwilliger algebras and modules

Let $C \subset X$. Let $\Gamma_i(C) = \{x \in X \mid \partial(x, C) = i\}$, i.e., the i th subconstituent of Γ with respect to C . We define the diagonal matrix $E_i^* \in \text{Mat}_X(C)$ so that

$$(E_i^*)_{xx} = \begin{cases} 1 & \text{if } x \in \Gamma_i(C), \\ 0 & \text{otherwise.} \end{cases}$$

The Terwilliger algebra $\mathcal{T}(C)$ of Γ with respect to C is defined as follows:

$$\mathcal{T}(C) = \langle A, E_0^*, \dots, E_D^* \rangle \subset \text{Mat}_X(C).$$

It is known that $\mathcal{T}(C)$ is semisimple, and non-commutative in general. If we set $C = \{x\}$ ($x \in X$), then $\mathcal{T}(C)$ is identical to the ordinary Terwilliger algebra $\mathcal{T}(x)$ or the *subconstituent algebra* introduced by Terwilliger [10]. Suzuki generalized the theory of subconstituent algebras to the case associated with subsets [6].

Let $W \subset C^X$ be an irreducible $\mathcal{T}(C)$ -module. There are two types of decompositions of W into subspaces which are invariant under the action of E_i^* and E_i respectively:

$$W = E_0^*W + \dots + E_D^*W \quad (\text{direct sum}),$$

$$W = E_0W + \dots + E_DW \quad (\text{direct sum}).$$

We define parameters for W to describe isomorphism classes of irreducible modules; The *endpoint* ν of W is defined as $\nu = \min\{i \mid E_i^*W \neq 0\}$, and the *dual endpoint* μ of W is $\mu = \min\{i \mid E_iW \neq 0\}$. The *diameter* of W is defined as $d = |\{i \mid E_i^*W \neq 0\}| - 1$. W is called *thin* if $\dim E_i^*W \leq 1$ for all i .

Suppose C satisfies $w + w^* = D$. We have a preceding result on irreducible modules of endpoint 0:

Theorem 4 [5] *Suppose C satisfies $w + w^* = D$. Let W be an irreducible $\mathcal{T}(C)$ -module of endpoint $\nu = 0$. Then W is thin with $d = w^*$.*

Our primary goal is to determine irreducible $\mathcal{T}(C)$ -modules of arbitrary endpoint ν . In this article, we discuss the case of Johnson graphs.

3 Johnson graphs

Definition 3.1 *The binary Hamming graph $\tilde{\Gamma} = H(N, 2)$ ($N \geq 2D$) has vertex set*

$$\tilde{X} = \{(\overbrace{x_1 \cdots x_N}^N) \mid x_i \in \{0, 1\}\},$$

i.e., the set of binary words of length N , and two vertices $x, y \in \tilde{X}$ are adjacent if x and y differ in exactly 1 coordinate.

Definition 3.2 *The Johnson graph $\Gamma = J(N, D)$ has vertex set*

$$X = \tilde{\Gamma}_D(\mathbf{0}) = \{(x_1 \cdots x_N) \in \tilde{X} \mid (\# \text{ of } 1\text{s}) = D\},$$

i.e., the set of binary words of length N and weight D , and two vertices $x, y \in X$ are adjacent if x and y differ in exactly 2 coordinates.

Theorem 5 [3] *Let $\Gamma = J(N, D)$ and $C \subset X$. Suppose C satisfies $w + w^* = D$. Then*

$$C \cong \{(\overbrace{1 \cdots 1}^{w^*} \overbrace{* \cdots *}^{N-w^*}) \mid (\# \text{ of } 1\text{s}) = D\},$$

i.e., the induced subgraph on C is isomorphic to the Johnson graph $J(N - w^, D - w^*)$.*

Let $C = \{(\overbrace{1 \cdots 1}^{w^*} \overbrace{* \cdots *}^{N-w^*}) \mid (\# \text{ of } 1\text{s}) = D\}$, and $\Gamma^{(1)} = H(w^*, 2)$, $\Gamma^{(2)} = H(N - w^*, 2)$. Then

$$C = \Gamma_{w^*}^{(1)}(\mathbf{0}) \times \Gamma_w^{(2)}(\mathbf{0}),$$

and we also have

$$\Gamma_i(C) = \Gamma_{w^*-i}^{(1)}(\mathbf{0}) \times \Gamma_{w+i}^{(2)}(\mathbf{0}).$$

Let $\mathcal{T}_1(\mathbf{0})$ be the Terwilliger algebra of $H(w^*, 2)$ with respect to $\mathbf{0}$, where $\mathbf{0}$ denotes the all zero word, and $\mathcal{T}_2(\mathbf{0})$ the Terwilliger algebra of $H(N - w^*, 2)$ with respect to $\mathbf{0}$. Let $\mathcal{T}(C)$ be the Terwilliger algebra of $J(N, D)$ with respect to C . Let \tilde{X} denote the vertex set of $H(N, 2)$. Recall that the vertex set X of $J(N, D)$ is a subset of \tilde{X} . For a subset \mathcal{A} of $\text{Mat}_{\tilde{X}}(C)$, let $\mathcal{A}|_{X \times X} \subset \text{Mat}_X(C)$ denote the set of principal submatrices of matrices in \mathcal{A} . The following is the key lemma.

Lemma 6

$$\mathcal{T}(C) \subseteq \mathcal{T}_1(\mathbf{0}) \otimes \mathcal{T}_2(\mathbf{0})|_{X \times X} \quad (\subset \text{Mat}_X(C))$$

Let W_i be an irreducible $\mathcal{T}_i(\mathbf{0})$ -module ($i = 1, 2$). Let

$$W := W_1 \otimes W_2|_X \subset C^X,$$

where the right hand side denotes the set of vectors from $W_1 \otimes W_2$ whose indices are restricted on X . Then

Lemma 7 W is a $\mathcal{T}(C)$ -module.

Go [4] gave an explicit description of W_1, W_2 . We will make use of results in [4] for the characterization of W .

Lemma 8 Let $\mathcal{B}_1, \mathcal{B}_2$ be standard bases for W_1, W_2 (see [4]). Then

- (i) $\mathcal{B} := \{u \otimes u' \mid u \in \mathcal{B}_1, u' \in \mathcal{B}_2, u \otimes u'|_X \neq 0\}$ is a basis for W .
- (ii) $\text{Span}\{u \otimes u'\} = E_i^* W$ for some i .
- (iii) W is thin.

We can determine the endpoint of W by comparing supports of W_1 and W_2 . For determination of the dual endpoint of W , the following will be useful:

Proposition 9 [11] Let $\mathcal{T}(\mathbf{0})$ be the Terwilliger algebra of the binary Hamming graph $H(N, 2)$ with respect to $\mathbf{0}$. Let U be an irreducible $\mathcal{T}(\mathbf{0})$ -module of endpoint r . Then $v(\neq \mathbf{0}) \in U|_X$ is an eigenvector of $J(N, D)$ for eigenvalue θ_r .

Next we will check that W is irreducible. To see that it is so, we consider a tridiagonal matrix. Let $[A]_{\mathcal{B}}$ be the matrix representing A with respect to the basis \mathcal{B} . Then $[A]_{\mathcal{B}}$ is tridiagonal since W is thin. Moreover, by calculation we can verify that the off-diagonal entries of $[A]_{\mathcal{B}}$ are nonzero. Hence we have the following:

Lemma 10 W is an irreducible $\mathcal{T}(C)$ -module.

4 Main results

Let $\Gamma = J(N, D)$ and $C \subset X$. Suppose C satisfies $w + w^* = D$. Let $\mathcal{T}(C)$ be the Terwilliger algebra of Γ with respect to C . Let W be an irreducible $\mathcal{T}(C)$ -module of endpoint ν , dual endpoint μ , diameter d .

Theorem 11 *There exist integers e, f satisfying*

$$\begin{aligned} 0 \leq e \leq \left\lfloor \frac{w^*}{2} \right\rfloor, \quad 0 \leq f \leq \left\lfloor \frac{N - w^*}{2} \right\rfloor, \\ \nu = \max\{e, f - w\}, \quad \mu = e + f, \\ d = \begin{cases} w^* - 2\nu & \text{if } \nu = e, \\ \min\{D - \mu, N - D - 2\nu - w\} & \text{if } \nu = f - w. \end{cases} \end{aligned}$$

Remarks. e, f comes from endpoints of W_1, W_2 .

Remarks. If $N \neq 2D$, then e, f are uniquely determined for given ν, μ, d . In this case,

$T(C) = T_1 \otimes T_2|_{X \times X}$ in Lemma 6.

Theorem 12 *W has a basis $\mathcal{B} = \{v_0, \dots, v_d\}$ satisfying*

$$v_i \in E_{i+\nu}^* W \quad (0 \leq i \leq d),$$

and with respect to which the matrix representing A is tridiagonal with entries

$$\begin{aligned} c_i(W) &= i(i + 2\nu - \mu + w), \\ a_i(W) &= D(N - D) + \mu(\mu + d - N - 1) + d(d - N \\ &\quad + 2\nu + w) + i(N - 4\nu - 2i - 2w), \\ b_i(W) &= (d - i)(N - d - 2\nu - \mu - i - w). \end{aligned}$$

Remarks. $c_i(W) + a_i(W) + b_i(W) = \theta_\mu$.

Remarks. If $w = 0$, the above $c_i(W), a_i(W), b_i(W)$ coincide with the results by Terwilliger [10].

Corollary 13 *Isomorphism classes are determined by (ν, μ, d) .*

5 Remark

Let $A^* = \text{diag}(E_1 \chi)$. Then (A, A^*) acts on W as a *Leonard pair* with parameter array $(h, r, s, s^*, r, d, \theta_0, \theta_0^*)$ (*Dual Hahn*):

$$\begin{aligned} \theta_i &= \theta_0 + hi(i + 1 + s), \\ \theta_i^* &= \theta_0^* + s^*i, \\ \varphi_i &= hs^*i(i - d - 1)(i + r), \\ \phi_i &= hs^*i(i - d - 1)(i + r - s - d - 1). \end{aligned}$$

Especially, we have

$$s = -N - 2 + 2\mu,$$

$$r = -N + d + 2\nu + \mu - 1 + w.$$

See [9] for details on Leonard pairs. If $w = 0$, the above parameters coincide with the results by Terwilliger [10].

References

- [1] E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin/Cummings, California, 1984.
- [2] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer Verlag, Berlin, Heidelberg, 1989.
- [3] A. E. Brouwer, C. D. Godsil, J. H. Koolen and W. J. Martin, Width and dual width of subsets in polynomial association schemes, *J. Combin. Th. (A)* 102 (2003), 255-271.
- [4] J. T. Go, The Terwilliger algebra of the Hypercube, *Europ. J. Combin.* 23 (2002), 399-430.
- [5] R. Hosoya and H. Suzuki, Tight distance-regular graphs with respect to subsets, *European J. Combin.* 28 (2007), 61-74.
- [6] H. Suzuki, The Terwilliger algebra associated with a set of vertices in a distance-regular graph, *J. Algebraic Combinatorics* 22 (2005), 5-38.
- [7] H. Tanaka, Classification of subsets with minimal width and dual width in Grassmann, bilinear forms and dual polar graphs, *J. Combin. Theory (A)* 113 (2006), 903-910.
- [8] H. Tanaka, On subsets with minimal width and dual width, in preparation.
- [9] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; an algebraic approach to the Askey scheme of orthogonal polynomials, arXiv:math/0408390.
- [10] P. Terwilliger, The subconstituent algebra of an association schemes, (part I, II, III), *J. Alg. Combin.* 1 (1992), 363-388, 2(1993), 73-103, 177-210.
- [11] P. Terwilliger, Lecture note on algebraic graph theory (Suzuki-version).