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Ultrafilters and Higson compactifications

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Abstract

We prove the following theorem: If there is a base $\mathcal{F}$ of a non-rapid ultrafilter on $\omega$, then we can approximate $\beta\omega$ by $|\mathcal{F}|$-many Higson compactifications of $\omega$ in a nontrivial way. It is still open whether we can eliminate the assumption that $\mathcal{F}$ is non-rapid.

MSC: Primary 03E17; Secondary 03E35, 54D35

1 Introduction

In this paper we give a partial answer to a question which was posed by Kada, Tomoyasu and Yoshinobu [3].

We refer the reader to the book [1] for undefined set-theoretic notions. For $X, Y \subseteq \omega^\omega$, we write $X \subseteq^* Y$ (or $Y \supseteq^* X$) if $X \setminus Y$ is finite. The symbol $\omega^\uparrow\omega$ denotes the set of all strictly increasing functions in $\omega^\omega$. For $f, g \in \omega^\omega$, we write $f \leq^* g$ if $f(n) \leq g(n)$ holds for all but finitely many $n \in \omega$. A dominating family is a cofinal subset of $\omega^\omega$ with respect to $\leq^*$. The dominating number $\mathfrak{d}$ is the smallest cardinality of a dominating family.

For compactifications $\alpha X$ and $\gamma X$ of a completely regular Hausdorff space $X$, we write $\alpha X \leq \gamma X$ if there is a continuous surjection $\varphi$ from $\gamma X$ onto $\alpha X$ such that $\varphi \upharpoonright X$ is the identity function on $X$, and $\alpha X \simeq \gamma X$ if $\alpha X \leq \gamma X \leq \alpha X$. The Stone–Čech compactification $\beta X$ of $X$ is the maximal compactification of $X$ in the sense of the order relation $\leq$ among compactifications of $X$ modulo the equivalence relation $\simeq$.

We introduce the following notation: For compactification $\alpha X$ of $X$ and disjoint closed subsets $A, B$ of $X$, we write $A \parallel B \ (\alpha X)$ if $\overline{cl}_{\alpha X} A \cap \overline{cl}_{\alpha X} B = \emptyset$, and otherwise we write $A \parallel B \ (\alpha X)$. It is not so hard to show that $A \parallel B \ (\alpha X)$ if and only if there is a bounded continuous function $f$ from $\alpha X$ to $\mathbb{R}$ such that $f'' A = \{0\}$ and $f'' B = \{1\}$. Note that $\alpha X \leq \gamma X$ is equivalent to the assertion that, for disjoint closed subsets $A, B$ of $X$, $A \parallel B \ (\alpha X)$ implies $A \parallel B \ (\gamma X)$. For a normal space $X$, $A \parallel B \ (\beta X)$ holds for any pair $A, B$ of disjoint closed subsets of $X$. 

We say a metric $d$ on a space $X$ is proper if each $d$-bounded subset of $X$ has a compact closure. We say a metric space is proper if its metric is proper. For a proper metric space $(X, d)$ and disjoint closed subsets $A, B$ of $X$, we say $A$ and $B$ diverge with respect to the metric $d$, or $A$ and $B$ $d$-diverge in short, if for every $R > 0$ there is a compact subset $K$ of $X$ such that $d(A \setminus K, B \setminus K) > R$ holds.

The Higson compactification $\overline{X}^d$ of $(X, d)$ is uniquely characterized (up to $\simeq$-equivalence) by the property that $A \| B (\overline{X}^d)$ if and only if $A$ and $B$ $d$-diverge. Note that Higson compactifications are metric-dependent.

In the paper [3] the authors introduced the following cardinal characteristics to investigate approximability of $\beta \omega$ by sets of Higson compactifications of $\omega$. For a metrizable space $X$, let $\text{PM}^\prime(\omega)$ be the set of proper metrics $d$ on $X$ such that $d$ is compatible with the topology on $X$ and $\overline{d} \neq \beta \omega$ holds. For $d_1, d_2 \in \text{PM}^\prime(\omega)$, we write $d_1 \subseteq d_2$ if $\overline{d_1} \leq \overline{d_2}$ holds.

**Definition 1.1.** $\mathfrak{h}p'$ is the smallest cardinality of a subset $D$ of $\text{PM}^\prime(\omega)$ such that $D$ is directed with respect to the order relation $\subseteq$ and $\sup\{\overline{d} : d \in D\} \simeq \beta \omega$, where the supremum is in the sense of the order relation $\leq$ among compactifications of $\omega$.

Throughout the present paper, an ultrafilter means a nonprincipal ultrafilter on $\omega$. The cardinal $u$ is the smallest cardinality of a subset of $[\omega]^\omega$ which generates an ultrafilter.

In the paper [3] the authors asked the following question.

**Question 1.2.** $\mathfrak{h}p' \leq u$?

This question is still open.

In Section 2 we prove that, if a subset $\mathcal{F}$ of $[\omega]^\omega$ generates a non-rapid ultrafilter, then $\mathfrak{h}p' \leq |\mathcal{F}|$ holds. We say a filter $\mathcal{F}$ on $\omega$ is rapid if for all $h \in \omega^\omega$ there is a set $X \in \mathcal{F}$ such that for all $n < \omega$ we have $|X \cap h(n)| \leq n$, or equivalently, if the set of increasing enumerations of sets in $\mathcal{F}$ is a dominating family. When an ultrafilter $\mathcal{U}$ is generated by a subset $\mathcal{F}$ of $[\omega]^\omega$, $\mathcal{U}$ is rapid if and only if the set of increasing enumerations of sets of $\mathcal{F}$ is a dominating family. As a consequence, we see that $u < \mathfrak{d}$ implies $\mathfrak{h}p' \leq u$, since an ultrafilter generated by a set of size less than $\mathfrak{d}$ cannot be rapid.

So the main result in Section 2 gives a partial answer to Question 1.2.

**Remark 1.3.** It is known that non-rapid ultrafilters can be constructed in ZFC, but we do not know if we can find a non-rapid ultrafilter which is generated by a subset of $[\omega]^\omega$ of size $u$ under ZFC. See Section 3 for further discussion.
2 The Main Result

First we prove a simple combinatorial lemma.

Lemma 2.1. Suppose that a subset $F$ of $\omega^\omega$ is not a dominating family. Then there is a function $h \in \omega^\omega$ such that, for all $f \in F$ there are infinitely many $m < \omega$ such that the interval $[h(m), h(m+1))$ contains two consecutive values of $f$.

Proof. Suppose that $F \subseteq \omega^\omega$, $g \in \omega^\omega$ and for all $f \in F$ there are infinitely many $n < \omega$ which satisfy $f(n) < g(n)$. Define $h \in \omega^\omega$ by letting $h(n) = g(2n)$ for each $n$. We show that $h$ satisfies the requirement. Suppose not. Find an $f \in F$ such that, for all but finitely many $m < \omega$, the interval $[h(m), h(m+1))$ contains at most one value of $f$. Then we can find a $k < \omega$ such that for all $n < \omega$ we have $f(n+k) > h(n)$. Since $h(n) = g(2n)$ and $g$ is increasing, for all $n > k$ we have $f(n+k) > h(n) = g(2n) > g(n+k)$. But it is impossible by the choice of $g$. \qed

Now we are going to prove the main theorem.

Theorem 2.2. Suppose that there is a subset $F$ of $[\omega]^{\omega}$ of size $\kappa$ which generates a non-rapid ultrafilter on $\omega$. Then $\hP' \leq \kappa$.

Proof. Let $F$ be a subset of $[\omega]^{\omega}$ of size $\kappa$ which generates a non-rapid ultrafilter. Then the set of increasing enumerations of sets in $F$ is not a dominating family. By the previous lemma, find a function $h \in \omega^\omega$ such that, for every $X \in F$, for infinitely many $m < \omega$ we have $|X \cap [h(m), h(m+1))| \geq 2$. We may assume that $h(0) = 0$. Define a function $\pi \in \omega^\omega$ by letting $\pi(k) = m$ if $h(m-1) \leq k < h(m)$.

For each $X \in F$, we define a function $\rho_X$ with domain $\omega \times \omega$ in the following way:

$$\rho_X(k,l) = \begin{cases} 0 & \text{if } k = l \\ 1 & \text{if } k, l \in X, k \neq l \text{ and } \pi(k) = \pi(l) \\ \pi(k) + \pi(l) & \text{otherwise.} \end{cases}$$

It is easily checked that $\rho_X$ is a metric on $\omega$ and any $\rho_X$-bounded subset of $\omega$ is finite, and so $\rho_X$ is a proper metric on $\omega$.

By the choice of $h$, For any $X \in F$ there are infinitely many pairs $k, l \in \omega$ for which $\rho_X(k,l) = 1$ holds, and so we can construct a pair $A, B$ of disjoint infinite subsets of $\omega$ so that $A \parallel B$ (\omega^{\rho_X}) holds. This ensures that $\rho_X \in \text{PM}'(\omega)$ for all $X \in F$.

Note that, for $X, Y \in F$, $X \supseteq^* Y$ implies $\rho_X \subseteq \rho_Y$. Since $F$ generates an ultrafilter, $F$ is $\supseteq^*$-directed (even $\supseteq$-directed), and so the set $\{\rho_X : X \in F\}$ is $\subseteq$-directed.

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We can easily see that, for $B \subseteq \omega$, if $X \subseteq^* B$ or $X \subseteq^* \omega \setminus B$, then $B \parallel \omega \setminus B$ ($\mathfrak{p}_x^\mathfrak{p}$). Since $\mathcal{F}$ generates an ultrafilter, for each $B \subseteq \omega$ we can find an $X \in \mathcal{F}$ such that $X \subseteq^* B$ or $X \subseteq^* \omega \setminus B$. This implies that, for any pair $A, B$ of disjoint subsets of $\omega$, there is an $X \in \mathcal{F}$ such that $A \parallel B$ ($\mathfrak{p}_x^\mathfrak{p}$) holds, which means that $\sup \{\mathfrak{p}_x^\mathfrak{p} : X \in \mathcal{F}\} \simeq \beta \omega$. By the definition of $\mathfrak{p}'$, we have $\mathfrak{p}' \leq |\mathcal{F}| = \kappa$. \hfill $\Box$

In the paper [3] the authors also introduced the following variant of the cardinal $\mathfrak{p}'$.

**Definition 2.3.** $\mathfrak{h}_t$ is the smallest cardinality of a subset $D$ of $\text{PM}'(\omega)$ such that $D$ is well-ordered by $\subseteq$ and $\sup \{\overline{w}^d : d \in D\} \simeq \beta \omega$ (if such a set $D$ exists; otherwise we write $\mathfrak{h}_t = \infty$).

An ultrafilter is called a *simple $p_\kappa$-point*, where $\kappa$ is a regular uncountable cardinal, if it is generated by a subset of $[\omega]^{\omega}$ which is well-ordered by $\supseteq^*$ in order type $\kappa$. The following result is obtained as a corollary of the previous theorem.

**Corollary 2.4.** Suppose that there is a subset $\mathcal{F}$ of $[\omega]^{\omega}$ of size $\kappa$ such that $\mathcal{F}$ is well-ordered by $\supseteq^*$ and generates a non-rapid ultrafilter on $\omega$ (so $\mathcal{F}$ generates a simple $p_\kappa$-point). Then $\mathfrak{h}_t \leq \kappa$.

## 3 Consequences of the main result

The cardinal $\mathfrak{p}_\kappa$, which was introduced in [3], is the smallest cardinal $\kappa$ for which a simple $p_\kappa$-point exists (if such a $\kappa$ exists; otherwise we write $\mathfrak{p}_\kappa = \infty$). Here we introduce more cardinal characteristics.

**Definition 3.1.** $\mathfrak{u}(\text{non-rapid})$ is the smallest cardinality of a subset $\mathcal{F}$ of $[\omega]^{\omega}$ which generates a non-rapid ultrafilter.

$\mathfrak{pp}(\text{non-rapid})$ is the smallest cardinality of a subset $\mathcal{F}$ of $[\omega]^{\omega}$ which is well-ordered by $\supseteq^*$ and generates a non-rapid ultrafilter (if such a set $\mathcal{F}$ exists; otherwise we write $\mathfrak{pp}(\text{non-rapid}) = \infty$).

Using the above cardinal characteristics, Theorem 2.2 and Corollary 2.4 are represented as follows.

**Corollary 3.2.** $\mathfrak{p}' \leq \mathfrak{u}(\text{non-rapid})$ and $\mathfrak{h}_t \leq \mathfrak{pp}(\text{non-rapid})$.

It is clear that $\mathfrak{u} \leq \mathfrak{pp}$, $\mathfrak{u} \leq \mathfrak{u}(\text{non-rapid})$ and $\mathfrak{pp} \leq \mathfrak{pp}(\text{non-rapid})$. Also it is easily observed that $\mathfrak{u} < \mathfrak{d}$ implies $\mathfrak{u}(\text{non-rapid}) = \mathfrak{u}$, and $\mathfrak{pp} < \mathfrak{d}$ implies $\mathfrak{pp}(\text{non-rapid}) = \mathfrak{pp}$. So we obtain the following result, which partially answers Question 1.2.

**Corollary 3.3.** If $\mathfrak{u} < \mathfrak{d}$, then $\mathfrak{p}' \leq \mathfrak{u}$. If $\mathfrak{pp} < \mathfrak{d}$, then $\mathfrak{p}' \leq \mathfrak{pp}$.

It is known that CH implies the existence of a simple $p_{\aleph_1}$-point. Since the
Miller forcing preserves p-points [1, Lemma 7.3.48] and the preservation of p-points is preserved under countable support iteration [1, Theorem 6.2.6], a generating set of a simple $p_{\aleph_1}$-point in the ground model still generates an ultrafilter in the forcing model by iterated Miller forcing. On the other hand, $\mathcal{d} = \aleph_2$ holds in the model obtained by a countable support iteration of Miller forcing of length $\omega_2$ over a model for CH. Hence $\mathcal{p}\mathcal{p} < \mathcal{d}$ is consistent with ZFC.

But the following question is still open.

**Question 3.4.** $u$(non-rapid) = $u$? $ \mathcal{p}\mathcal{p}$(non-rapid) = $\mathcal{p}\mathcal{p}$?

In the paper [3], another upper bound for $\mathfrak{h}\mathfrak{p}'$ is given.

**Definition 3.5 ([2, Section 5]).** For a function $h \in \omega^\omega$, $I_h$ is the smallest size of a subset $\Phi$ of $\prod_{n<\omega} [w]^{\leq 2^n}$ such that for every $f \in \prod_{n<\omega} h(n)$ there is a $\varphi \in \Phi$ such that $f(n) \in \varphi(n)$ for all but finitely many $n$. Let $l = \sup \{I_h : h \in \omega^\omega\}$.

**Theorem 3.6 ([3, Theorem 6.11]).** $\mathfrak{h}\mathfrak{p}' \leq l$.

Now we can see that the above inequality is consistently strict.

**Corollary 3.7.** $\mathfrak{h}\mathfrak{p}' < l$ (moreover, $\mathfrak{h}\mathfrak{t} < l$) is consistent with ZFC.

**Proof.** We know that there is a proper forcing notion $\mathbb{P}$ which satisfies the following two properties (see Remark 3.8).

- $\mathbb{P}$ preserves p-points.
- In the forcing model by $\mathbb{P}$, for any function $H \in \omega^\omega \cap V$, there is a function $g \in \prod_{n<\omega} H(n)$ such that, for every function $x \in \prod_{n<\omega} H(n) \cap V$ there are infinitely many $n < \omega$ with $x(n) = g(n)$, where $V$ denotes a ground model.

We consider a forcing model obtained by a countable support iteration of alternation of Miller forcing and the above forcing notion $\mathbb{P}$ of length $\omega_2$ over a model for CH.

Since every iterand preserves p-points and the preservation of p-points is preserved under countable support iteration, a generating set of a simple $p_{\aleph_1}$-point in the ground model still generates an ultrafilter in our forcing model, and so $\mathcal{p}\mathcal{p} = \aleph_1$ holds. On the other hand, it is easily observed that $\mathcal{d} = l = \aleph_2 = \mathfrak{c}$ holds in the same model. By Corollary 3.3, $\aleph_1 = \mathfrak{h}\mathfrak{p}' = \mathfrak{h}\mathfrak{t} < l = \aleph_2$ holds in this model. 

**Remark 3.8.** The book [1] tells us in Subsection 7.4.C that the infinitely equal forcing EE meets the requirements which appear in the proof of Corollary 3.7. But Brendle pointed out (in private communication) that EE does not preserve p-points, and the following "tree-like infinitely equal forcing" TEE is what we actually need.
$p \in \text{TEE}$ if:

1. $p$ is a subtree of $\bigcup_{m<w} \prod_{n<m} 2^n$ without endpoints,
2. there is a $C \in [\omega]^{\omega}$ such that, for $s \in p$, if $|s| = n \in C$ then $\text{succ}_p(s) = 2^n$,

and TEE is ordered by inclusion.

Appendix: Ultrafilter number for non-q-points

After the submission of the first version of this article, Blass pointed out that the proof of the main theorem (Theorem 2.2) works under the assumption that $\mathcal{F}$ generates an ultrafilter which is not a q-point.

An ultrafilter $\mathcal{U}$ is called a q-point if for any finite-to-one function $f$ with domain $\omega$ there is an element $X$ of $\mathcal{U}$ such that $f \upharpoonright X$ is a one-to-one function.

It is easy to see that a q-point is a rapid ultrafilter, so the assumption that $\mathcal{F}$ generates a non-q-point ultrafilter is weaker than that $\mathcal{F}$ generates a non-rapid ultrafilter.

To modify the proof of Theorem 2.2 to fit in the weaker assumption, just take a function $\pi$ from $\omega$ to $\omega \setminus \{0\}$ which witnesses that the ultrafilter generated by $\mathcal{F}$ is not a q-point. Then for any $X \in \mathcal{F}$ there are infinitely many $m \in \omega \setminus \{0\}$ for which $\pi^{-1}(\{m\}) \cap X$ has at least two elements. Define $\rho_X$ for each $X \in \mathcal{F}$ in the same way as the original proof.

Let $u(\text{non-q-point})$ be the smallest size of a subset $\mathcal{F}$ of $[\omega]^{\omega}$ which generates a non-q-point ultrafilter. Clearly we have the inequality $u \leq u(\text{non-q-point}) \leq u(\text{non-rapid})$, and so $u < \mathfrak{d}$ implies $u = u(\text{non-q-point})$.

Now we can refine the first inequality of Corollary 3.2 to the inequality $\mathfrak{h} \mathfrak{p}' \leq u(\text{non-q-point})$. Also, instead of the first equality of Question 3.4, we should ask whether $u(\text{non-q-point}) = u$ is proved under ZFC.

References

