A generalization of a problem of Fremlin

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1 Fremlin-Miller Covering Principle

The following result is stated in A. Miller [3] as an answer to a question by David Fremlin:

Theorem 1. (Theorem 3.7 in A. Miller [3]) The following holds in the generic extension obtained by adding at least \( \aleph_3 \) Cohen reals to a model of CH:

\[
\text{(1.1) \quad For any family } \mathcal{F} \text{ of Borel sets with } |\mathcal{F}| = \aleph_2 \text{ such that } \bigcap \mathcal{F} = \emptyset, \text{ there is a subfamily } \mathcal{F}' \subseteq \mathcal{F} \text{ with } |\mathcal{F}'| \leq \aleph_1 \text{ such that } \bigcap \mathcal{F}' = \emptyset.
\]

Note that by moving to complements of elements of \( \mathcal{F} \), the assertion (1.1) can be also conceived as a covering property resembling Lindelöf property of topological spaces. Thus we shall call here the property (1.1) the Fremlin-Miller Covering Principle. More generally, for cardinals \( \kappa \geq \lambda \), let us denote with FMCP(\( \kappa, \lambda \)) the following parametrized Fremlin-Miller Covering Principle:

\[
\text{FMCP}(\kappa, \lambda): \quad \text{For any family } \mathcal{F} \text{ of Borel sets with } |\mathcal{F}| < \kappa \text{ such that } \bigcap \mathcal{F} = \emptyset \text{ there is } \mathcal{F}' \in [\mathcal{F}]^{< \lambda} \text{ such that } \bigcap \mathcal{F}' = \emptyset.
\]

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Lemma 2. ([3]) (0) For cardinals $\kappa \geq \kappa' \geq \lambda' \geq \lambda$, FMCP($\kappa, \lambda$) implies FMCP($\kappa', \lambda'$).

(1) FMCP($\kappa, \kappa$) holds for any cardinal $\kappa$.

(2) FMCP($c^+, c$) does not hold.

(3) FMCP($\aleph_2, \aleph_1$) does not hold.

(4) If $\kappa$ is one of $a$, $b$, \ldots or $b^*$ then FMCP($\kappa^+, \kappa$) does not hold.

Proof. (0), (1): Trivial by definition.

(2): Let $\mathcal{A}$ be a maximal almost disjoint family $\subseteq [\omega]^\omega$ of cardinality $c$. For each $a \in \mathcal{A}$, let

$$X_a = \{x \in P(\omega) : x \text{ is almost disjoint from } a\}.$$ Then $X_a \in \text{Borel}(P(\omega))$ for all $a \in \mathcal{A}$ and $\bigcap_{a \in \mathcal{A}} X_a = \emptyset$ by the maximality of $\mathcal{A}$ but $\bigcap_{a \in \mathcal{A'}} X_a \neq \emptyset$ for any $\mathcal{A}' \subsetneq \mathcal{A}$.

(3): Let $(f_\alpha)_{\alpha < \omega_1}, (g_\beta)_{\beta < \omega_1}$ be a Hausdorff gap. For each $\alpha < \omega_1$, let

$$X_\alpha = \{f \in \omega^\omega : f_\alpha \leq^* f \leq^* g_\alpha\}.$$ Then $X_\alpha$'s are Borel sets and $\bigcap_{\alpha < \omega_1} X_\alpha = \emptyset$ but $\bigcap_{\alpha \in I} X_\alpha \neq \emptyset$ for any countable $I \subseteq \omega_1$.

(4): Similarly to (2) and (3).

By Lemma 2, "$\aleph_2 < \kappa \leq c$ and FMCP($\kappa, \aleph_2$)" is the first non-trivial instance of the principle FMCP($\kappa, \lambda$).

It is easy to show that the following principle for cardinals $\kappa \leq \lambda$ is a generalization of the corresponding parametrized Fremlin-Miller Covering Principle:

GFMCP($\kappa, \lambda$): For any projective relation $R \subseteq \mathbb{R}^2$, and $X \in [\mathbb{R}]^{<\kappa}$, if $X$ is unbounded in $\langle \mathbb{R}, R \rangle$, there is $X_0 \in [X]^{<\lambda}$ such that $X_0$ is unbounded in $\langle \mathbb{R}, R \rangle$.

Here we say $X$ is unbounded in $\langle \mathbb{R}, R \rangle$ if

$$\forall r \in \mathbb{R} \exists x \in X - (x R r)$$

holds.

Proposition 3. GFMCP($\kappa, \lambda$) implies FMCP($\kappa, \lambda$) for any cardinals $\kappa \geq \lambda$.

Proof. Assume that GFMCP($\kappa, \lambda$) holds and suppose that $\langle X_\alpha : \alpha < \delta \rangle$ is a sequence of Borel subsets of $\mathbb{R}$ for some $\delta < \kappa$ such that $\bigcap_{\alpha < \delta} X_\alpha = \emptyset$.

For $\alpha < \delta$, let $c_\alpha$ be a Borel code of $X_\alpha$ and let $X^* = \{c_\alpha : \alpha < \delta\}$.

For any $x \in \mathbb{R}$, let
\[(1.2) \quad B_x = \begin{cases} \text{the Borel set coded by } x, & \text{if } x \text{ is a Borel code} \\ \emptyset, & \text{otherwise.} \end{cases} \]

Let \( R \subseteq \mathbb{R}^2 \) be defined by

\[ x R y \iff B_y \text{ is a non empty subset of } B_x \]

for \( x, y \in \mathbb{R} \). The relation \( R \) is easily seen to be \( \Pi^1_1 \). Clearly, we have

\[(1.3) \quad \text{X is unbounded in } \langle \mathbb{R}, R \rangle \iff \bigcap \{B_x : x \in X\} = \emptyset \]

for any \( X \subseteq \mathbb{R} \). In particular, \( X^* \) above is unbounded in \( \langle \mathbb{R}, R \rangle \). By GFMCP(\( \kappa, \lambda \)), there is \( X^{**} \subseteq X^* \) of cardinality \( < \lambda \) such that \( X^{**} \) is already unbounded in \( \langle \mathbb{R}, R \rangle \).

Thus, again by \( (1.3) \), \( \bigcap_{\alpha \in I} X_\alpha = \emptyset \) for \( I = \{\alpha < \delta : c_\alpha \in X^{**}\} \). \( \square \) (Proposition 3)

The proof of Theorem 1 in [3] can be recast to show the following consistency result on GFMCP(\( \kappa, \aleph_2 \)):

**Theorem 4.** Let \( \kappa < \mu \) be regular cardinals. Suppose that \( \mathbb{P}_{\{\alpha\}}, \alpha < \mu \) are posets such that

\[(1.4) \quad \mathbb{P}_{\{\alpha\}} \cong \mathbb{P}_{\{0\}} \text{ for all } \alpha < \mu; \]

\[(1.5) \quad \mathbb{P} = \prod_{\alpha < \mu}^{fin} \mathbb{P}_{\alpha} \text{ satisfies the c.c.c.}; \]

\[(1.6) \quad |\mathbb{P}_{\{0\}}| \leq \kappa = \kappa^{\aleph_0}, \kappa^+ < \mu. \]

Then \( \vdash \text{GFMCP}(\mu, \kappa^+) \).

We shall give the details of the proof of Theorem 4 in the next section.

The formulation of GFMCP(\( \kappa, \aleph_2 \)) has a certain resemblance to that of HP(\( \aleph_2 \)) of J. Brendle and S. Fuchino [1]. This feeling is also supported by the fact that they both hold in Cohen models. The following proposition shows however that these principles are rather independent to each other:

**Proposition 5.** (1) \( c \geq \aleph_3 \land \text{GFMCP}(\kappa, \aleph_2) \land \neg \text{HP}(\aleph_2) \) is consistent.

(2) \( \neg \text{GFMCP}(\aleph_3, \aleph_2) \land \text{HP}(\aleph_2) \) is consistent.

**Proof.** (1): The arguments used in the proof of Theorem 4 are also valid for the generic extension with (measure theoretic) side-by-side product of random forcing. It is known that \( \text{HP}(\aleph_2) \) does not hold in a random extension (see [1]).

(2): In a model of \( \text{HP}(\aleph_2) \land c = \aleph_2 \) we have \( \neg \text{GFMCP}(\aleph_3, \aleph_2) \) by Lemma 2, (2). \( \square \) (Proposition 5)

**Problem 1.** Is \( \neg \text{GFMCP}(\kappa, \aleph_2) \land \text{HP}(\aleph_2) \) consistent under \( c \geq \aleph_3 \)?
2 Proof of the consistency result

In this section we prove Theorem 4.

Let \( \kappa < \mu \) be regular cardinals and \( P(\alpha), \alpha < \mu \) satisfy (1.4), (1.5) and (1.6).

For \( X \subseteq \mu \), we denote
\[
(2.1) \quad P_X = \prod_{\alpha \in X}^{fin} P_\alpha.
\]

Thus \( P = P_\mu \). We assume that finite support product is introduced just as in [1]. In particular, we have \( P_X \leq P_Y \leq P \) for all \( X \subseteq Y \subseteq \mu \).

A bijection \( f : \mu \rightarrow \mu \) induces an automorphism of \( P \) and this induces in turn an automorphism on \( P \)-names. We shall denote both of these automorphisms by \( \tilde{f} \).

All of the following Lemmas 6, 7 and 8 are folklore:

Lemma 6. Suppose that \( X \subseteq \mu \) and \( \dot{x}_\xi, \xi < \delta \) are \( P \)-names of elements of \( H(\aleph_1) \) (in the sense of \( V^P \)) such that \( supp(\dot{x}_\xi) \subseteq X \) for all \( \xi < \delta \). If
\[
(2.2) \quad X \setminus \bigcup \{supp(\dot{x}_\xi) : \xi < \delta\} \text{ is uncountable,}
\]
then we have
\[
(2.3) \quad \models_P " < \langle H(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, \ldots, \epsilon \rangle \triangleright \langle H(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, \ldots, \epsilon \rangle " .
\]

Proof. Suppose that \( p \models " < \langle H(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, \ldots, \epsilon \rangle \models \exists x \varphi(x, \dot{a}_1, \ldots, \dot{a}_n) " \) for a \( \mathcal{L}_{ZF} \)-formula \( \varphi \) and \( P_X \)-names \( \dot{a}_1, \ldots, \dot{a}_n \) of elements of \( H(\aleph_1) \). By the Tarski-Vaught criterion, it is enough to show that
\[
p \models " < \langle H(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, \ldots, \epsilon \rangle \models \varphi(\dot{c}, \dot{a}_1, \ldots, \dot{a}_n) "
\]
for some \( P_X \)-name \( \dot{c} \) of an element of \( H(\aleph_1) \).

By (1.5), we may assume without loss of generality that
\[
(2.4) \quad supp(\dot{a}_1), \ldots, supp(\dot{a}_n) \text{ are all countable.}
\]

By (2.2), we may assume that \( supp(p) \subseteq X \). Let
\[
(2.5) \quad X' = \bigcup \{supp(\dot{x}_\xi) : \xi < \delta\} \cup \bigcup \{supp(\dot{a}_i) : i \in n+1 \setminus 1\} \cup supp(p).
\]

By the assumptions above, we have \( X' \subseteq X \). By (2.2) and (2.4), \( X \setminus X' \) is still uncountable. By Maximal Principle, there is a \( P \)-name \( \dot{b} \) of an element of \( H(\aleph_1) \) such that
\[
p \models " < \langle H(\aleph_1), \{\dot{x}_\xi : \xi < \delta\}, \ldots, \epsilon \rangle \models \varphi(\dot{b}, \dot{a}_1, \ldots, \dot{a}_n) " .
\]
By (1.5), we can find such \( b \) with countable \( \text{supp}(b) \).

Let \( f : \mu \to \mu \) be a bijection such that

\[
 f \upharpoonright X' = id_{X'} \text{ and } f'' \text{supp}(b) \subseteq X.
\]

Let \( \hat{c} = \tilde{f}(\dot{b}) \). Then \( \hat{c} \) is a \( P \)-name and

\[
p \models \varphi(\hat{c}, \dot{a}_1, \ldots, \dot{a}_n).
\]

\[\square \text{ (Lemma 6)}\]

**Lemma 7.** Suppose that \( X \subseteq \mu \), \( \mu \setminus X \) is infinite and \( X_0 \subseteq \mu \setminus X \) is countable. Let \( \dot{x}_\xi \), \( \xi < \delta \) be \( P \)-names of elements of \( \mathcal{H}(\aleph_1) \) (in the sense of \( V^P \)) such that \( \text{supp}({\dot{x}_\xi}) \subseteq X \) for all \( \xi < \delta \).

If \( p \models \varphi(\dot{x}_\xi, \ldots, \dot{a}_n) \) for some \( p \in \mathbb{P}_X \) and \( L_{\text{ZF}} \)-sentence \( \varphi \) then we have \( p \models \exists \dot{\eta}(\dot{x}_\xi, \ldots, \dot{a}_n) \).

Thus we have

\[
p \models \exists \dot{\eta}(\dot{x}_\xi, \ldots, \dot{a}_n).
\]

**Proof.** It is enough to show the following (2.6) for all \( L_{\text{ZF}} \)-formula \( \psi \) by induction on \( \psi \):

(2.6) \( \psi \) For any \( P \)-names \( \dot{a}_1, \ldots, \dot{a}_n \) of elements of \( \mathcal{H}(\aleph_1) \) such that

\[
(2.6a) \text{ supp}(\dot{a}_i) \subseteq X \cup X_0 \text{ for } i \in n + 1 \setminus 1 \text{ and}
\]

\[
(2.6b) X_0 \setminus \{\text{supp} \dot{a}_i : i \in n + 1 \setminus 1\} \text{ is infinite},
\]

if \( q \in \mathbb{P}_{X \cup X_0} \) and \( q \leq_p p \), then

\[
q \models \varphi(\dot{a}_1, \ldots, \dot{a}_n).
\]

if and only if

\[
q \models \exists \dot{\eta}(\dot{x}_\xi, \ldots, \dot{a}_n) \models \psi(\dot{a}_1, \ldots, \dot{a}_n).
\]

The crucial step in the induction proof of (2.6) \( \psi \) is when \( \psi(x_1, \ldots, x_n) \) is of the form \( \exists x \varphi(x, x_1, \ldots, x_n) \).

Suppose that \( \dot{a}_1, \ldots, \dot{a}_n \) are \( P \)-names of elements of \( \mathcal{H}(\aleph_1) \) satisfying (2.6a) and (2.6b), \( q \in \mathbb{P}_{X \cup X_0} \), \( q \leq_p p \) and

\[
q \models \varphi(\dot{a}_1, \ldots, \dot{a}_n).
\]

Then there is a \( P \)-name \( \dot{a} \) of an element of \( \mathcal{H}(\aleph_1) \) such that

\[
q \models \eta(\dot{a}, \dot{a}_1, \ldots, \dot{a}_n).
\]
By (1.5), we may assume that \( \text{supp}(\dot{a}) \) is countable. Let \( f : \mu \rightarrow \mu \) be a bijection such that

\[(2.7) \quad f \upharpoonright X' = id_{X'},\]

where \( X' = X \cup \bigcup \{\text{supp}(\dot{a}_{i}) : i \in n + 1 \setminus 1\} \cup \text{supp}(q) \);

\[(2.8) \quad f''(\text{supp}(r) \cup \text{supp}(\dot{a})) \subseteq X \cup X_{0} \text{ and}\]

\[(2.9) \quad X_{0} \setminus (\bigcup \{\text{supp}(\dot{a}_{i}) : i \in n + 1 \setminus 1\} \cup \text{supp}(\dot{a})) \text{ is infinite.}\]

Then by induction's hypothesis, we have

\[q \models \exists_{\mathcal{P}_{X \cup X_{0}}} \langle \mathcal{H}(\aleph_{1}), \{\dot{x}_{\xi} : \xi < \delta\}, \ldots, \in \rangle \models \eta(f(\dot{a}), \dot{a}_{1}, \ldots, a_{n}).\]

It follows that

\[q \models \exists_{\mathcal{P}_{X \cup X_{0}}} \langle \mathcal{H}(\aleph_{1}), \{\dot{x}_{\xi} : \xi < \delta\}, \ldots, \in \rangle \models \psi(\dot{a}_{1}, \ldots, \dot{a}_{n}).\]

The "only if" direction of this induction step can be shown similarly and more easily. \(\square\) (Lemma 7)

If \( G \) is a \((V, \mathbb{Q})\)-generic set for a poset \( \mathbb{Q} \) and \( M \) is a set, we denote with \( M[G] \) the set \( \{\dot{x}^{G} : \dot{x} \in V^{\mathbb{Q}} \cap M\} \).

**Lemma 8.** Suppose that \( \mathbb{Q} \) is a poset and \( P \in M \prec \mathcal{H}(\theta) \) for sufficiently large regular \( \theta \). If \( G \) is a \((V, \mathbb{Q})\)-generic set then we have

\[(2.10) \quad M[G] \prec \mathcal{H}(\theta)[G].\]

**Proof.** Note that \( \mathcal{H}(\theta)[G] = \mathcal{H}(\theta)^{V[G]} \). We check again the forcing version of Tarski-Vaught criterion.

Suppose that

\[(2.11) \quad p \models_{\mathbb{Q}} "\mathcal{H}(\theta) \models \exists x \varphi(x, \dot{a}_{1}, \ldots, \dot{a}_{n})"\]

for \( \mathcal{L}_{ZF} \)-formula \( \varphi \) and \( \mathbb{Q} \)-names \( \dot{a}_{1}, \ldots, \dot{a}_{n} \) of elements of \( M \). We may assume that \( \dot{a}_{1}, \ldots, \dot{a}_{n} \in M \). (2.11) is equivalent to

\[\mathcal{H}(\theta) \models p \models_{\mathbb{Q}} "\exists x \varphi(x, \dot{a}_{1}, \ldots, \dot{a}_{n})".\]

Then by elementarity we have

\[M \models p \models_{\mathbb{Q}} "\exists x \varphi(x, \dot{a}_{1}, \ldots, \dot{a}_{n})".\]
It follows that there is some \( a \in V^P \cap M \) such that \( M \models p \forces \varphi(\dot{a}, \dot{a}_1, ..., \dot{a}_n) \).

By elementarity of \( M \) this is equivalent to \( \mathcal{H}(\theta) \models p \forces \varphi(\dot{a}, \dot{a}_1, ..., \dot{a}_n) \). This, in turn, is equivalent to \( p \forces \varphi(\dot{a}, \dot{a}_1, ..., \dot{a}_n) \). \( \square \) (Lemma 8)

**Proof of Theorem 4:** Suppose that \( \kappa, \mu, P_{\dot{\alpha}}, \alpha < \mu, P \) are as in Theorem 4, \( p \in P \) and

\[
(2.12) \quad p \forces \{ \dot{x}_\alpha : \alpha < \delta \} \text{ is unbounded in } \mathcal{H}(\aleph_1) \text{ with respect to } R = \{(x, y) : \mathcal{H}(\aleph_1) \models \varphi(x, y, \dot{a}) \}
\]

where \( \delta \leq \kappa \), \( \varphi \) is a \( L_{ZF} \)-formula and \( \dot{a} \) is a \( P \)-name of an element of \( \mathcal{H}(\aleph_1) \).

Let \( X \subseteq \lambda \) be such that \( X \supseteq \{\text{supp}(\dot{x}_\alpha) : \alpha < \delta \} \cup \text{supp}(p) \cup \text{supp}(\dot{a}) \). Then \( |X| < \kappa \) and \( X \setminus \{\text{supp}(\dot{x}_\alpha) : \alpha < \delta \} \) is uncountable.

Let \( G \) be a \( (V, P_X) \)-generic filter with \( p \in G \) and let \( \theta \) be a sufficiently large regular cardinal. By Lemma 7, we have

\[
(2.13) \quad \mathcal{H}(\theta)[G] \models p \forces \{ \dot{x}_\alpha^G : \alpha < \delta \} \text{ is unbounded in } \mathcal{H}(\aleph_1) \text{ with respect to } R''.
\]

Let \( M \prec \mathcal{H}(\theta) \) be such that

\[
(2.14) \quad P, \{ \dot{x}_\alpha : \alpha < \delta \} \in M; \\
(2.15) \quad [M]^\kappa \subseteq M; \text{ and} \\
(2.16) \quad |M| \leq \kappa.
\]

The last two conditions are possible since \( \kappa^\kappa = \kappa \). By Lemma 8, we have

\[
(2.17) \quad M[G] \prec \mathcal{H}(\theta)[G]
\]

and hence

\[
(2.18) \quad M[G] \models P \forces \{ \dot{x}_\alpha^G : \alpha < \delta \} \text{ is unbounded in } \mathcal{H}(\aleph_1) \text{ with respect to } R''.
\]

Note that \( P \) is an element of \( M \) but not \( P_{\mu \setminus Y} \) for \( Y \) as below and thus we cannot apply the elementary submodel argument to the latter poset.

Let \( Y = \delta \cap M \). Since \( |Y| \leq \kappa \) by (2.16), it is enough to show the following claim:

**Claim 8.1.** \( \mathcal{H}(\theta)[G] \models \forces P_{\mu \setminus X} \{ \dot{x}_\alpha^G : \alpha \in Y \} \text{ is unbounded in } \mathcal{H}(\aleph_1) \text{ with respect to } R'' \).

\( \vdash \) In the following we work always in \( \mathcal{H}(\theta)[G] \). Suppose that \( q \in P_{\mu \setminus X} \) and \( \dot{x} \) is a \( P_{\mu \setminus X} \)-name of an element of \( \mathcal{H}(\aleph_1) \). Let \( Z = \text{supp}(\dot{x}) \cup \text{supp}(p) \). Let \( X_0 \in M \) be a countable subset of \( \mu \) disjoint from \( Y \cup Z \). \( f : \mu \setminus X \to \mu \setminus X \) be a bijection such that
(2.19) \( f''Z \subseteq Y \cup X_0 \) and \( f \restriction Y = id_Y \).

Note that \( \tilde{f}(\dot{x}) \) is a \( \mathbb{P}_X \)-name of an element of \( \mathcal{H}(\mathcal{N}_1) \). By (1.5) and (2.15), we may assume that \( \tilde{f}(\dot{x}) \in M \). Also note that \( \mathbb{P}_{X_0} \cong \mathbb{P}_\omega \).

By (2.18), there are \( \tilde{r} \leq \mathbb{P}_{X_0} \tilde{f}(q) \) and \( \alpha^* \in \delta \cap M(= Y) \) such that

(2.20) \( M[G] \models \tilde{r} \models_{\mathbb{P}_X} \neg(\dot{x}_{\alpha^*} \ast \mathcal{R} \tilde{f}(\dot{x})) \).

By (2.17), it follows that \( \tilde{r} \models_{\mathbb{P}_X} \neg(\dot{x}_{\alpha^*} \ast \mathcal{R} \tilde{f}(\dot{x})) \).

By Lemma 6, it follows that

(2.21) \( \tilde{r} \models_{\mathbb{P}_\mu \setminus X} \neg(\dot{x}_{\alpha^*} \ast \mathcal{R} \tilde{f}(\dot{x})) \).

Let \( r = \tilde{f}^{-1}(\tilde{r}) \). Then \( r \leq \mathbb{P}_\mu \setminus X q \). By mapping the parameters in (2.21) by \( \tilde{f}^{-1} \), we obtain

(2.22) \( r \models_{\mathbb{P}_\mu \setminus X} \neg(\dot{x}_{\alpha^*} \ast \mathcal{R} \dot{x}) \).

Since \( q \) and \( \dot{x} \) were arbitrary, it follows that

(2.23) \( \models_{\mathbb{P}_\mu \setminus X} \{\dot{x}_{\alpha^*} : \alpha \in Y\} \) is unbounded in \( \mathcal{H}(\mathcal{N}_1) \) with respect to \( \mathcal{R} \).

\( \vdash \) (Claim 8.1)
\( \Box \) (Theorem 4)

References

