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On Terwilliger algebras with respect to subsets in Hamming graphs and Johnson graphs

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In this talk, we determine irreducible modules of the Terwilliger algebra of a Q-polynomial distance-regular graph Γ with respect to a subset with a special condition. Here we focus on the case where Γ is the Johnson graph. We construct irreducible modules of the Terwilliger algebra of Γ from those of binary Hamming graphs. This is a joint work with Hajime Tanaka.

1 Width and dual width

Let Γ be a Q-polynomial distance-regular graph of diameter D with vertex set X. We refer the reader to [1], [2] for terminology and background materials on Q-polynomial distance-regular graphs. Let C be a nonempty subset of X. Let $\chi \in \mathbf{C}^X$ be the *characteristic vector* of C, i.e.,

$$(\chi)_x = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_0, \ldots, A_D$ be distance matrices of Γ. We write $A = A_1$. Let $E_0, \ldots, E_D$ be primitive idempotents of Γ. Brouwer, Godsil, Koolen and Martin [3] introduced two parameters of C. The *width* w of C is defiend as

$$w = \max\{i \mid \chi^T A_i \chi \neq 0\}.$$

Dually, the *dual width* w^* of C is defined as

$$w^* = \max\{i \mid \chi^T E_i \chi \neq 0\}.$$

We can verify that $w = \max\{\partial(x, y) \mid x, y \in C\}$, i.e., the maximum distance between two vertices in C. Obviously, $w = 0$ if and only if $C = \{x\}$ $(x \in X)$. The following fundamental bound holds.

Theorem 1 [3]

$$w + w^* \geq D.$$

When the above bound is attained, Brouwer et.al. showed that some good properties hold:

Theorem 2 [3] *Suppose $w + w^* = D$. Then*

(i) *C is completely regular.*

(ii) *C induces a Q-polynomial distance-regular graph whenever C is connected.*

Recently, Tanaka proved the following:

Theorem 3 [8] *Suppose $w + w^* = D$. Then*

(i) *C induces a Q-polynomial distance-regular graph whenever $q \neq -1$.*

(ii) *C is convex if and only if Γ has classical parameters.*

The subsets with $w + w^* = D$ were classified for some Q-polynomial distance-regular graphs (see [3], [8]). Our current goal is to characterize Q-polynomial distance-regular graphs having subsets with $w + w^* = D$ in terms of Terwilliger algebras. We will see the definitions and basic terminology on Terwilliger algebras in the next section.

2 Terwilliger algebras and modules

Let $C \subset X$. Let $\Gamma_i(C) = \{x \in X \mid \partial(x, C) = i\}$, i.e., the i th subconstituent of Γ with respect to C. We define the diagonal matirx $E_i^* \in \mathrm{Mat}_X(\boldsymbol{C})$ so that

$$(E_i^*)_{xx} = \begin{cases} 1 & \text{if } x \in \Gamma_i(C), \\ 0 & \text{otherwise.} \end{cases}$$

The *Terwilliger algebra* $\mathcal{T}(C)$ of Γ with respect to C is defined as follows:

$$\mathcal{T}(C) = < A, E_0^*, \ldots, E_D^* > \quad \subset \mathrm{Mat}_X(\boldsymbol{C}).$$

It is known that $\mathcal{T}(C)$ is semisimple, and non-commutative in general. If we set $C = \{x\}$ $(x \in X)$, then $\mathcal{T}(C)$ is identical to the ordinary Terwilliger algebra $\mathcal{T}(x)$ or the *subconstituent algebra* introduecd by Terwilliger [10]. Suzuki generalized the theory of subconstituent algebras to the case associated with subsets [6].

Let $W \subset \boldsymbol{C}^X$ be an irreducible $\mathcal{T}(C)$-module. There are two types of decompositions of W into subspaces which are invariant under the action of E_i^* and E_i respectively:

$$W = E_0^* W + \cdots + E_D^* W \quad \text{(direct sum)},$$

$$W = E_0 W + \cdots + E_D W \quad \text{(direct sum)}.$$

We define parameters for W to describe isomorphism classes of irreducible modules; The *endpoint* ν of W is defined as $\nu = \min\{i \mid E_i^* W \neq 0\}$, and the *dual endpoint* μ of W is $\mu = \min\{i \mid E_i W \neq 0\}$. The *diameter* of W is defined as $d = |\{i \mid E_i^* W \neq 0\}| - 1$. W is called *thin* if $\dim E_i^* W \leq 1$ for all i.

Suppose C satisfies $w + w^* = D$. We have a preceeding result on irreducible modules of endpoint 0:

Theorem 4 [5] *Suppose C satisfies $w + w^* = D$. Let W be an irreducible $\mathcal{T}(C)$-module of endpoint $\nu = 0$. Then W is thin with $d = w^*$.*

Our primary goal is to determine irreducible $\mathcal{T}(C)$-modules of arbitrary endpoint ν. In this article, we discuss the case of Johnson graphs.

3 Johnson graphs

Definition 3.1 *The binary Hamming graph $\tilde{\Gamma} = H(N, 2)$ $(N \geq 2D)$ has vertex set*

$$\tilde{X} = \{(\overbrace{x_1 \cdots x_N}^{N}) \mid x_i \in \{0, 1\}\},$$

i.e., the set of binary words of length N, and two vertices $x, y \in \tilde{X}$ are adjacent if x and y differ in exactly 1 coordinate.

Definition 3.2 *The Johnson graph $\Gamma = J(N, D)$ has vertex set*

$$X = \tilde{\Gamma}_D(\mathbf{0}) = \{(x_1 \cdots x_N) \in \tilde{X} \mid (\#\ of\ 1s) = D\},$$

i.e., the set of binary words of length N and weight D, and two vertices $x, y \in X$ are adjacent if x and y differ in exactly 2 coordinates.

Theorem 5 [3] *Let $\Gamma = J(N, D)$ and $C \subset X$. Suppose C satisfies $w + w^* = D$. Then*

$$C \cong \{(\overbrace{1 \cdots 1}^{w^*}\overbrace{* \cdots *}^{N-w^*}) \mid (\#\ of\ 1s) = D\},$$

i.e., the induced subgraph on C is isomophic to the Johnson graph $J(N - w^*, D - w^*)$.

Let $C = \{(\overbrace{1 \cdots 1}^{w^*}\overbrace{* \cdots *}^{N-w^*}) \mid (\#\text{ of 1s}) = D\}$, and $\Gamma^{(1)} = H(w^*, 2)$, $\Gamma^{(2)} = H(N - w^*, 2)$. Then

$$C = \Gamma^{(1)}_{w^*}(\mathbf{0}) \times \Gamma^{(2)}_{w}(\mathbf{0}),$$

and we also have

$$\Gamma_i(C) = \Gamma^{(1)}_{w^*-i}(\mathbf{0}) \times \Gamma^{(2)}_{w+i}(\mathbf{0}).$$

Let $\mathcal{T}_1(\mathbf{0})$ be the Terwilliger algebra of $H(w^*, 2)$ with respect to $\mathbf{0}$, where $\mathbf{0}$ denotes the all zero word, and $\mathcal{T}_2(\mathbf{0})$ the Terwilliger algebra of $H(N - w^*, 2)$ with respect to $\mathbf{0}$. Let $\mathcal{T}(C)$ be the Terwilliger algebra of $J(N, D)$ with respect to C. Let $\tilde{X}$ denote the vertex set of $H(N, 2)$. Recall that the vertex set X of $J(N, D)$ is a subset of $\tilde{X}$. For a subset $\mathcal{A}$ of $\mathrm{Mat}_{\tilde{X}}(\boldsymbol{C})$, let $\mathcal{A}|_{X \times X} \subset \mathrm{Mat}_X(\boldsymbol{C})$ denote the set of principal submatrices of matrices in $\mathcal{A}$. The following is the key lemma.

Lemma 6

$$\mathcal{T}(C) \subseteq \mathcal{T}_1(\mathbf{0}) \otimes \mathcal{T}_2(\mathbf{0})|_{X\times X} \quad (\subset \mathrm{Mat}_X(\boldsymbol{C}))$$

Let W_i be an irreducible $\mathcal{T}_i(\mathbf{0})$-module ($i = 1, 2$). Let

$$W := W_1 \otimes W_2|_X \subset \boldsymbol{C}^X,$$

where the right hand side denotes the set of vectors from $W_1 \otimes W_2$ whose indices are restricted on X. Then

Lemma 7 *W is a $\mathcal{T}(C)$-module.*

Go [4] gave an explicit description of W_1, W_2. We will make use of results in [4] for the characterization of W.

Lemma 8 *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be standard bases for W_1, W_2 (see [4]). Then*

(i) $\mathcal{B} := \{u \otimes u' \mid u \in \mathcal{B}_1, u' \in \mathcal{B}_2, u \otimes u'|_X \neq 0\}$ *is a basis for W.*

(ii) $\mathrm{Span}\{u \otimes u'\} = E_i^* W$ *for some i.*

(iii) *W is thin.*

We can determine the endpoint of W by comparing suppots of W_1 and W_2. For determination of the dual enpoint of W, the following will be useful:

Proposition 9 [11] *Let $\mathcal{T}(\mathbf{0})$ be the Terwilliger algebra of the binary Hamming graph $H(N, 2)$ with respect to $\mathbf{0}$. Let U be an irreducible $\mathcal{T}(\mathbf{0})$-module of endpoint r. Then $\boldsymbol{v}(\neq \mathbf{0}) \in U|_X$ is an eigenvector of $J(N, D)$ for eigenvalue θ_r.*

Next we will check that W is irreducible. To see that it is so, we consider a tridiagonal matrix. Let $[A]_{\mathcal{B}}$ be the matrix representing A with respect to the basis $\mathcal{B}$. Then $[A]_{\mathcal{B}}$ is tridiagonal since W is thin. Moeover, by calculation we can verify that the off-diagonal entries of $[A]_{\mathcal{B}}$ are nonzero. Hence we have the following:

Lemma 10 *W is an irreducible $\mathcal{T}(C)$-module.*

4 Main results

Let $\Gamma = J(N, D)$ and $C \subset X$. Suppose C satisfies $w + w^* = D$. Let $\mathcal{T}(C)$ be the Terwilliger algebra of Γ with respect to C. Let W be an irreducible $\mathcal{T}(C)$-module of endpoint ν, dual endpoint μ, diameter d.

Theorem 11 *There exist integers e, f satisfying*

$$0 \le e \le \left\lfloor \frac{w^*}{2} \right\rfloor, \quad 0 \le f \le \left\lfloor \frac{N-w^*}{2} \right\rfloor,$$

$$\nu = \max\{e, f-w\}, \quad \mu = e+f,$$

$$d = \begin{cases} w^* - 2\nu & \text{if } \nu = e, \\ \min\{D-\mu, N-D-2\nu-w\} & \text{if } \nu = f-w. \end{cases}$$

Remarks. e, f comes from endpoints of W_1, W_2.
Remarks. If $N \neq 2D$, then e, f are uniquely determined for given ν, μ, d. In this case, $\mathcal{T}(C) = \mathcal{T}_1 \otimes \mathcal{T}_2|_{X\times X}$ in Lemma 6.

Theorem 12 *W has a basis $\mathcal{B} = \{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d\}$ satisfying*

$$\boldsymbol{v}_i \in E^*_{i+\nu}W \quad (0 \le i \le d),$$

and with respect to which the matrix representing A is tridiagonal with entries

$$\begin{aligned} c_i(W) &= i(i+2\nu-\mu+w), \\ a_i(W) &= D(N-D)+\mu(\mu+d-N-1)+d(d-N \\ &\quad +2\nu+w)+i(N-4\nu-2i-2w), \\ b_i(W) &= (d-i)(N-d-2\nu-\mu-i-w). \end{aligned}$$

Remarks. $c_i(W) + a_i(W) + b_i(W) = \theta_\mu$.
Remarks. If $w = 0$, the above $c_i(W)$, $a_i(W)$, $b_i(W)$ coincide with the results by Terwilliger [10].

Corollary 13 *Isomophism classes are determined by (ν, μ, d).*

5 Remark

Let $A^* = diag(E_1\chi)$. Then (A, A^*) acts on W as a *Leonard pair* with parameter array $(h, r, s, s^*, r, d, \theta_0, \theta^*_0)$ (*Dual Hahn*):

$$\begin{aligned} \theta_i &= \theta_0 + hi(i+1+s), \\ \theta^*_i &= \theta^*_0 + s^*i, \\ \varphi_i &= hs^*i(i-d-1)(i+r), \\ \phi_i &= hs^*i(i-d-1)(i+r-s-d-1). \end{aligned}$$

Especially, we have

$$s = -N - 2 + 2\mu,$$

$$r = -N + d + 2\nu + \mu - 1 + w.$$

See [9] for details on Leonard pairs. If $w = 0$, the above parameters coincide with the results by Terwilliger [10].

References

[1] E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin/Cummings, California, 1984.

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer Verlag, Berlin, Heidelberg, 1989.

[3] A. E. Brouwer, C. D. Godsil, J. H. Koolen and W. J. Martin, Width and dual width of subsets in polynomial association schemes, J. Combin. Th. (A) 102 (2003), 255-271.

[4] J. T. Go, The Terwilliger algebra of the Hypercube, Europ. J. Combin. 23 (2002), 399-430.

[5] R. Hosoya and H. Suzuki, Tight distance-regular graphs with respect to subsets, European J. Combin. 28 (2007), 61-74.

[6] H. Suzuki, The Terwilliger algebra associated with a set of vertices in a distance-regular graph, J. Algebraic Combinatorics 22 (2005), 5-38.

[7] H. Tanaka, Classification of subsets with minimal width and dual width in Grassmann, bilinear forms and dual polar graphs, J. Combin. Theory (A) 113 (2006), 903-910.

[8] H. Tanaka, On subsets with minimal width and dual width, in preparation.

[9] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; an algebraic approach to the Askey scheme of orthogonal polynomials, arXive:math/0408390.

[10] P. Terwilliger, The subconstituent algebra of an association schemes, (part I, II, III), J. Alg. Combin. 1 (1992), 363-388, 2(1993), 73-103, 177-210.

[11] P. Terwilliger, Lecture note on algebraic graph theory (Suzuki-version).