THE INEQUALITY $\mathfrak{b} > \aleph_1$ CAN BE CONSIDERED AS AN ANALOGUE OF SUSLIN’S HYPOTHESIS (Axiomatic Set Theory and Set-theoretic Topology)

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THE INEQUALITY $b > \aleph_1$ CAN BE CONSIDERED AS AN ANALOGUE OF SUSLIN'S HYPOTHESIS

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ABSTRACT. In [3], the author introduced a chain condition, called the anti-rectangle refining property, of forcing notions and the statement $\neg C(arec)$ that We show that every forcing notion with the anti-rectangle refining property has an uncountable antichain. Since a typical example of a forcing notion with the anti-rectangle refining property is an Aronszajn tree, $\neg C(arec)$ is a generalization of Suslin's Hypothesis. We show that $\neg C(arec)$ implies that the bounding number is larger than $\aleph_1$, that is, this statement can be considered as an analogue of Suslin's Hypothesis.

1. INTRODUCTION

The author investigated several fragments of Martin's Axiom in [3]. Fragments of Martin's Axiom were studied mainly by Stevo Todorcević in 1980's, and many applications are discovered (see [2] and his many other articles). In this manuscript, we give a proof of one question in this area as follows.

We explain some notions in [3]. A forcing notion $P$ has the anti-rectangle refining property if for any uncountable subset $I$ and $J$ of $P$, there exists uncountable subsets $I'$ and $J'$ of $I$ and $J$ respectively such that for every $p \in I'$ and $q \in J'$, $p$ and $q$ are incompatible in $P$. $\neg C(arec)$ is the statement that every forcing notion with the anti-rectangle refining property has an uncountable antichain. Since an Aronszajn tree has the anti-rectangle refining property, $\neg C(arec)$ can be considered a generalization of Suslin's Hypothesis. In fact, $\neg C(arec)$ implies Suslin's Hypothesis and that every $(\omega_1, \omega_1)$-gaps are indestructible. The author would like to find other examples of a generalization of Suslin's Hypothesis, that is, other statements about combinatorics on $\omega_1$ which is deduced from $\neg C(arec)$. One candidate is the statement that the bounding number $b$ is larger than $\aleph_1$.

We had already known that $K_2(\mathfrak{b})$, which is a weak fragments of Martin's Axiom and implies $\neg C(arec)$, implies that $b > \aleph_1$. So it is naturally arisen a question that $\neg C(arec)$ implies $b > \aleph_1$. In this manuscript, we show a positive answer of this question, that is $\neg C(arec)$ implies that $b > \aleph_1$ in section 3.

A proof of the theorem is self contained in this manuscript, however I omit some proofs of well known results in section 2. All of them are written in [3] or [1].

2. A REASON WHY WE WILL PROVE AS BELOW

At first, we will see a proof that $K_2(\mathfrak{b})$ implies $b > \aleph_1$. A partition $[\omega_1]^2 = \mathcal{K}_0 \cup \mathcal{K}_1$ has the rectangle refining property if for any uncountable subset $I$ and

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J of $\omega_1$, there exist uncountable subsets $I'$ and $J'$ of $I$ and $J$ respectively such that for every $\alpha \in I'$ and $\beta \in J'$, if $\alpha < \beta$, then $\{\alpha, \beta\} \in K_0$. We note that the rectangle refining property is a strong property than the countable chain condition. $\mathcal{K}_2$(rec) is the statement that every partition $[\omega_1]^2 = K_0 \cup K_1$ with the rectangle refining property has an uncountable $K_0$-homogeneous set. We note that $\mathcal{K}_2$(rec) is deduced from Martin’s Axiom for $\aleph_1$-dense sets, and $\mathcal{K}_2$(rec) implies $\neg\mathcal{C}$(arec).

Let $F = \{f_\xi; \xi \in \omega_1\}$ be a set of strictly increasing functions from $\omega$ into $\omega$ such that for every $\xi$ and $\eta$ in $\omega_1$, if $\xi < \eta$, then $f_\xi \leq^* f_\eta$, i.e., there exists $m \in \omega$ such that for all $n \geq m$, $f_\xi(n) \leq f_\eta(n)$. For this family, we define a partition $[\omega_1]^2 = K_0 \cup K_1$ by letting $\{\xi, \eta\} \in K_0$ iff there exists $m$ and $n$ in $\omega$ such that $f_\xi(m) < f_\eta(m)$ and $f_\eta(n) < f_\xi(n)$. We call that $F$ is unbounded when for every function $g$ in $\omega^\omega$, there exists $f \in F$ such that $f \not\leq^* g$. We note that if $F$ is unbounded, then this partition has the rectangle refining property. (This follows from Lemma 3.2 below.) However, in [1, Lemma 16], if $F$ is unbounded, since an uncountable subset of $F$ is also unbounded, for every uncountable subset $F'$ of $F$, there are two functions $f$ and $g$ in $F$ such that $g$ dominates $f$ everywhere, i.e., for every $n \in \omega$, $f(n) \leq g(n)$. Therefore, $\mathcal{K}_2$(rec) implies $b > \aleph_1$.

So to try to prove that $\neg\mathcal{C}$(arec) implies $b > \aleph_1$, it seems to be natural to modify the argument above. Let $\mathbb{P}'$ be a forcing notion which consists of finite subsets $\sigma$ of $\omega_1$ such that the set $\{f_\xi; \xi \in \sigma\}$ is totally ordered by the dominance everywhere, i.e., for every $\xi \in \sigma$ and $n \in \omega$, max $\{f_\xi(n); \xi \in \sigma \cap \xi\} \leq f_\xi(n)$, ordered by the reverse inclusion. As the above partition has the rectangle refining property, we note that $\mathbb{P}'$ has the anti-rectangle refining property if $F$ is unbounded. So if we show that $\mathbb{P}'$ is ccc whenever $F$ is unbounded, we conclude that $F$ doesn’t have to be unbounded. However, unfortunately, in general, $\mathbb{P}'$ does not have the ccc even if $F$ is unbounded. For example, if the set $\{\{\xi_\zeta, \eta_\zeta\}; \zeta \in \omega_1\}$ is a subset of $\mathbb{P}'$ such that

- for any $\zeta < \zeta'$ in $\omega_1$, $\xi_\zeta < \eta_\zeta < \xi_{\zeta'}$, and
- for any $\zeta \in \omega_1$, $f_{\xi_\zeta}(0) = 0$ and $f_{\eta_\zeta}(1) = 1$,

then it is an uncountable antichain in $\mathbb{P}'$.

In section 3, we define a forcing notion $\mathbb{P}$ which is a modification of $\mathbb{P}'$ and show that (Lemma 3.2) $\mathbb{P}$ has the anti-rectangle refining property whenever $F$ is unbounded, and (Lemma 3.3) $\mathbb{P}$ has the countable chain condition whenever $F$ is unbounded. This completes the proof of our theorem.

3. A PROOF

Throughout this section, let $F = \{f_\xi; \xi \in \omega_1\}$ be a set of strictly increasing functions from $\omega$ into $\omega$ such that for every $\xi$ and $\eta$ in $\omega_1$, if $\xi < \eta$, then $f_\xi \leq^* f_\eta$. We define a forcing notion $\mathbb{P}$ which consists of finite subsets $\sigma$ of $\omega_1$ such that for every $\xi \in \sigma$ and $n \in \omega$, either max $\{f_\xi(n); \xi \in \sigma \cap \xi\} < f_\xi(n)$ or $f_\xi(n) \in \{f_\xi(n); \xi \in \sigma \cap \xi\}$, ordered by the reverse inclusion.

**Proposition 3.1.** Suppose that $F = \{f_\xi; \xi \in \omega_1\}$ is unbounded. Then there exists $e \in \omega$ such that for every $n \in \omega \setminus e$ and $k \in \omega$, the set $\{\xi \in \omega_1; f_\xi(n) \geq k\}$ is uncountable.
$\neg C(\text{arec}) \Rightarrow b > \aleph_1$

Proof. Assume not, i.e. there exists an infinite set $Z$ of natural numbers such that for every $n \in Z$, there exists $k_n \in \omega$ such that the set $\{\xi \in \omega_1; f^*_\xi(n) \geq k_n\}$ is countable. Let $\delta \in \omega_1$ be such that for all $n \in Z$, $\{\xi \in \omega_1; f^*_\xi(n) \geq k_n\}$ is a subset of $\delta$. Let $\{n_i; i \in \omega\}$ be an increasing enumeration of $Z$, and we define a function $g$ on $\omega$ by

$$g(m) := \max \{\{f^*_\delta(m)\} \cup \{k_{n_i}; i \in m+1\} \cup \{g(i)+1; i \in m\}\}$$

for each $m \in \omega$. We notice that for each $\xi \in \delta$, $f^*_\xi \leq^* g$. Moreover for each $\xi \in \omega_1 \setminus \delta$ and $m \in \omega$, since $m \leq n_m$,

$$f^*_\xi(m) \leq f^*_\xi(n_m) \leq k_{n_m} \leq g(m).$$

So $F$ is bounded by $g$, which is a contradiction. \hfill \square

Lemma 3.2. If $F = \{f^*_\xi; \xi \in \omega_1\}$ is unbounded, then $\mathbb{P}$ has the anti-rectangle refining property.

Proof. Let $I$ and $J$ be uncountable subsets of $\mathbb{P}$. By shrinking $I$ and $J$ if necessary, we may assume that

- $I$ forms a $\Delta$-system with a root $\mu$, and $J$ also forms a $\Delta$-system with a root $\nu$,
- all members of $I$ has the same size, and all members of $J$ also has the same size,
- for any $\sigma \in I$ and $\tau \in J$,

$$\max(\mu \cup \nu) < \min(\sigma \setminus \mu), \quad \max(\mu \cup \nu) < \min(\tau \setminus \nu), \quad (\sigma \setminus \mu) \cap (\tau \setminus \nu) = \emptyset,$$

- there exists $e \in \omega$, such that for every $\sigma \in I$ and $\tau \in J$ and $n \geq e$,

$$\max(\{f^*_\xi(n); \xi \in \mu \cup \nu\}) < \min(\{f^*_\xi(n); \xi \in \sigma \setminus \mu\})$$

and

$$\max(\{f^*_\xi(n); \xi \in \mu \cup \nu\}) < \min(\{f^*_n(n); \eta \in \tau \setminus \nu\}).$$

We notice that for every $A \in [\omega_1]^{\aleph_1}$, the set $\{f^*_\xi; \xi \in A\}$ is unbounded. So by the previous lemma, there exists $e_0 \geq e$ such that for every $k \in \omega$, the set

$$\{\sigma \in I; \min(\{f^*_\xi(e_0); \xi \in \sigma \setminus \mu\}) \geq k\}$$

is uncountable. Let $J'$ be uncountable subset of $J$ and $k_0 \in \omega$ such that for every $\tau \in J'$,

$$\max(\{f^*_\eta(e_0); \eta \in \tau\}) \leq k_0,$$

and then we take an uncountable subset $I'$ of $I$ such that for every $\sigma \in I'$,

$$\min(\{f^*_\xi(e_0); \xi \in \sigma \setminus \mu\}) > k_0.$$

Then we notice that for any $\sigma \in I'$ and $\tau \in J'$, since $e_0 \geq e$, if $\tau \not\subseteq \max(\sigma) + 1$, then $\sigma$ and $\tau$ are incompatible in $\mathbb{P}$.

Conversely, by the previous lemma, there exists $e_1 > e_0$ such that for every $k \in \omega$, the set

$$\{\tau \in J'; \min(\{f^*_\eta(e_1); \eta \in \tau \setminus \nu\}) \geq k\}$$

is uncountable. Let $I''$ be uncountable subset of $I'$ and $k_1 \in \omega$ such that for every $\sigma \in I''$,

$$\max(\{f^*_\xi(e_1); \xi \in \sigma\}) \leq k_1,$$
and then we take an uncountable subset $J''$ of $J'$ such that for every $\tau \in J''$,

$$\min \left(\{f_{\eta}(e_{1}); \eta \in \tau \setminus \nu\}\right) > k_{1}.$$  

Then we notice that, since $e_{1} \geq e$, for any $\sigma \in I''$ and $\tau \in J''$, if $\sigma \not\subseteq \max(\tau) + 1$, then $\sigma$ and $\tau$ are incompatible in $P$.

By shrinking $I''$ and $J''$ if necessary, we may assume that for any $\sigma \in I''$ and $\tau \in J''$, either $\tau \not\subseteq \max(\sigma) + 1$ or $\sigma \not\subseteq \max(\tau) + 1$. Then for every $\sigma \in I''$ and $\tau \in J''$, $\sigma$ and $\tau$ are incompatible in $P$. \hfill \Box

**Lemma 3.3.** If $F = \{f_{\xi}; \xi \in \omega_{1}\}$ is unbounded, then $P$ has the countable chain condition.

**Proof.** Here, for each $\sigma \in P$, letting $\langle \xi_{i}; i \in |\sigma|\rangle$ be an increasing enumeration of $\sigma$, we denote

$$\bar{\sigma} := \langle f_{\xi_{i}}; i \in |\sigma|\rangle,$$

which is a member of the set $(\omega^{\omega})^{|\sigma|}$. Let $I$ be an uncountable subset of $P$. Without loss of generality, we may assume that

- $I$ forms a $\Delta$-system with a root $\mu$,
- for every $\sigma$ and $\tau$ in $I$, either $\max(\sigma) < \min(\tau \setminus \mu)$ or $\max(\tau) < \min(\sigma \setminus \mu)$,
- there exist $n_{0} \in \omega$ such that for every $n \geq n_{0}$, $\sigma \in I$ and $\xi \in \sigma \setminus \mu$,

$$\max \{f_{\xi}(n); \xi \in \mu\} < f_{\xi}(n),$$

- there exist $k \in \omega$ such that for every $\sigma \in I$, $|\sigma| = k$,
- for every $\sigma$ and $\tau$ in $I$, $\bar{\sigma} \upharpoonright n_{0} = \bar{\tau} \upharpoonright n_{0}$, i.e. for each $j \in k$, the initial segment of the $j$-th element of $\bar{\sigma}$ of length $n_{0}$ is equal to the initial segment of the $j$-th element of $\bar{\tau}$ of length $n_{0}$.

Then there exists $\gamma \in \omega_{1}$ such that the set $\{\bar{\sigma}; \sigma \in I \cap [\gamma]^{<\omega_{0}}\}$ is dense in the set $\{\bar{\sigma}; \sigma \in I\}$ as a subspace of the space $(\omega^{\omega})^{k}$. We fix some (any) $\nu \in I \setminus [\gamma]^{<\omega_{0}}$. For each $\sigma \in I$, we define two functions $g_{\sigma}$ and $h_{\sigma}$ on $\omega$ as follows: For each $n \in \omega$,

$$g_{\sigma}(n) := \max \{f_{\xi}(n); \xi \in \sigma\} \quad (= \max \{f_{\xi}(n); \xi \in \sigma \setminus \mu\}),$$

and

$$h_{\sigma}(n) := \min \{f_{\xi}(n); \xi \in \sigma \setminus \mu\}.$$  

We notice that for $\sigma$ and $\tau$ in $I$, if $\max(\sigma) < \min(\tau \setminus \mu)$, then $g_{\sigma} \leq^{*} h_{\tau}$. So we can find $n_{1} \geq n_{0}$ and $I' \in [I \setminus [\gamma]^{<\omega_{0}}]^{n_{1}}$ such that for every $\tau \in I'$ and $n \geq n_{1}$, $g_{\sigma}(n) \leq h_{\tau}(n)$, and for every $\tau$ and $\tau'$ in $I'$, $\bar{\tau} \upharpoonright n_{1} = \bar{\tau}' \upharpoontright n_{1}$. Since $F$ is unbounded and $I'$ is uncountable, the set $\{h_{\tau}; \tau \in I'\}$ is unbounded. Hence there exists $n \geq n_{1}$ such that the set $\{h_{\tau}(n); \tau \in I'\}$ is infinite. Let

$$n_{2} := \min \{n \in [n_{1}, \omega); \{h_{\tau}(n); \tau \in I'\} \text{ is infinite}\}.$$  

By the minimality of $n_{2}$, we can take $\bar{\tau} \in (\omega^{\omega})^{k}$ and infinite $I'' \subseteq I'$ such that

- for all $\tau \in I''$, $\bar{\tau} \subseteq \bar{\tau}'$, i.e. for every $j \in k$, the $j$-th member of $\bar{\tau}$ is an initial segment of the $j$-th member of $\bar{\tau}'$,
- the set $\{h_{\tau}(n); \tau \in I''\}$ is infinite.
By our assumption, there exists $\sigma \in I \cap [\gamma]^\omega_0$ such that $\vec{t} \subseteq \vec{\sigma}$. Then there is $n_3 \geq n_2$ such that for every $n \geq n_3$, $g_\sigma(n) \leq g_\nu(n)$, and take $\tau \in I''$ such that $g_\nu(n_3) < h_\tau(n_2)$.

We will show that for every $n \geq n_2$, $g_\sigma(n) \leq h_\tau(n)$ holds. If $n_2 \leq n < n_3$, then $g_\sigma(n) < g_\sigma(n_3) \leq g_\nu(n_3) < h_\tau(n_2) \leq h_\tau(n)$, so it is ok. If $n \geq n_3$, then since $n \geq n_3 \geq n_1$ and $\tau \in I'' \subseteq I'$,

$$g_\sigma(n) \leq g_\nu(n) \leq h_\tau(n).$$

We recall that $\vec{t} \in (\omega^{n_2})^k$ is an initial segment of both $\vec{\sigma}$ and $\vec{\tau}$, for every $n \geq n_2$, $g_\sigma(n) \leq h_\tau(n)$, and both $\sigma$ and $\tau$ are members of $\mathbb{P}$. Therefore $\sigma \cup \tau$ is also a condition of $\mathbb{P}$, i.e. $\sigma$ and $\tau$ are compatible in $\mathbb{P}$. ☐

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