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<th>Title</th>
<th>Evaluation modules for the three-point $\mathfrak{sl}_2$ loop algebra (Finite Groups and Algebraic Combinatorics)</th>
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The three-point \( \mathfrak{g}_2 \) loop algebra

The equitable basis for \( \mathfrak{g}_2 \)

polynomials in two variables

Reduction of the evaluation modules by

\( 2 \gamma \) bases for evaluation module

The \( 2 \gamma \) action on the evaluation modules

The evaluation modules

The C\( \ell \) irreducible modules

Three-point \( \mathfrak{g}_2 \) loop algebra

The transformation algebra realization of the

Overview
The sequence (dim(%) p) can be decomposed into a direct sum (\( \bigoplus \)).

We call the direct sum of the decompositions \( \bigoplus \).

Let \( V \) denote the free \( \mathbb{K} \)-module.

Decompositions

\[
\text{Finite-dimensional \textit{F}}_{\mathbb{K}} \text{-modules}
\]

After some general remarks we focus on the evaluation modules.

If \( \text{dim} \mathbb{K} \) is a tensor product of \( \text{evaluation modules} \),

then \( \text{dim} \mathbb{K} \) is a tensor product of \( \text{evaluation modules} \).

For these modules, there is a special case called \( \text{loop algebra} \).

Our goal is to describe the \( \text{loop algebra} \).

Theorem (Harthong) there exists an isomorphism of \( \mathbb{K} \)-modules

\[
\phi: \mathbb{K} \rightarrow \mathbb{K}
\]

and the three-point \( \mathbb{K} \)-loop algebra

The \( \text{loop algebra} \).

\[
\oplus
\]

For mutually distinct \( \phi, f \in \mathbb{K}
\]

\[
\oplus
\]

For mutually distinct \( \phi, f \in \mathbb{K}
\]

\[
\oplus
\]

For distinct \( \phi, f \in \mathbb{K}
\]
We call the evaluation parameter for $\lambda^a_k$.

The $\mathbb{Z}$-module $\Lambda_k^a$ is not trivial and reducible.

(i) If $a$ is a positive integer.

(ii) $\Lambda_k^a$ is the reducible $\mathbb{Z}$-module with dimension $d$.

For an evaluation module $\Lambda_k^a$, we mean the module $\Lambda_k^a$.

Every element of $\Lambda_k^a$ is a unique $\mathbb{Z}$-module.

The trivial $\mathbb{Z}$-module.

The evaluation modules for $\Lambda_k^a$.

We now define the evaluation modules for $\Lambda_k^a$.

The shape of $\Lambda_k^a$.

How the decompositions $[\lambda^a_k]$ are related.
Theorem I: Characterizing the Evaluation Module

Let $\mathbb{A}$ be an arbitrary algebra, and let $V$ be a finitely generated free $\mathbb{A}$-module. Then $V$ is isomorphic to an evaluation module for some $\mathbb{A}$-module $\Lambda$.

### Proof

1. **Existence:**
   - Choose a basis $\{e_1, e_2, \ldots, e_n\}$ for $V$.
   - Define a module homomorphism $\phi: \Lambda \to V$ by setting $\phi(e_i) = e_i$ for $1 \leq i \leq n$.
   - The map $\phi$ is surjective because $V$ is free.
   - Let $V' = \ker(\phi)$. Then $V' = \{0\}$ by the rank-nullity theorem.
   - Therefore, $\phi$ is injective.
   - Hence, $\Lambda$ is isomorphic to $V$ as a module.

2. **Uniqueness:**
   - Assume $\Lambda' \to V$ is another module homomorphism.
   - Then we must have $\phi = \phi'$.
   - Hence, $\Lambda' = \Lambda$.

### Modules II

Theorem I: Characterizing the Evaluation Module

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### Proof

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   - Assume $\Lambda' \to V$ is another module homomorphism.
   - Then we must have $\phi = \phi'$.
   - Hence, $\Lambda' = \Lambda$.
The orbits of $S^4$ on $R^4 \setminus \{0\}$ cont.}

Here is another way to view the relative func-

In this section, we will return to the subgroup $G$ later in the

(ii) The $G$-module $V(p_0)$, twisted via $\alpha$.

(i) The $G$-module $V(p_0)$ twisted via $\alpha$.

For $a \in (0,1)$ the following are isomorphic:

- (i) The subgroups $G$ of $S^4$, cont.

- (ii) The $G$-module $V(p_0)$ twisted via $\alpha$.

- (iii) The $G$-module $V(p_0)$ twisted via $\alpha$.

Let $G$ denote the kernel of this action.

Either we gave an action of $G$ on the set $R^4 \setminus \{0\}$.

A subgroup $G$ of $S^4$.

We now describe the orbits for the action

The orbits of $S^4$ on $R^4 \setminus \{0\}$.
We have now defined $2n$ bases for $V^n(a)$.

We denote this basis by $x_{a}^{(j)}$.

(i) $x_{a}^{(j)}$ is a component of the decomposition $(x_{a}^{(j)})$.

(ii) for $0 \leq n < d$ the vector $w_{i}$ is contained in

there exists a unique basis $(w_{i})_{i=1}^{d}$ for $V^n(a)$ such

Lemma 4: Mutually disjoint $(x_{a}^{(j)})$ exist

The basis $(x_{a}^{(j)}|_{i}^{(k)})$ for $V^n(a)$

Location of $(x_{a}^{(j)})$ (I)

The vectors $x_{a}^{(j)}|_{i}^{(k)}$ in $V^n(a)$

For notational convenience, let $i \leq j$.

We are about to define $2n$ bases for this mod-

We consider the submodule $V^n(a)$.

(iii) $x_{a}^{(j)}$ is a scalar $\neq 0$.

For the time being we fix an integer $d \geq 2$ and

24 bases for $V^n(a)$

The orbits of $S_4$ on $V^n(a)$, cont.
Theorem 1.2: The transition matrix is the matrix $(I + \mathbf{a} \cdot \mathbf{b}^T)$ relative to $\mathbf{a}$ for $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$. When $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$, the form is symmetric (parallel to the modulus). The form is unique up to multiplication by a nonzero scalar in $\mathbb{F}_q$.

The transition matrices are now considered in more detail. In order to describe these, it is convenient to introduce a certain bilinear form on $V(q)$.

We now consider the transition matrices of some transition matrices...

In the following, we consider the matrix representation of the groups $\mathbb{F}_q^k$, for $q$ prime. The entries of the matrices are given in the form $[\alpha]_{i,j} = \alpha_{i,j}$, where $\alpha_{i,j}$ is the constant term of the polynomial $\alpha(x)$ of degree $\leq q-1$ with respect to the basis $\{x, x^2, \ldots, x^{q-1}\}$.

Theorem 1.3: For the moment, assume that $q$ is a prime. Then the transition matrices are determined by the bilinear form.

We denote by $\mathfrak{a}(\beta)$ the bilinear form on $V(q)$.
By the definition of $\mathbf{A}$ we mean an $F$-linear map $D: \mathbf{A} \rightarrow \mathbf{A}$ such that

We will use the following terms:

Definition of $\mathbf{A}$

The next goal is to display a $G$-module structure.

Example: Some basis for $\mathbf{A}$

Lemma: For distinct $f, g \in \mathbf{A}$ the elements

Some basis for $\mathbf{A}$

Comment: on the $\mathbf{A}$ (i.e. 1)

Revising the evaluation modules

$D = m \phi + n \psi$
For any field \( K \) and for \( 0 \leq n \leq m \) the nil component is described by the following:

\[
\frac{\mu(x^n)}{\nu(x^n)} = \frac{\mu(x)}{\nu(x)}^{m-n}
\]

where \( \mu, \nu \) are polynomials in \( K[x] \) and \( m \geq n \).

The basis \( \mathcal{B} \) of \( V \) is given by \( \{ e_i \} \) for \( 0 \leq i \leq m \).

The elements \( \alpha \in \mathcal{B} \) for \( V \).  

The decomposition \( \mathcal{D}_{\mathcal{B}} \) for \( V \).
THE END

Theorem 6.1

We have explained this fact using A.

Some automorphisms of A

We saw earlier that if we exist the 8-module $V(a)$ for an element of C then the result is isomorphic to $V(a)$.