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Evaluation modules for the three-point \mathfrak{sl}_2 loop algebra

Overview

- The tetrahedron algebra realization of the three-point \mathfrak{sl}_2 loop algebra
- The f.d. irreducible modules
- The evaluation modules
- The S_4 -action on the evaluation modules
- 24 bases for an evaluation module
- Realization of the evaluation modules by polynomials in two variables

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Warmup: The Lie algebra \mathfrak{sl}_2

Throughout, \mathbb{F} will denote an algebraically closed field with characteristic 0.

Recall that \mathfrak{sl}_2 is the Lie algebra over \mathbb{F} with a basis e, f, h and Lie bracket

$$[h, e] = 2e, \quad [h, f] = -2f, \\ [e, f] = h.$$

The equitable basis for \mathfrak{sl}_2

The equitable basis for \mathfrak{sl}_2

Define

$$x = 2e - h, \quad y = -2f - h, \quad z = h.$$

Then x, y, z is a basis for \mathfrak{sl}_2 and

$$[x, y] = 2x + 2y, \\ [y, z] = 2y + 2z, \\ [z, x] = 2z + 2x.$$

We call x, y, z the equitable basis for \mathfrak{sl}_2 .

The three-point \mathfrak{sl}_2 loop algebra

The three-point \mathfrak{sl}_2 loop algebra is the Lie algebra over \mathbb{F} consisting of the vector space

$$\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}], \quad \otimes = \otimes_{\mathbb{F}}$$

where t is indeterminate, and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab$$

The equitable presentation for the three-point \mathfrak{sl}_2 loop algebra

We now recall the equitable presentation for the three-point \mathfrak{sl}_2 loop algebra.

To give the presentation we define a Lie algebra \mathfrak{X} by generators and relations, and display an isomorphism from \mathfrak{X} to the three-point \mathfrak{sl}_2 loop algebra.

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\mathfrak{X} and the three-point \mathfrak{sl}_2 loop algebra

Theorem [Hartwig +T] There exists an isomorphism of Lie algebras

$$\psi : \mathfrak{X} \rightarrow \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$$

that sends

$$\begin{aligned} x_{12} &\mapsto x \otimes 1, & x_{03} &\mapsto y \otimes (1 + z \otimes (t-1)), \\ x_{23} &\mapsto y \otimes 1, & x_{01} &\mapsto z \otimes (1 - t^{-1}) - x \otimes t^{-1}, \\ x_{31} &\mapsto z \otimes 1, & x_{02} &\mapsto x \otimes (1 - t)^{-1} + y \otimes (1 - t)^{-1} \end{aligned}$$

where x, y, z is the equitable basis for \mathfrak{sl}_2 .

From now on we work with \mathfrak{X} .

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The tetrahedron algebra \mathfrak{X}

Definition [Hartwig+T] The tetrahedron algebra \mathfrak{X} is the Lie algebra over \mathbb{F} that has generators

$$\{x_{ij} \mid i, j \in \mathbb{I}, i \neq j\} \quad \mathbb{I} = \{0, 1, 2, 3\}$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{ij} + x_{ji} = 0.$$

(ii) For mutually distinct $i, j, k \in \mathbb{I}$,

$$[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}.$$

(iii) For mutually distinct $h, i, j, k \in \mathbb{I}$,

$$[x_{hi}, [x_{hi}, [x_{hi}, x_{jk}]]] = 4[x_{hi}, x_{jk}].$$

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The algebra \mathfrak{X}

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Finite-dimensional irred. \mathfrak{X} -modules

Our goal is to describe the f.d. irreducible \mathfrak{X} -modules.

For these modules there is a special case called an **evaluation module**.

It turns out that every f.d. irreducible \mathfrak{X} -module is a tensor product of evaluation modules.

After some general remarks we focus on the evaluation modules.

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Decompositions

Let V denote a f.d. irreducible \mathfrak{X} -module.

By a **decomposition** of V we mean a sequence $\{V_n\}_{n=0}^d$ of nonzero subspaces of V such that

$$V = \sum_{n=0}^d V_n \quad (\text{direct sum}).$$

We call d the **diameter** of the decomposition.

By the **shape** of this decomposition we mean the sequence $\{\dim(V_n)\}_{n=0}^d$.

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The decompositions $[i, j]$

Hartwig showed:

- (i) Each generator x_{ij} is semisimple on V .
- (ii) There exists an integer $d \geq 0$ such that for each generator x_{ij} , the set of distinct eigenvalues on V is $\{2n - d \mid 0 \leq n \leq d\}$.

We let $[i, j]$ denote the eigenspace decomposition for x_{ij} on V associated with the above ordering of the eigenvalues.

How the decompositions $[i, j]$ are related

The shape of V

Hartwig showed that the shape of the decomposition $[i, j]$ is independent of the pair i, j .

We call this common shape the **shape of V** .

The evaluation modules for \mathfrak{sl}_2

We now define the evaluation modules for \mathfrak{sl}_2 .

For $a \in \mathbb{F} \setminus \{0, 1\}$ we define a Lie algebra homomorphism

$$E\mathbb{V}_a : \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}] \xrightarrow{\psi} \mathfrak{sl}_2$$

$$u \otimes f(t) \mapsto u f(a)$$

For an \mathfrak{sl}_2 -module V we pull back the \mathfrak{sl}_2 -module structure via $E\mathbb{V}_a$: this turns V into a \mathfrak{sl}_2 -module which we call $V(a)$.

The evaluation modules for \mathfrak{sl}_2 , cont.

By an **evaluation module** for \mathfrak{sl}_2 we mean the module $V_d(a)$ where

- (i) d is a positive integer;
- (ii) V_d is the irreducible \mathfrak{sl}_2 -module with dimension $d + 1$.

The \mathfrak{sl}_2 -module $V_d(a)$ is nontrivial and irreducible.

We call a the **evaluation parameter** for $V_d(a)$.

The trivial \mathfrak{sl}_2 -module

Up to isomorphism there exists a unique \mathfrak{sl}_2 -module V with dimension 1.

Every element of \mathfrak{sl}_2 is 0 on V .

We call V the **trivial \mathfrak{sl}_2 -module**.

Characterizing the evaluation modules, I

Theorem For a nontrivial f.d. irreducible \mathbb{K} -module V TFAE:

- (i) V is isomorphic to an evaluation module for \mathbb{K} .
- (ii) V has shape $(1, 1, \dots, 1)$.

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Characterizing the evaluation modules, II

Theorem Let V denote a nontrivial f.d. irreducible \mathbb{K} -module.

Then for $a \in \mathbb{F} \setminus \{0, 1\}$ TFAE:

- (i) V is isomorphic to an evaluation module with evaluation parameter a .
- (ii) Each of the following vanishes on V :

$$\begin{aligned} ax_{01} + (1 - a)x_{02} - x_{03}, \\ ax_{10} + (1 - a)x_{13} - x_{12}, \\ ax_{23} + (1 - a)x_{20} - x_{21}, \\ ax_{32} + (1 - a)x_{31} - x_{30}. \end{aligned}$$

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An S_4 -action on \mathbb{K} -modules

For a \mathbb{K} -module V and $\sigma \in S_4$ there exists a \mathbb{K} -module structure on V , called V twisted via σ , that behaves as follows:

For $u \in \mathbb{K}$ and $v \in V$, the vector $u \cdot v$ computed in V twisted via σ coincides with the vector $\sigma^{-1}(u) \cdot v$ computed in the original \mathbb{K} -module V .

Sometimes we abbreviate ${}^\sigma V$ for V twisted via σ .

S_4 acts on the set of \mathbb{K} -modules, with σ sending V to ${}^\sigma V$ for all $\sigma \in S_4$ and all \mathbb{K} -modules V .

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An S_4 -action on \mathbb{K}

We identify the symmetric group S_4 with the group of permutations of I .

S_4 acts on the set of generators for \mathbb{K} by permuting the indices:

$$\sigma(x_{ij}) = x_{\sigma(i)\sigma(j)} \quad \sigma \in S_4.$$

This action leaves invariant the defining relations and therefore induces an action of S_4 on \mathbb{K} as a group of automorphisms.

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The S_4 -action on \mathbb{K} -modules, cont.

The above S_4 -action on \mathbb{K} -modules sends evaluation modules to evaluation modules.

The effect of this action on the evaluation parameter is described in the following two slides.

An action of S_4 on $\mathbb{F} \setminus \{0, 1\}$

Lemma There exists an action of S_4 on the set $\mathbb{F} \setminus \{0, 1\}$ that does the following.

For $a \in \mathbb{F} \setminus \{0, 1\}$,

- $(2, 0)$ sends $a \mapsto a^{-1}$;
- $(0, 1)$ sends $a \mapsto a(a - 1)^{-1}$;
- $(1, 3)$ sends $a \mapsto a^{-1}$.

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The effect of S_4 on the evaluation parameter

Theorem For an integer $d \geq 1$, $\sigma \in S_4$, and $a \in \mathbb{F} \setminus \{0, 1\}$ the following are isomorphic:

- (i) The \mathbb{B} -module $V_d(a)$ twisted via σ ;
- (ii) The \mathbb{B} -module $V_d(\sigma(a))$.

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A subgroup G of S_4

Earlier we gave an action of S_4 on the set $\mathbb{F} \setminus \{0, 1\}$.

Let G denote the kernel of this action.

It turns out that G consists of

- (01)(23), (02)(13), (03)(12)

together with the identity element.

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The subgroup G of S_4 , cont.

Corollary For an integer $d \geq 1$, for $\sigma \in G$, and for $a \in \mathbb{F} \setminus \{0, 1\}$ the following are isomorphic:

- (i) The \mathbb{B} -module $V_d(a)$ twisted via σ ;
- (ii) The \mathbb{B} -module $V_d(a)$.

We will return to the subgroup G later in the talk.

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The orbits of S_4 on $\mathbb{F} \setminus \{0, 1\}$

We now describe the orbits for the S_4 -action on $\mathbb{F} \setminus \{0, 1\}$.

Pick $a \in \mathbb{F} \setminus \{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$.

By the (i, j, k, ℓ) -relative of a we mean the scalar $\sigma(a)$ where $\sigma \in S_4$ sends the sequence (i, j, k, ℓ) to $(2, 0, 1, 3)$.

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The orbits of S_4 on $\mathbb{F} \setminus \{0, 1\}$, cont.

The relative function satisfies this recursion:

Lemma Pick $a \in \mathbb{F} \setminus \{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$.

Let α denote the (i, j, k, ℓ) -relative of a . Then

- α^{-1} is the (j, i, k, ℓ) -relative of a ;
- $\alpha(\alpha - 1)^{-1}$ is the (i, k, j, ℓ) -relative of a ;
- α^{-1} is the (i, j, ℓ, k) -relative of a .

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The orbits of S_4 on $\mathbb{F} \setminus \{0, 1\}$, cont.

Here is another way to view the relative function.

Lemma For $a \in \mathbb{F} \setminus \{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$ the following (i), (ii) coincide:

- (i) the (i, j, k, ℓ) -relative of a ;
- (ii) the scalar

$$\frac{i - \ell j - k}{i - k j - \ell}$$

where we define $\frac{0}{0} = a$, $\frac{1}{1} = 0$, $\frac{2}{2} = 1$, $\frac{3}{3} = \infty$.

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The orbits of S_4 on $\mathbb{F}\{0,1\}$, cont.

Here is an explicit description of the relative function.

Theorem Pick $a \in \mathbb{F}\{0,1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$.

Then the (i, j, k, ℓ) -relative of a is given in the following table.

(i, j, k, ℓ)	(i, j, k, ℓ) -relative
(2,0,1,3)	a
(0,2,1,3)	$1-a$
(1,0,2,3)	a^2
(0,1,0,2)	a^2-1
(0,1,2,3)	$a(a-1)^{-1}$
(2,1,0,3)	$1-a^2$
(1,2,0,3)	$1-a^{-2}$
(0,2,3,1)	a^{-1}
(1,3,0,2)	$a^{-1}-a$
(3,1,0,2)	$a^{-1}-a^{-2}$
(3,1,2,0)	$a^{-1}-a^{-2}-1$
(2,3,0,1)	$a^{-1}-a^{-2}-1$
(2,3,1,0)	$a^{-1}-a^{-2}-1$
(2,3,1,2)	$a^{-1}-a^{-2}-1$
(3,0,2,1)	$a^{-1}-a^{-2}-1$
(3,0,2,1)	$a^{-1}-a^{-2}-1$

24 bases for $V_d(a)$

For the time being we fix an integer $d \geq 1$ and a scalar $a \in \mathbb{F}\{0,1\}$.

We consider the \mathbb{Q} -module $V_d(a)$.

We are about to define 24 bases for this module.

The vectors η_i ($i \in \mathbb{I}$) in $V_d(a)$

For notational convenience, for $i \in \mathbb{I}$ we fix a nonzero vector $\eta_i \in V_d(a)$ which is a common eigenvector for $\{x_{ij} \mid j \in \mathbb{I}, j \neq i\}$.

Location of η_i ($i \in \mathbb{I}$)

The basis $[i, j, k, \ell]$ for $V_d(a)$

Lemma For mutually distinct $i, j, k, \ell \in \mathbb{I}$ there exists a unique basis $\{u_n\}_{n=0}^d$ for $V_d(a)$ such that:

(i) for $0 \leq n \leq d$ the vector u_n is contained in component n of the decomposition $[k, \ell]$:

$$(ii) \eta_i = \sum_{n=0}^d u_n.$$

We denote this basis by $[i, j, k, \ell]$.

We have now defined 24 bases for $V_d(a)$.

The basis $[i, j, k, \ell]$ for $V_d(a)$

How the generators x_r act on the 24 bases

Theorem For mutually distinct $i, j, k, \ell \in \mathbb{I}$ and distinct $r, s \in \mathbb{I}$ consider the matrix representing x_r with respect to the basis $\{i, j, k, \ell\}$ of $V_d(\alpha)$. The entries of this matrix are given in the following table. All entries not displayed are zero.

gen.	$(n, n-1)$ -entry	(n, n) -entry	$(n-1, n)$ -entry
x_1	0	$\frac{d}{2n-2}$	0
x_2	0	$\frac{2n-d}{2n-d}$	0
x_3	0	$\frac{2d-2n+2}{2n-d}$	0
x_4	0	$\frac{d}{2n}$	0
x_5	$\frac{2n}{2n}$	$\frac{d-2n}{2n}$	0
x_6	$\frac{2n}{2n}$	$\frac{d-2n}{2n}$	0
x_7	0	$\frac{d-2n}{2n}$	0
x_8	0	$\frac{2n-d}{2n-d}$	$\frac{2(n-d-1)\alpha^{-1}}{2(d-n+1)(\alpha-1)}$
x_9	$\frac{2n(\alpha-1)^{-1}}{2n(1-\alpha)^{-1}}$	$\frac{(d-2n)(\alpha+1)(\alpha-1)^{-1}}{(d-2n)(\alpha+1)(1-\alpha)^{-1}}$	$\frac{2(d-n+1)(1-\alpha)^{-1}}{2(d-n+1)(\alpha-1)^{-1}}$

In the above table the scalar α denotes the (i, j, k, ℓ) -relative of α .

The matrix Z

The following matrix will play a role in our discussion.

For an integer $d \geq 0$ let $Z = Z(d)$ denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with entries

$$Z_{ij} = \begin{cases} 1, & \text{if } i+j=d, \\ 0, & \text{if } i+j \neq d. \end{cases} \quad (0 \leq i, j \leq d).$$

We observe

$$Z^2 = I.$$

Some transition matrices

We now consider the transition matrices between our 24 bases.

In order to describe these, it is convenient to introduce a certain bilinear form on $V_d(\alpha)$.

The transition matrices

Theorem Referring to $V_d(\alpha)$, pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and consider the transition matrices from the basis $\{i, j, k, \ell\}$ to the bases

$$[i, i, k, \ell], \quad [i, k, j, \ell], \quad [i, j, \ell, k].$$

(i) The first transition matrix is diagonal with (r, r) -entry

$$\frac{\langle \eta_j, \eta_\ell \rangle}{\langle \eta_i, \eta_\ell \rangle} \alpha^r$$

for $0 \leq r \leq d$, where α is the (i, j, k, ℓ) -relative of α .

(ii) The second transition matrix is lower triangular with (r, s) -entry

$$\binom{r}{s} \alpha^{r-s} (1-\alpha)^s$$

for $0 \leq s \leq r \leq d$, where α is the (i, j, k, ℓ) -relative of α .

(iii) The third transition matrix is the matrix Z .

A bilinear form on $V_d(\alpha)$

Lemma There exists a nonzero bilinear form (\cdot, \cdot) on $V_d(\alpha)$ such that

$$(w, z, v) = -(u, w, v) \quad w \in \mathbb{B}, \quad u, v \in V.$$

The form is nondegenerate.

The form is unique up to multiplication by a nonzero scalar in \mathbb{F} .

The form is symmetric (resp. antisymmetric) when d is even (resp. d is odd).

We call (\cdot, \cdot) a **standard bilinear form** for $V_d(\alpha)$.

Realizing the evaluation modules for \boxtimes using polynomials in two variables

Let z_0, z_1 denote commuting indeterminates.

let $\mathbb{F}[z_0, z_1]$ denote the \mathbb{F} -algebra of all polynomials in z_0, z_1 that have coefficients in \mathbb{F} .

We abbreviate $\mathcal{A} = \mathbb{F}[z_0, z_1]$.

We often view \mathcal{A} as a vector space over \mathbb{F} .

For an integer $d \geq 0$ let \mathcal{A}_d denote the subspace of \mathcal{A} consisting of the homogeneous polynomials in z_0, z_1 that have total degree d .

Thus $\{z_0^{d-n} z_1^n\}_{n=0}^d$ is a basis for \mathcal{A}_d .

Realizing the evaluation modules

Note that

$$A = \sum_{n=0}^{\infty} A_d \quad (\text{direct sum})$$

and that

$$A_r A_s = A_{r+s} \quad (r, s \geq 0).$$

We fix mutually distinct $\beta_i \in \mathbb{F}$ ($i \in \mathbb{I}$).

Then there exist unique $z_2, z_3 \in A$ such that

$$\sum_{i \in \mathbb{I}} z_i = 0, \quad \sum_{i \in \mathbb{I}} \beta_i z_i = 0.$$

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Comments on the z_i ($i \in \mathbb{I}$)

Lemma For mutually distinct $i, j, k, \ell \in \mathbb{I}$ we have

$$z_k = \frac{\beta_\ell - \beta_i}{\beta_k - \beta_\ell} z_i + \frac{\beta_\ell - \beta_j}{\beta_k - \beta_\ell} z_j,$$

$$z_\ell = \frac{\beta_i - \beta_k}{\beta_\ell - \beta_k} z_i + \frac{\beta_j - \beta_k}{\beta_\ell - \beta_k} z_j.$$

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Example: Some bases for A_3

Some bases for A_d

Lemma For an integer $d \geq 0$ and distinct $i, j \in \mathbb{I}$ the elements $\{z_i^k - z_j^k\}_{k=0}^d$ form a basis for A_d .

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Some bases for A

Lemma For distinct $i, j \in \mathbb{I}$ the elements $z_i^r z_j^s$ $0 \leq r, s < \infty$ form a basis for A .

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Derivations of A

Our next goal is to display a \mathbb{B} -module structure on A .

We will use the following terms.

By a derivation of A we mean an \mathbb{F} -linear map $D : A \rightarrow A$ such that

$$D(uv) = D(u)v + uD(v) \quad (u, v \in A).$$

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A is a \mathbb{B} -module

Theorem There exists a unique \mathbb{B} -module structure on A such that:

- (i) each element of \mathbb{B} acts as a derivation on A ;
- (ii) $x_j z_i = -z_i$ and $x_j z_j = z_j$ for distinct $i, j \in \mathbb{I}$.

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The decomposition $[i, j]$ for A_d

Earlier in the talk we described the \mathbb{B} -module $V_d(a)$.

We now consider how things look from the point of view of A_d .

Proposition For an integer $d \geq 0$ and for distinct $i, j \in \mathbb{I}$ the decomposition $[i, j]$ on A_d is described as follows.

For $0 \leq n \leq d$ the n th component is spanned by $z_i^d z_j^n$.

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The eigenvectors for the x_j on A

Lemma For distinct $i, j \in \mathbb{I}$ and integers $r, s \geq 0$ the element $z_i^r z_j^s$ is an eigenvector for x_j with eigenvalue $s - r$.

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The elements η_i ($i \in \mathbb{I}$) for A_d

For an integer $d \geq 1$ and $i \in \mathbb{I}$ the element z_i^d is a scalar multiple of η_i .

Recall η_i is defined up to scalar multiplication.

For the rest of talk we choose $\eta_i = z_i^d$.

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The irreducible \mathbb{B} -submodules of A

Proposition Referring to the \mathbb{B} -module A ,

- (i) For $d \geq 0$ the subspace A_d is an irreducible \mathbb{B} -submodule of A .
- (ii) The \mathbb{B} -module A_0 is trivial.
- (iii) For $d \geq 1$ the \mathbb{B} -module A_d is isomorphic to $V_d(a)$ where

$$a = \frac{\beta_0 - \beta_1 \beta_2 - \beta_3}{\beta_0 - \beta_3 \beta_2 - \beta_1}$$

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The basis $[i, j, k, \ell]$ for A_d

Proposition For an integer $d \geq 1$ and for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the basis $[i, j, k, \ell]$ of A_d is described as follows.

For $0 \leq n \leq d$ the n th component is

$$z_i^{d-n} z_j^n \binom{d}{n} \frac{(\beta_j - \beta_k)^{d-n} (\beta_j - \beta_\ell)^n}{(\beta_i - \beta_j)^n}$$

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The group G revisited

We saw earlier that if we twist the \mathbb{Q} -module $V_A(\alpha)$ via an element of G then the result is isomorphic to $V_A(\alpha)$.

We now explain this fact using A .

Some automorphisms of A

Lemma For mutually distinct $i, j, k, \ell \in I$ there exists a unique automorphism of A that sends

$$\begin{array}{l} z_i \mapsto \frac{\beta_j - \beta_k}{\beta_i - \beta_k} z_j \\ z_k \mapsto \frac{\beta_\ell - \beta_k}{\beta_i - \beta_k} z_\ell \\ z_\ell \mapsto \frac{\beta_i - \beta_k}{\beta_j - \beta_k} z_i \\ z_j \mapsto \frac{\beta_i - \beta_k}{\beta_j - \beta_k} z_j \end{array}$$

Some automorphisms of A

Theorem The following hold for $\sigma \in G$:

- (i) There exists an automorphism g_σ of A that sends z_r to a scalar multiple of $z_\sigma(r)$ for all $r \in I$.
- (ii) For $u \in \mathbb{Q}$ the equation $\sigma(u) = g_\sigma u g_\sigma^{-1}$ holds on A .
- (iii) The map g_σ is an isomorphism of \mathbb{Q} -modules from A to A twisted via σ .

THE END