数理解析研究所講究録
第 1593 巻 2008 年 51-60

Evaluation modules for the three-point $\mathfrak{sl}_2$ loop algebra

Tatsuro Ito

Kanazawa University

Paul Terwilliger

University of Wisconsin

1

Overview

- The tetrahedron algebra realization of the three-point $\mathfrak{sl}_2$ loop algebra
- The f.d. irreducible modules
- The evaluation modules
- The S_4-action on the evaluation modules
- 24 bases for an evaluation module
- Realization of the evaluation modules by polynomials in two variables

2

Warmup: The Lie algebra $\mathfrak{sl}_2$

Throughout, $\mathbb{F}$ will denote an algebraically closed field with characteristic 0.

Recall that $\mathfrak{sl}_2$ is the Lie algebra over $\mathbb{F}$ with a basis e, f, h and Lie bracket

$$[h,e]=2e, \qquad [h,f]=-2f,$$
$$[e,f]=h.$$

3

The equitable basis for $\mathfrak{sl}_2$

Define

$$x=2e-h, \qquad y=-2f-h, \qquad z=h.$$

Then x,y,z is a basis for $\mathfrak{sl}_2$ and

$$\begin{aligned}[x,y] &= 2x+2y,\\ [y,z] &= 2y+2z,\\ [z,x] &= 2z+2x.\end{aligned}$$

We call x,y,z the **equitable basis** for $\mathfrak{sl}_2$.

4

The equitable basis for $\mathfrak{sl}_2$

5

The three-point $\mathfrak{sl}_2$ loop algebra

The **three-point $\mathfrak{sl}_2$ loop algebra** is the Lie algebra over $\mathbb{F}$ consisting of the vector space

$$\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}], \qquad \otimes = \otimes_{\mathbb{F}}$$

where t is indeterminate, and Lie bracket

$$[u \otimes a, v \otimes b] = [u,v] \otimes ab.$$

6

The equitable presentation for the three-point $\mathfrak{sl}_2$ loop algebra

We now recall the equitable presentation for the three-point $\mathfrak{sl}_2$ loop algebra.

To give the presentation we define a Lie algebra $\boxtimes$ by generators and relations, and display an isomorphism from $\boxtimes$ to the three-point $\mathfrak{sl}_2$ loop algebra.

The tetrahedron algebra $\boxtimes$

Definition [Hartwig+T] The tetrahedron algebra $\boxtimes$ is the Lie algebra over $\mathbb{F}$ that has generators

$$\{x_{ij} \mid i, j \in \mathbb{I}, i \neq j\} \qquad \mathbb{I} = \{0, 1, 2, 3\}$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{ij} + x_{ji} = 0.$$

(ii) For mutually distinct $h, i, j \in \mathbb{I}$,

$$[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}.$$

(iii) For mutually distinct $h, i, j, k \in \mathbb{I}$,

$$[x_{hi}, [x_{hi}, [x_{hi}, x_{jk}]]] = 4[x_{hi}, x_{jk}].$$

The algebra $\boxtimes$

$\boxtimes$ and the three-point $\mathfrak{sl}_2$ loop algebra

Theorem [Hartwig +T] There exists an isomorphism of Lie algebras

$$\psi : \boxtimes \to \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$$

that sends

$$\begin{aligned} x_{12} &\mapsto x \otimes 1, & x_{03} &\mapsto y \otimes t + z \otimes (t-1), \\ x_{23} &\mapsto y \otimes 1, & x_{01} &\mapsto z \otimes (1 - t^{-1}) - x \otimes t^{-1}, \\ x_{31} &\mapsto z \otimes 1, & x_{02} &\mapsto x \otimes (1-t)^{-1} + y \otimes t(1-t)^{-1} \end{aligned}$$

where x, y, z is the equitable basis for $\mathfrak{sl}_2$.

From now on we work with $\boxtimes$.

Finite-dimensional irred. $\boxtimes$-modules

Our goal is to describe the f.d. irreducible $\boxtimes$-modules.

For these modules there is a special case called an **evaluation module**.

It turns out that every f.d. irreducible $\boxtimes$-module is a tensor product of evaluation modules.

After some general remarks we focus on the evaluation modules.

Decompositions

Let V denote a f.d. irreducible $\boxtimes$-module.

By a **decomposition** of V we mean a sequence $\{V_n\}_{n=0}^d$ of nonzero subspaces of V such that

$$V = \sum_{n=0}^{d} V_n \qquad \text{(direct sum)}.$$

We call d the **diameter** of the decomposition.

By the **shape** of this decomposition we mean the sequence $\{\dim(V_n)\}_{n=0}^d$.

The decompositions $[i,j]$

Hartwig showed:

(i) Each generator x_{ij} is semisimple on V.

(ii) There exists an integer $d \geq 0$ such that for each generator x_{ij} the set of distinct eigenvalues on V is

$$\{2n - d \mid 0 \leq n \leq d\}.$$

We let $[i,j]$ denote the eigenspace decomposition for x_{ij} on V associated with the above ordering of the eigenvalues.

How the decompositions $[i,j]$ are related

The shape of V

Hartwig showed that the shape of the decomposition $[i,j]$ is independent of the pair i, j.

We call this common shape **the shape of V**.

The trivial $\boxtimes$-module

Up to isomorphism there exists a unique $\boxtimes$-module V with dimension 1.

Every element of $\boxtimes$ is 0 on V.

We call V the **trivial** $\boxtimes$-module.

The evaluation modules for $\boxtimes$

We now define the evaluation modules for $\boxtimes$.

For $a \in \mathbb{F}\backslash\{0,1\}$ we define a Lie algebra homomorphism

$$\begin{array}{cccccc} EV_a : \boxtimes & \to & \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}] & \to & \mathfrak{sl}_2 \\ \psi & & u \otimes f(t) & \to & uf(a) \end{array}$$

For an $\mathfrak{sl}_2$-module V we pull back the $\mathfrak{sl}_2$-module structure via EV_a; this turns V into a $\boxtimes$-module which we call $V(a)$.

The evaluation modules for $\boxtimes$, cont.

By an **evaluation module** for $\boxtimes$ we mean the module $V_d(a)$ where

(i) d is a positive integer;

(ii) V_d is the irreducible $\mathfrak{sl}_2$-module with dimension $d+1$.

The $\boxtimes$-module $V_d(a)$ is nontrivial and irreducible.

We call a the **evaluation parameter** for $V_d(a)$.

Characterizing the evaluation modules, I

Theorem For a nontrivial f.d. irreducible ⊠-module V TFAE:

(i) V is isomorphic to an evaluation module for ⊠.

(ii) V has shape $(1, 1, \ldots, 1)$.

Characterizing the evaluation modules, II

Theorem Let V denote a nontrivial f.d. irreducible ⊠-module.

Then for $a \in \mathbb{F}\backslash\{0,1\}$ TFAE:

(i) V is isomorphic to an evaluation module with evaluation parameter a.

(ii) Each of the following vanishes on V:

$$ax_{01} + (1-a)x_{02} - x_{03},$$
$$ax_{10} + (1-a)x_{13} - x_{12},$$
$$ax_{23} + (1-a)x_{20} - x_{21},$$
$$ax_{32} + (1-a)x_{31} - x_{30}.$$

An S_4-action on ⊠

We identify the symmetric group S_4 with the group of permutations of $\mathbb{I}$.

S_4 acts on the set of generators for ⊠ by permuting the indices:

$$\sigma(x_{ij}) = x_{\sigma(i),\sigma(j)} \qquad \sigma \in S_4.$$

This action leaves invariant the defining relations and therefore induces an action of S_4 on ⊠ as a group of automorphisms.

An S_4-action on ⊠-modules

For a ⊠-module V and $\sigma \in S_4$ there exists a ⊠-module structure on V, called V **twisted via** σ, that behaves as follows:

For $u \in$ ⊠ and $v \in V$, the vector $u.v$ computed in V twisted via σ coincides with the vector $\sigma^{-1}(u).v$ computed in the original ⊠-module V.

Sometimes we abbreviate ${}^{\sigma}V$ for V twisted via σ.

S_4 acts on the set of ⊠-modules, with σ sending V to ${}^{\sigma}V$ for all $\sigma \in S_4$ and all ⊠-modules V.

The S_4-action on ⊠-modules, cont.

The above S_4-action on ⊠-modules sends evaluation modules to evaluation modules.

The effect of this action on the evaluation parameter is described in the following two slides.

An action of S_4 on $\mathbb{F}\backslash\{0,1\}$

Lemma There exists an action of S_4 on the set $\mathbb{F}\backslash\{0,1\}$ that does the following.

For $a \in \mathbb{F}\backslash\{0,1\}$,

- $(2,0)$ sends $a \mapsto a^{-1}$;
- $(0,1)$ sends $a \mapsto a(a-1)^{-1}$;
- $(1,3)$ sends $a \mapsto a^{-1}$.

The effect of S_4 on the evaluation parameter

Theorem For an integer $d \geq 1$, $\sigma \in S_4$, and $a \in \mathbb{F}\backslash\{0,1\}$ the following are isomorphic:

(i) The ⊠-module $V_d(a)$ twisted via σ;

(ii) The ⊠-module $V_d(\sigma(a))$.

A subgroup G of S_4

Earlier we gave an action of S_4 on the set $\mathbb{F}\backslash\{0,1\}$.

Let G denote the kernel of this action.

It turns out that G consists of

$$(01)(23), \qquad (02)(13), \qquad (03)(12)$$

together with the identity element.

The subgroup G of S_4, cont.

Corollary For an integer $d \geq 1$, for $\sigma \in G$, and for $a \in \mathbb{F}\backslash\{0,1\}$ the following are isomorphic:

(i) The ⊠-module $V_d(a)$ twisted via σ;

(ii) The ⊠-module $V_d(a)$.

We will return to the subgroup G later in the talk.

The orbits of S_4 on $\mathbb{F}\backslash\{0,1\}$

We now describe the orbits for the S_4-action on $\mathbb{F}\backslash\{0,1\}$.

Pick $a \in \mathbb{F}\backslash\{0,1\}$ and mutually distinct $i,j,k,\ell \in \mathbb{I}$.

By the (i,j,k,ℓ)-relative of a we mean the scalar $\sigma(a)$ where $\sigma \in S_4$ sends the sequence (i,j,k,ℓ) to $(2,0,1,3)$.

The orbits of S_4 on $\mathbb{F}\backslash\{0,1\}$, cont.

The relative function satisfies this recursion:

Lemma Pick $a \in \mathbb{F}\backslash\{0,1\}$ and mutually distinct $i,j,k,\ell \in \mathbb{I}$.

Let α denote the (i,j,k,ℓ)-relative of a. Then

- α^{-1} is the (j,i,k,ℓ)-relative of a;
- $\alpha(\alpha-1)^{-1}$ is the (i,k,j,ℓ)-relative of a;
- α^{-1} is the (i,j,ℓ,k)-relative of a.

The orbits of S_4 on $\mathbb{F}\backslash\{0,1\}$, cont.

Here is another way to view the relative function.

Lemma For $a \in \mathbb{F}\backslash\{0,1\}$ and mutually distinct $i,j,k,\ell \in \mathbb{I}$ the following (i), (ii) coincide:

(i) the (i,j,k,ℓ)-relative of a;

(ii) the scalar

$$\frac{\hat{i}-\hat{\ell}}{\hat{i}-\hat{k}}\frac{\hat{j}-\hat{k}}{\hat{j}-\hat{\ell}}$$

where we define

$$\hat{0}=a, \; \hat{1}=0, \; \hat{2}=1, \; \hat{3}=\infty.$$

The orbits of S_4 on $\mathbb{F}\backslash\{0,1\}$, cont.

Here is an explicit description of the relative function.

Theorem Pick $a \in \mathbb{F}\backslash\{0,1\}$ and mutually distinct $i,j,k,\ell \in \mathbb{I}$.

Then the (i,j,k,ℓ)-relative of a is given in the following table.

(i,j,k,ℓ)				(i,j,k,ℓ)-relative
(2,0,1,3)	(0,2,3,1)	(1,3,2,0)	(3,1,0,2)	a
(0,2,1,3)	(2,0,3,1)	(1,3,0,2)	(3,1,2,0)	a^{-1}
(1,0,2,3)	(0,1,3,2)	(2,3,1,0)	(3,2,0,1)	$1-a$
(0,1,2,3)	(1,0,3,2)	(2,3,0,1)	(3,2,1,0)	$(1-a)^{-1}$
(2,1,0,3)	(1,2,3,0)	(0,3,2,1)	(3,0,1,2)	$a(a-1)^{-1}$
(1,2,0,3)	(2,1,3,0)	(0,3,1,2)	(3,0,2,1)	$1-a^{-1}$

24 bases for $V_d(a)$

For the time being we fix an integer $d \geq 1$ and a scalar $a \in \mathbb{F}\backslash\{0,1\}$.

We consider the $\boxtimes$-module $V_d(a)$.

We are about to define 24 bases for this module.

The vectors η_i $(i \in \mathbb{I})$ in $V_d(a)$

For notational convenience, for $i \in \mathbb{I}$ we fix a nonzero vector $\eta_i \in V_d(a)$ which is a common eigenvector for $\{x_{ij} \mid j \in \mathbb{I}, j \neq i\}$.

Location of η_i $(i \in \mathbb{I})$

The basis $[i,j,k,\ell]$ for $V_d(a)$

Lemma For mutually distinct $i,j,k,\ell \in \mathbb{I}$ there exists a unique basis $\{u_n\}_{n=0}^{d}$ for $V_d(a)$ such that:

(i) for $0 \leq n \leq d$ the vector u_n is contained in component n of the decomposition $[k,\ell]$;

(ii) $\eta_i = \sum_{n=0}^{d} u_n$.

We denote this basis by $[i,j,k,\ell]$.

We have now defined 24 bases for $V_d(a)$.

The basis $[i,j,k,\ell]$ for $V_d(a)$

How the generators x_{rs} act on the 24 bases

Theorem For mutually distinct $i,j,k,\ell \in \mathbb{I}$ and distinct $r,s \in \mathbb{I}$ consider the matrix representing x_{rs} with respect to the basis $[i,j,k,\ell]$ of $V_d(a)$. The entries of this matrix are given in the following table. All entries not displayed are zero.

gen.	$(n,n-1)$-entry	(n,n)-entry	$(n-1,n)$-entry
$x_{\ell k}$	0	$d-2n$	0
$x_{k\ell}$	0	$2n-d$	0
x_{ki}	0	$2n-d$	$2d-2n+2$
x_{ik}	0	$d-2n$	$2n-2d-2$
$x_{i\ell}$	$-2n$	$2n-d$	0
$x_{\ell i}$	$2n$	$d-2n$	0
$x_{\ell j}$	$2\alpha n$	$d-2n$	0
$x_{j\ell}$	$-2\alpha n$	$2n-d$	0
x_{jk}	0	$d-2n$	$2(n-d-1)\alpha^{-1}$
x_{kj}	0	$2n-d$	$2(d-n+1)\alpha^{-1}$
x_{ji}	$2\alpha n(\alpha-1)^{-1}$	$(d-2n)(\alpha+1)(\alpha-1)^{-1}$	$2(d-n+1)(1-\alpha)^{-1}$
x_{ij}	$2\alpha n(1-\alpha)^{-1}$	$(d-2n)(\alpha+1)(1-\alpha)^{-1}$	$2(d-n+1)(\alpha-1)^{-1}$

In the above table the scalar α denotes the (i,j,k,ℓ)-relative of a.

Some transition matrices

We now consider the transition matrices between our 24 bases.

In order to describe these, it is convenient to introduce a certain bilinear form on $V_d(a)$.

A bilinear form on $V_d(a)$

Lemma There exists a nonzero bilinear form $\langle\, , \rangle$ on $V_d(a)$ such that

$$\langle w.u, v\rangle = -\langle u, w.v\rangle \qquad w \in \boxtimes, \quad u,v \in V.$$

The form is nondegenerate.

The form is unique up to multiplication by a nonzero scalar in $\mathbb{F}$.

The form is symmetric (resp. antisymmetric) when d is even (resp. d is odd).

We call $\langle\, , \rangle$ a **standard bilinear form** for $V_d(a)$.

The matrix Z

The following matrix will play a role in our discussion.

For an integer $d \geq 0$ let $Z = Z(d)$ denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$ with entries

$$Z_{ij} = \begin{cases} 1, & \text{if } i+j=d; \\ 0, & \text{if } i+j \neq d \end{cases} \qquad (0 \leq i,j \leq d).$$

We observe

$$Z^2 = I.$$

The transition matrices

Theorem Referring to $V_d(a)$, pick mutually distinct $i,j,k,\ell \in \mathbb{I}$ and consider the transition matrices from the basis $[i,j,k,\ell]$ to the bases

$$[j,i,k,\ell], \qquad [i,k,j,\ell], \qquad [i,j,\ell,k].$$

(i) The first transition matrix is diagonal with (r,r)-entry

$$\frac{\langle \eta_j, \eta_\ell\rangle}{\langle \eta_i, \eta_\ell\rangle}\alpha^r$$

for $0 \leq r \leq d$, where α is the (i,j,k,ℓ)-relative of a.

(ii) The second transition matrix is lower triangular with (r,s)-entry

$$\binom{r}{s}\alpha^{r-s}(1-\alpha)^s$$

for $0 \leq s \leq r \leq d$, where α is the (i,j,k,ℓ)-relative of a.

(iii) The third transition matrix is the matrix Z.

Realizing the evaluation modules for $\boxtimes$ using polynomials in two variables

Let z_0, z_1 denote commuting indeterminates.

let $\mathbb{F}[z_0,z_1]$ denote the $\mathbb{F}$-algebra of all polynomials in z_0, z_1 that have coefficients in $\mathbb{F}$.

We abbreviate $\mathcal{A} = \mathbb{F}[z_0,z_1]$.

We often view $\mathcal{A}$ as a vector space over $\mathbb{F}$.

For an integer $d \geq 0$ let $\mathcal{A}_d$ denote the subspace of $\mathcal{A}$ consisting of the homogeneous polynomials in z_0, z_1 that have total degree d.

Thus $\{z_0^{d-n}z_1^n\}_{n=0}^d$ is a basis for $\mathcal{A}_d$.

Realizing the evaluation modules

Note that

$$\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_d \qquad \text{(direct sum)}$$

and that

$$\mathcal{A}_r \mathcal{A}_s = \mathcal{A}_{r+s} \qquad (r, s \geq 0).$$

We fix mutually distinct $\beta_i \in \mathbb{F}$ $(i \in \mathbb{I})$.

Then there exist unique $z_2, z_3 \in \mathcal{A}$ such that

$$\sum_{i \in \mathbb{I}} z_i = 0, \qquad \sum_{i \in \mathbb{I}} \beta_i z_i = 0.$$

Comments on the z_i $(i \in \mathbb{I})$

Lemma For mutually distinct $i, j, k, \ell \in \mathbb{I}$ we have

$$z_k = \frac{\beta_\ell - \beta_i}{\beta_k - \beta_\ell} z_i + \frac{\beta_\ell - \beta_j}{\beta_k - \beta_\ell} z_j,$$
$$z_\ell = \frac{\beta_i - \beta_k}{\beta_k - \beta_\ell} z_i + \frac{\beta_j - \beta_k}{\beta_k - \beta_\ell} z_j.$$

Some bases for $\mathcal{A}$

Lemma For distinct $i, j \in \mathbb{I}$ the elements

$$z_i^r z_j^s \qquad 0 \leq r, s < \infty$$

form a basis for $\mathcal{A}$.

Some bases for $\mathcal{A}_d$

Lemma For an integer $d \geq 0$ and distinct $i, j \in \mathbb{I}$ the elements $\{z_i^{d-n} z_j^n\}_{n=0}^d$ form a basis for $\mathcal{A}_d$.

Example: Some bases for $\mathcal{A}_3$

Derivations of $\mathcal{A}$

Our next goal is to display a $\boxtimes$-module structure on $\mathcal{A}$.

We will use the following terms.

By a **derivation** of $\mathcal{A}$ we mean an $\mathbb{F}$-linear map $D : \mathcal{A} \to \mathcal{A}$ such that

$$D(uv) = D(u)v + uD(v) \qquad (u, v \in \mathcal{A}).$$

$\mathcal{A}$ is a ⊠-module

Theorem There exists a unique ⊠-module structure on $\mathcal{A}$ such that:

(i) each element of ⊠ acts as a derivation on $\mathcal{A}$;

(ii) $x_{ij}.z_i = -z_i$ and $x_{ij}.z_j = z_j$ for distinct $i, j \in \mathbb{I}$.

The eigenvectors for the x_{ij} on $\mathcal{A}$

Lemma for distinct $i, j \in \mathbb{I}$ and integers $r, s \geq 0$ the element $z_i^r z_j^s$ is an eigenvector for x_{ij} with eigenvalue $s - r$.

The irreducible ⊠-submodules of $\mathcal{A}$

Proposition Referring to the ⊠-module $\mathcal{A}$,

(i) For $d \geq 0$ the subspace $\mathcal{A}_d$ is an irreducible ⊠-submodule of $\mathcal{A}$.

(ii) The ⊠-module $\mathcal{A}_0$ is trivial.

(iii) For $d \geq 1$ the ⊠-module $\mathcal{A}_d$ is isomorphic to $V_d(a)$ where

$$a = \frac{\beta_0 - \beta_1}{\beta_0 - \beta_3} \frac{\beta_2 - \beta_3}{\beta_2 - \beta_1}.$$

The decomposition $[i, j]$ for $\mathcal{A}_d$

Earlier in the talk we described the ⊠-module $V_d(a)$.

We now consider how things look from the point of view of $\mathcal{A}_d$.

Proposition For an integer $d \geq 0$ and for distinct $i, j \in \mathbb{I}$ the decomposition $[i, j]$ on $\mathcal{A}_d$ is described as follows.

For $0 \leq n \leq d$ the nth component is spanned by $z_i^{d-n} z_j^n$.

The elements η_i $(i \in \mathbb{I})$ for $\mathcal{A}_d$

For an integer $d \geq 1$ and $i \in \mathbb{I}$ the element z_i^d is a scalar multiple of η_i.

Recall η_i is defined up to scalar multiplication.

For the rest of talk we choose $\eta_i = z_i^d$.

The basis $[i, j, k, \ell]$ for $\mathcal{A}_d$

Proposition For an integer $d \geq 1$ and for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the basis $[i, j, k, \ell]$ of $\mathcal{A}_d$ is described as follows.

For $0 \leq n \leq d$ the nth component is

$$z_k^{d-n} z_\ell^n \binom{d}{n} \frac{(\beta_j - \beta_k)^{d-n} (\beta_j - \beta_\ell)^n}{(\beta_i - \beta_j)^d}.$$

The group G revisited

We saw earlier that if we twist the $\boxtimes$-module $V_d(a)$ via an element of G then the result is isomorphic to $V_d(a)$.

We now explain this fact using $\mathcal{A}$.

Some automorphisms of $\mathcal{A}$

Lemma For mutually distinct $i, j, k, \ell \in \mathbb{I}$ there exists a unique automorphism of $\mathcal{A}$ that sends

$$z_i \mapsto \frac{\beta_j - \beta_k}{\beta_i - \beta_k} z_j, \qquad z_j \mapsto \frac{\beta_i - \beta_\ell}{\beta_j - \beta_\ell} z_i,$$

$$z_k \mapsto \frac{\beta_\ell - \beta_i}{\beta_i - \beta_k} z_\ell, \qquad z_\ell \mapsto \frac{\beta_k - \beta_j}{\beta_j - \beta_\ell} z_k.$$

Some automorphisms of $\mathcal{A}$

Theorem The following hold for $\sigma \in G$:

(i) There exists an automorphism g_σ of $\mathcal{A}$ that sends z_r to a scalar multiple of $z_{\sigma(r)}$ for all $r \in \mathbb{I}$.

(ii) For $u \in \boxtimes$ the equation

$$\sigma(u) = g_\sigma u g_\sigma^{-1}$$

holds on $\mathcal{A}$.

(iii) The map g_σ is an isomorphism of $\boxtimes$-modules from $\mathcal{A}$ to $\mathcal{A}$ twisted via σ.

THE END