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Ultrafilters and Higson compactifications

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Abstract

We prove the following theorem: If there is a base $\mathcal{F}$ of a non-rapid ultrafilter on ω, then we can approximate $\beta\omega$ by $|\mathcal{F}|$-many Higson compactifications of ω in a nontrivial way. It is still open whether we can eliminate the assumption that $\mathcal{F}$ is non-rapid.

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1 Introduction

In this paper we give a partial answer to a question which was posed by Kada, Tomoyasu and Yoshinobu [3].

We refer the reader to the book [1] for undefined set-theoretic notions. For $X, Y \in [\omega]^{\omega}$, we write $X \subseteq^* Y$ (or $Y \supseteq^* X$) if $X \setminus Y$ is finite. The symbol $\omega^{\uparrow\omega}$ denotes the set of all strictly increasing functions in ω^{ω}. For $f, g \in \omega^{\omega}$, we write $f \leq^* g$ if $f(n) \leq g(n)$ holds for all but finitely many $n \in \omega$. A *dominating family* is a cofinal subset of ω^{ω} with respect to $\leq^*$. The *dominating number* $\mathfrak{d}$ is the smallest cardinality of a dominating family.

For compactifications αX and γX of a completely regular Hausdorff space X, we write $\alpha X \leq \gamma X$ if there is a continuous surjection φ from γX onto αX such that $\varphi \restriction X$ is the identity function on X, and $\alpha X \simeq \gamma X$ if $\alpha X \leq \gamma X \leq \alpha X$. The Stone–Čech compactification βX of X is the maximal compactification of X in the sense of the order relation $\leq$ among compactifications of X modulo the equivalence relation $\simeq$.

We introduce the following notation: For compactification αX of X and disjoint closed subsets A, B of X, we write $A \parallel B\ (\alpha X)$ if $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$, and otherwise we write $A \nparallel B\ (\alpha X)$. It is not so hard to show that $A \parallel B\ (\alpha X)$ if and only if there is a bounded continuous function f from αX to $\mathbb{R}$ such that $f''A = \{0\}$ and $f''B = \{1\}$. Note that $\alpha X \leq \gamma X$ is equivalent to the assertion that, for disjoint closed subsets A, B of X, $A \parallel B\ (\alpha X)$ implies $A \parallel B\ (\gamma X)$. For a normal space X, $A \parallel B\ (\beta X)$ holds for any pair A, B of disjoint closed subsets of X.

We say a metric d on a space X is *proper* if each d-bounded subset of X has a compact closure. We say a metric space is proper if its metric is proper. For a proper metric space (X, d) and disjoint closed subsets A, B of X, we say A and B *diverge with respect to the metric* d, or A and B *d-diverge* in short, if for every $R > 0$ there is a compact subset K of X such that $d(A \setminus K, B \setminus K) > R$ holds.

The *Higson compactification* $\overline{X}^d$ of (X, d) is uniquely characterized (up to $\simeq$-equivalence) by the property that $A \parallel B$ $(\overline{X}^d)$ if and only if A and B d-diverge. Note that Higson compactifications are metric-dependent.

In the paper [3] the authors introduced the following cardinal characteristics to investigate approximability of $\beta\omega$ by sets of Higson compactifications of ω. For a metrizable space X, let $\mathrm{PM}'(\omega)$ be the set of proper metrics d on X such that d is compatible with the topology on X and $\overline{\omega}^d \not\simeq \beta\omega$ holds. For $d_1, d_2 \in \mathrm{PM}'(\omega)$, we write $d_1 \sqsubseteq d_2$ if $\overline{\omega}^{d_1} \leq \overline{\omega}^{d_2}$ holds.

Definition 1.1. $\mathfrak{hp}'$ is the smallest cardinality of a subset D of $\mathrm{PM}'(\omega)$ such that D is directed with respect to the order relation $\sqsubseteq$ and $\sup\{\overline{\omega}^d : d \in D\} \simeq \beta\omega$, where the supremum is in the sense of the order relation $\leq$ among compactifications of ω.

Throughout the present paper, an *ultrafilter* means a nonprincipal ultrafilter on ω. The cardinal $\mathfrak{u}$ is the smallest cardinality of a subset of $[\omega]^\omega$ which generates an ultrafilter.

In the paper [3] the authors asked the following question.

Question 1.2. $\mathfrak{hp}' \leq \mathfrak{u}$*?*

This question is still open.

In Section 2 we prove that, if a subset $\mathcal{F}$ of $[\omega]^\omega$ generates a *non-rapid ultrafilter*, then $\mathfrak{hp}' \leq |\mathcal{F}|$ holds. We say a filter $\mathcal{F}$ on ω is *rapid* if for all $h \in \omega^{\uparrow\omega}$ there is a set $X \in \mathcal{F}$ such that for all $n < \omega$ we have $|X \cap h(n)| \leq n$, or equivalently, if the set of increasing enumerations of sets in $\mathcal{F}$ is a dominating family. When an ultrafilter $\mathcal{U}$ is generated by a subset $\mathcal{F}$ of $[\omega]^\omega$, $\mathcal{U}$ is rapid if and only if the set of increasing enumerations of sets of $\mathcal{F}$ is a dominating family. As a consequence, we see that $\mathfrak{u} < \mathfrak{d}$ implies $\mathfrak{hp}' \leq \mathfrak{u}$, since an ultrafilter generated by a set of size less than $\mathfrak{d}$ cannot be rapid. So the main result in Section 2 gives a partial answer to Question 1.2.

Remark 1.3. It is known that non-rapid ultrafilters can be constructed in ZFC, but we do not know if we can find a non-rapid ultrafilter which is generated by a subset of $[\omega]^\omega$ of size $\mathfrak{u}$ under ZFC. See Section 3 for further discussion.

2 The Main Result

First we prove a simple combinatorial lemma.

Lemma 2.1. *Suppose that a subset $\mathcal{F}$ of $\omega^{\uparrow\omega}$ is not a dominating family. Then there is a function $h \in \omega^{\uparrow\omega}$ such that, for all $f \in \mathcal{F}$ there are infinitely many $m < \omega$ such that the interval $[h(m), h(m+1))$ contains two consecutive values of f.*

Proof. Suppose that $\mathcal{F} \subseteq \omega^{\uparrow\omega}$, $g \in \omega^{\uparrow\omega}$ and for all $f \in \mathcal{F}$ there are infinitely many $n < \omega$ which satisfy $f(n) < g(n)$. Define $h \in \omega^{\uparrow\omega}$ by letting $h(n) = g(2n)$ for each n. We show that h satisfies the requirement. Suppose not. Find an $f \in \mathcal{F}$ such that, for all but finitely many $m < \omega$, the interval $[h(m), h(m+1))$ contains at most one value of f. Then we can find a $k < \omega$ such that for all $n < \omega$ we have $f(n+k) > h(n)$. Since $h(n) = g(2n)$ and g is increasing, for all $n > k$ we have $f(n+k) > h(n) = g(2n) > g(n+k)$. But it is impossible by the choice of g. $\square$

Now we are going to prove the main theorem.

Theorem 2.2. *Suppose that there is a subset $\mathcal{F}$ of $[\omega]^\omega$ of size κ which generates a non-rapid ultrafilter on ω. Then $\mathfrak{hp}' \leq \kappa$.*

Proof. Let $\mathcal{F}$ be a subset of $[\omega]^\omega$ of size κ which generates a non-rapid ultrafilter. Then the set of increasing enumerations of sets in $\mathcal{F}$ is not a dominating family. By the previous lemma, find a function $h \in \omega^{\uparrow\omega}$ such that, for every $X \in \mathcal{F}$, for infinitely many $m < \omega$ we have $|X \cap [h(m), h(m+1))| \geq 2$. We may assume that $h(0) = 0$. Define a function $\pi \in \omega^\omega$ by letting $\pi(k) = m$ if $h(m-1) \leq k < h(m)$.

For each $X \in \mathcal{F}$, we define a function ρ_X with domain $\omega \times \omega$ in the following way:

$$\rho_X(k,l) = \begin{cases} 0 & \text{if } k = l \\ 1 & \text{if } k, l \in X,\ k \neq l \text{ and } \pi(k) = \pi(l) \\ \pi(k) + \pi(l) & \text{otherwise.} \end{cases}$$

It is easily checked that ρ_X is a metric on ω and any ρ_X-bounded subset of ω is finite, and so ρ_X is a proper metric on ω.

By the choice of h, For any $X \in \mathcal{F}$ there are infinitely many pairs $k, l \in \omega$ for which $\rho_X(k,l) = 1$ holds, and so we can construct a pair A, B of disjoint infinite subsets of ω so that $A \not\parallel B\ (\overline{\omega}^{\rho_X})$ holds. This ensures that $\rho_X \in \mathrm{PM}'(\omega)$ for all $X \in \mathcal{F}$.

Note that, for $X, Y \in \mathcal{F}$, $X \supseteq^* Y$ implies $\rho_X \sqsubseteq \rho_Y$. Since $\mathcal{F}$ generates an ultrafilter, $\mathcal{F}$ is $\supseteq^*$-directed (even $\supseteq$-directed), and so the set $\{\rho_X : X \in \mathcal{F}\}$ is $\sqsubseteq$-directed.

We can easily see that, for $B \subseteq \omega$, if $X \subseteq^* B$ or $X \subseteq^* \omega \setminus B$, then $B \parallel \omega \setminus B$ $(\overline{\omega}^{\rho_X})$. Since $\mathcal{F}$ generates an ultrafilter, for each $B \subseteq \omega$ we can find an $X \in \mathcal{F}$ such that $X \subseteq^* B$ or $X \subseteq^* \omega \setminus B$. This implies that, for any pair A, B of disjoint subsets of ω, there is an $X \in \mathcal{F}$ such that $A \parallel B$ $(\overline{\omega}^{\rho_X})$ holds, which means that $\sup\{\overline{\omega}^{\rho_X} : X \in \mathcal{F}\} \simeq \beta\omega$. By the definition of $\mathfrak{hp}'$, we have $\mathfrak{hp}' \leq |\mathcal{F}| = \kappa$. □

In the paper [3] the authors also introduced the following variant of the cardinal $\mathfrak{hp}'$.

Definition 2.3. $\mathfrak{ht}$ is the smallest cardinality of a subset D of $\mathrm{PM}'(\omega)$ such that D is well-ordered by $\sqsubseteq$ and $\sup\{\overline{\omega}^d : d \in D\} \simeq \beta\omega$ (if such a set D exists; otherwise we write $\mathfrak{ht} = \infty$).

An ultrafilter is called a *simple p_κ-point*, where κ is a regular uncountable cardinal, if it is generated by a subset of $[\omega]^\omega$ which is well-ordered by $\supseteq^*$ in order type κ. The following result is obtained as a corollary of the previous theorem.

Corollary 2.4. *Suppose that there is a subset $\mathcal{F}$ of $[\omega]^\omega$ of size κ such that $\mathcal{F}$ is well-ordered by $\supseteq^*$ and generates a non-rapid ultrafilter on ω (so $\mathcal{F}$ generates a simple p_κ-point). Then $\mathfrak{ht} \leq \kappa$.*

3 Consequences of the main result

The cardinal $\mathfrak{pp}$, which was introduced in [3], is the smallest cardinal κ for which a simple p_κ-point exists (if such a κ exists; otherwise we write $\mathfrak{pp} = \infty$). Here we introduce more cardinal characteristics.

Definition 3.1. $\mathfrak{u}$(non-rapid) is the smallest cardinality of a subset $\mathcal{F}$ of $[\omega]^\omega$ which generates a non-rapid ultrafilter.

$\mathfrak{pp}$(non-rapid) is the smallest cardinality of a subset $\mathcal{F}$ of $[\omega]^\omega$ which is well-ordered by $\supseteq^*$ and generates a non-rapid ultrafilter (if such a set $\mathcal{F}$ exists; otherwise we write $\mathfrak{pp}(\text{non-rapid}) = \infty$).

Using the above cardinal characteristics, Theorem 2.2 and Corollary 2.4 are represented as follows.

Corollary 3.2. $\mathfrak{hp}' \leq \mathfrak{u}(\text{non-rapid})$ *and* $\mathfrak{ht} \leq \mathfrak{pp}(\text{non-rapid})$.

It is clear that $\mathfrak{u} \leq \mathfrak{pp}$, $\mathfrak{u} \leq \mathfrak{u}(\text{non-rapid})$ and $\mathfrak{pp} \leq \mathfrak{pp}(\text{non-rapid})$. Also it is easily observed that $\mathfrak{u} < \mathfrak{d}$ implies $\mathfrak{u}(\text{non-rapid}) = \mathfrak{u}$, and $\mathfrak{pp} < \mathfrak{d}$ implies $\mathfrak{pp}(\text{non-rapid}) = \mathfrak{pp}$. So we obtain the following result, which partially answers Question 1.2.

Corollary 3.3. *If* $\mathfrak{u} < \mathfrak{d}$, *then* $\mathfrak{hp}' \leq \mathfrak{u}$. *If* $\mathfrak{pp} < \mathfrak{d}$, *then* $\mathfrak{hp}' \leq \mathfrak{pp}$.

It is known that CH implies the existence of a simple $p_{\aleph_1}$-point. Since the

Miller forcing preserves p-points [1, Lemma 7.3.48] and the preservation of p-points is preserved under countable support iteration [1, Theorem 6.2.6], a generating set of a simple $\mathrm{p}_{\aleph_1}$-point in the ground model still generates an ultrafilter in the forcing model by iterated Miller forcing. On the other hand, $\mathfrak{d} = \aleph_2$ holds in the model obtained by a countable support iteration of Miller forcing of length ω_2 over a model for CH. Hence $\mathfrak{pp} < \mathfrak{d}$ is consistent with ZFC.

But the following question is still open.

Question 3.4. $\mathfrak{u}(\text{non-rapid}) = \mathfrak{u}$? $\mathfrak{pp}(\text{non-rapid}) = \mathfrak{pp}$?

In the paper [3], another upper bound for $\mathfrak{hp}'$ is given.

Definition 3.5 ([2, Section 5]). For a function $h \in \omega^\omega$, $\mathfrak{l}_h$ is the smallest size of a subset Φ of $\prod_{n<\omega}[\omega]^{\leq 2^n}$ such that for every $f \in \prod_{n<\omega} h(n)$ there is a $\varphi \in \Phi$ such that $f(n) \in \varphi(n)$ for all but finitely many n. Let $\mathfrak{l} = \sup\{\mathfrak{l}_h : h \in \omega^\omega\}$.

Theorem 3.6 ([3, Theorem 6.11]). $\mathfrak{hp}' \leq \mathfrak{l}$.

Now we can see that the above inequality is consistently strict.

Corollary 3.7. $\mathfrak{hp}' < \mathfrak{l}$ *(moreover,* $\mathfrak{ht} < \mathfrak{l}$*) is consistent with ZFC.*

Proof. We know that there is a proper forcing notion $\mathbb{P}$ which satisfies the following two properties (see Remark 3.8).

- $\mathbb{P}$ preserves p-points.
- In the forcing model by $\mathbb{P}$, for any function $H \in \omega^\omega \cap \mathbf{V}$, there is a function $g \in \prod_{n<\omega} H(n)$ such that, for every function $x \in \prod_{n<\omega} H(n) \cap \mathbf{V}$ there are infinitely many $n < \omega$ with $x(n) = g(n)$, where $\mathbf{V}$ denotes a ground model.

We consider a forcing model obtained by a countable support iteration of alternation of Miller forcing and the above forcing notion $\mathbb{P}$ of length ω_2 over a model for CH.

Since every iterand preserves p-points and the preservation of p-points is preserved under countable support iteration, a generating set of a simple $\mathrm{p}_{\aleph_1}$-point in the ground model still generates an ultrafilter in our forcing model, and so $\mathfrak{pp} = \aleph_1$ holds. On the other hand, it is easily observed that $\mathfrak{d} = \mathfrak{l} = \aleph_2 = \mathfrak{c}$ holds in the same model. By Corollary 3.3, $\aleph_1 = \mathfrak{hp}' = \mathfrak{ht} < \mathfrak{l} = \aleph_2$ holds in this model. □

Remark 3.8. The book [1] tells us in Subsection 7.4.C that the *infinitely equal forcing* $\mathbb{EE}$ meets the requirements which appear in the proof of Corollary 3.7. But Brendle pointed out (in private communication) that $\mathbb{EE}$ does not preserve p-points, and the following "tree-like infinitely equal forcing" $\mathbb{TEE}$ is what we actually need.

$p \in \mathbb{TEE}$ if:

1. p is a subtree of $\bigcup_{m<\omega} \prod_{n<m} 2^n$ without endpoints,
2. there is a $C \in [\omega]^\omega$ such that, for $s \in p$, if $|s| = n \in C$ then $\mathrm{succ}_p(s) = 2^n$,

and $\mathbb{TEE}$ is ordered by inclusion.

Appendix: Ultrafilter number for non-q-points

After the submission of the first version of this article, Blass pointed out that the proof of the main theorem (Theorem 2.2) works under the assumption that $\mathcal{F}$ generates an ultrafilter which is not a q-point.

An ultrafilter $\mathcal{U}$ is called a *q-point* if for any finite-to-one function f with domain ω there is an element X of $\mathcal{U}$ such that $f \restriction X$ is a one-to-one function.

It is easy to see that a q-point is a rapid ultrafilter, so the assumption that $\mathcal{F}$ generates a non-q-point ultrafilter is weaker than that $\mathcal{F}$ generates a non-rapid ultrafilter.

To modify the proof of Theorem 2.2 to fit in the weaker assumption, just take a function π from ω to $\omega \setminus \{0\}$ which witnesses that the ultrafilter generated by $\mathcal{F}$ is not a q-point. Then for any $X \in \mathcal{F}$ there are infinitely many $m \in \omega \setminus \{0\}$ for which $\pi^{-1}(\{m\}) \cap X$ has at least two elements. Define ρ_X for each $X \in \mathcal{F}$ in the same way as the original proof.

Let $\mathfrak{u}$(non-q-point) be the smallest size of a subset $\mathcal{F}$ of $[\omega]^\omega$ which generates a non-q-point ultrafilter. Clearly we have the inequality $\mathfrak{u} \leq \mathfrak{u}$(non-q-point) $\leq \mathfrak{u}$(non-rapid), and so $\mathfrak{u} < \mathfrak{d}$ implies $\mathfrak{u} = \mathfrak{u}$(non-q-point). Now we can refine the first inequality of Corollary 3.2 to the inequality $\mathfrak{hp}' \leq \mathfrak{u}$(non-q-point). Also, instead of the first equality of Question 3.4, we should ask whether $\mathfrak{u}$(non-q-point) $= \mathfrak{u}$ is proved under ZFC.

References

[1] T. Bartoszyński and H. Judah, *Set Theory: On the Structure of the Real Line*, A. K. Peters, Wellesley, Massachusetts, 1995.

[2] M. Kada, *More on Cichoń's diagram and infinite games*, J. Symbolic Logic **65** (2000), 1713–1724.

[3] M. Kada, K. Tomoyasu, and Y. Yoshinobu, *How many miles to $\beta\omega$? — Approximating $\beta\omega$ by metric-dependent compactifications*, Topology Appl. **145** (2004), 277–292.